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Relative trace formulas and subconvexity estimates for L-functions of Hilbert modular forms

by

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1. Introduction. In this paper, by the method developed in [21] and [19], we explicitly compute Jacquet's relative trace formula for the toral periods along the diagonal split torus in PGL(2), which encodes the central L-values for the quadratic base change of holomorphic Hilbert cusp forms on its spectral side. By introducing new techniques in a broader setting, we elaborate an explicit relative trace formula partly obtained by Ramakrishnan and Rogawski [16] for the elliptic modular case.

Let F be a totally real number field and $\mathbb{A}$ its adele ring. Let Σ_∞ denote the set of archimedean places of F, and Σ_{fin} the set of finite places of F. We consider a family $l = (l_v)_{v\in\Sigma_\infty}$ of positive even integers, calling it a *weight*. Given a weight l and an integral ideal $\mathfrak{n}$ of F, let $\Pi_{\mathrm{cus}}(l,\mathfrak{n})$ be the set of all irreducible cuspidal representations π of the adele group $\mathrm{PGL}(2,\mathbb{A})$ such that its vth local component π_v is isomorphic to the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ of weight l_v if $v\in\Sigma_\infty$, and has a non-zero vector invariant under the local Hecke congruence subgroup

$$\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathfrak{o}_v) \;\middle|\; c \in \mathfrak{n}\mathfrak{o}_v \right\}$$

if $v\in\Sigma_{\mathrm{fin}}$, where $\mathfrak{o}$ is the maximal order of F, and $\mathfrak{o}_v$ its completion at v.

The standard L-function $L(s,\pi)$ of $\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})$ is defined to be the Euler product of the local factors $L(s,\pi_v)$ over all places v if $\mathrm{Re}(s)$ is sufficiently large. Recall that for a finite place v not dividing the ideal $\mathfrak{n}$,

$$L(s,\pi_v) = (1-q_v^{\nu(\pi_v)/2-s})^{-1}(1-q_v^{-\nu(\pi_v)/2-s})^{-1},$$

where q_v is the order of the residue field at v and $q_v^{\pm\nu(\pi_v)/2}$ is the Satake parameter of π at v. We remark that the L-function in our sense is

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completed by the archimedean local factors $L(s,\pi_v) = \Gamma_{\mathbb{C}}(s + (l_v - 1)/2)$ for $v \in \Sigma_\infty$, and the Euler product with all the gamma factors removed from $L(s,\pi)$ is denoted by $L_{\mathrm{fin}}(s,\pi)$. It is well known that $L(s,\pi)$, originally defined on a half-plane $\mathrm{Re}(s) \gg 0$, has a holomorphic continuation to the whole complex plane with the self-dual functional equation $L(s,\pi) = \epsilon(s,\pi)L(1-s,\pi)$.

The central value $L_{\mathrm{fin}}(1/2,\pi)$ and its twist $L_{\mathrm{fin}}(1/2,\pi\otimes\eta)$ by a real valued idele class character η of $F^\times$ have been studied extensively from several different points of view. For example, when the base field F is $\mathbb{Q}$, Iwaniec and Sarnak [7] announced a number of asymptotic formulas for the first and the second moments of the central L-values $L_{\mathrm{fin}}(1/2,\pi\otimes\eta)$ for $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$ twisted by the Hecke operators and by suitably designed mollifiers. Combining such asymptotic formulas, they proved that among L-functions whose functional equations have even signs, at least $50-\varepsilon$ percent do not vanish at the central point in a quantitative sense for any given $\varepsilon > 0$ if the weight l (or the square-free level $\mathfrak{n}$) is sufficiently large; moreover, they claimed that the quantitative non-vanishing of more than 50 percent of them eliminates the possibility of a Landau–Siegel zero of the quadratic Dirichlet L-function $L(s,\eta)$.

Among several twisted means of L-values considered in [7], one of the most basic means is

$$\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L(1,\pi,\mathrm{Ad})}\alpha(\nu_S(\pi)), \tag{1.1}$$

where S is a finite set of finite places coprime to both $\mathfrak{n}$ and the conductor $\mathfrak{f}$ of η, $\nu_S(\pi) = \{\nu(\pi_v)\}_{v\in S}$ is the collection of the exponents in the Satake parameters of π over S, and $\alpha(\{\nu_v\}_{v\in S})$ is a polynomial in the functions $q_v^{-\nu_v/2} + q_v^{\nu_v/2}$ of the variable ν_v. When $F = \mathbb{Q}$, Ramakrishnan and Rogawski [16] studied the asymptotic behavior of the twisted second moment (1.1) for an odd Dirichlet character η as the level $\mathfrak{n}$, to be kept prime and coprime to $\mathfrak{f}$, grows to infinity when the weight $l \geq 4$ is fixed. In the same setting, Michel and Ramakrishnan [11] obtained an explicit closed formula for the average (1.1), and observed that the formula simplifies significantly in a certain range of the parameters $(\mathfrak{n},\mathfrak{f},\deg(\alpha))$ called the stable range. Later, Feigon and Whitehouse [4] extended the results of [16] and [11] to Hilbert modular forms in a more general setting of [8], but still keeping the square-free condition on the level $\mathfrak{n}$ and the oddness conditions on η at all archimedean places.

In this article, we consider the twisted second moment (1.1) in our general Hilbert modular setting without assuming those conditions on $\mathfrak{n}$ and η, and obtain a formula in a computable form (Theorems 9.1–9.3); thus, we generalize some results of [4] and [11] in several directions.

As a first application of our formula, we prove an equidistribution theorem for the Satake parameters weighted by the central L-values $L(1/2,\pi)L(1/2,\pi\otimes\eta)$ as in [4]; we work with a more general sign condition on η than in [4] at archimedean places, allowing the level $\mathfrak{n}$ to be a general ideal, not necessarily square-free. We remark that a similar asymptotic result for spectral averages of L-values of non-holomorphic modular forms was proved first in [21] for square-free levels, and later in [19] for arbitrary levels.

In what follows, $\mathrm{N}(\mathfrak{n})$ denotes the absolute norm of an ideal $\mathfrak{n}\subset\mathfrak{o}$.

THEOREM 1.1. *Let $l=(l_v)_{v\in\Sigma_\infty}$ be a weight such that $l_v\geq 6$ for all $v\in\Sigma_\infty$. Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$ and $\eta=\bigotimes_v\eta_v$ a quadratic idele class character of $F^\times$ with conductor $\mathfrak{f}$ prime to $\mathfrak{n}$, and let S be a finite set of finite places relatively prime to $\mathfrak{n}\mathfrak{f}$. Assume that η_v is non-trivial for all prime divisors v of $\mathfrak{n}$, and $(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})=1$, where $\epsilon(\eta)$ is the number of $v\in\Sigma_\infty$ such that η_v is non-trivial, and $\tilde{\eta}$ is the character of the group of ideals prime to $\mathfrak{f}$ induced by η. Then, for any even holomorphic function $\alpha(\mathbf{s})$ on the complex manifold $\mathfrak{X}_S=\prod_{v\in S}\bigl(\mathbb{C}/\frac{4\pi i}{\log q_v}\mathbb{Z}\bigr)$, we have the asymptotic formula*

$$\begin{aligned}(2\pi)^{[F:\mathbb{Q}]}\prod_{v\in\Sigma_\infty}\frac{(l_v-2)!}{\{(l_v/2-1)!\}^2}&\\ \times\frac{1}{\mathrm{N}(\mathfrak{n})}\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}&\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}\alpha(\nu_S(\pi))\\ =4D_F^{3/2}\nu(\mathfrak{n})L_{\mathrm{fin}}(1,\eta)&\int_{\mathfrak{X}_S^0}\alpha(\mathbf{s})\,d\mu_S^\eta(\mathbf{s})+\mathcal{O}_{\epsilon,l,\eta,\alpha}(\mathrm{N}(\mathfrak{n})^{-\delta})\end{aligned}$$

for some $\delta>0$; if $\mathfrak{n}$ is restricted to square-free ideals, then the asymptotic formula is true with a smaller error term $\mathcal{O}_{\epsilon,l,\eta,\alpha}(\mathrm{N}(\mathfrak{n})^{-\inf_{v\in\Sigma_\infty}l_v/2+1+\epsilon})$ for any $\epsilon>0$. Here on the right-hand side of the formula,

$$\nu(\mathfrak{n})=\prod_{\substack{v\in\Sigma_{\mathrm{fin}}\\ \mathrm{ord}_v(\mathfrak{n})\geq 3}}(1-q_v^{-2})\prod_{\substack{v\in\Sigma_{\mathrm{fin}}\\ \mathrm{ord}_v(\mathfrak{n})=2}}(1-(q_v^2-q_v)^{-1}),$$

$\mathfrak{X}_S^0$ denotes the purely imaginary locus of $\mathfrak{X}_S$ and $d\mu_S^\eta(i\mathbf{y})=\prod_{v\in S}d\mu_v^{\eta_v}(iy_v)$ with

$$d\mu_v^{\eta_v}(iy_v)=\begin{cases}\dfrac{q_v-1}{(q_v^{1/2}+q_v^{-1/2}-x_v)^2}\,d\mu^{\mathrm{ST}}(x_v) & (\eta_v(\varpi_v)=1),\\ \dfrac{q_v+1}{(q_v^{1/2}+q_v^{-1/2})^2-x_v^2}\,d\mu^{\mathrm{ST}}(x_v) & (\eta_v(\varpi_v)=-1),\end{cases}$$

where $x_v=q_v^{iy_v/2}+q_v^{-iy_v/2}$, $d\mu^{\mathrm{ST}}(x)=\frac{\sqrt{4-x^2}}{2\pi}\,dx$ is the Sato–Tate measure and ϖ_v is a prime element of $\mathfrak{o}_v$. On the left-hand side of the formula,

$\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ denotes the set of $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$ whose conductor $\mathfrak{f}_\pi$ is $\mathfrak{n}$, and $L^{S_\pi}(s,\pi;\mathrm{Ad})$ is the adjoint square L-function of π whose local v-factors are removed for all v belonging to $S_\pi = \{v \in \Sigma_{\mathrm{fin}} \mid \mathrm{ord}_v(\mathfrak{f}_\pi) \geq 2\}$.

We remark that our relative trace formula yields an exact formula for the L-value average (1.1), which reduces to a finite expression for some $(\mathfrak{n},\eta,\alpha)$ (see Corollary 9.4). In our forthcoming paper [20], we show that the error term in Theorem 1.1 can be improved to $\mathcal{O}(\mathrm{N}(\mathfrak{n})^{-1+\epsilon})$. As a corollary to this theorem, we have the following result (cf. [16, Corollary B], [19, Theorem 3]).

COROLLARY 1.2. *Let $l = (l_v)_{v\in\Sigma_\infty}$ be a weight such that $l_v \geq 6$ for all $v \in \Sigma_\infty$. Let η be a quadratic idele class character of $F^\times$ with conductor $\mathfrak{f}$. Let S be a finite set of finite places relatively prime to $\mathfrak{f}$, and $\{J_v\}_{v\in S}$ a collection of subintervals of $[-2,2]$. Given a sequence $\{\mathfrak{n}_k\}_{k\in\mathbb{N}}$ of $\mathfrak{o}$-ideals relatively prime to $\mathfrak{f}$ and S such that $(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}_k) = 1$, $\eta_v(\varpi_v) = -1$ for all prime divisors v of $\mathfrak{n}_k$ and $\lim_{k\to\infty}\mathrm{N}(\mathfrak{n}_k) = \infty$, there exists k_0 with the following property: For any $k \geq k_0$, there exists $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n}_k)$ such that $L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta) \neq 0$ and $q_v^{\nu(\pi_v)/2} + q_v^{-\nu(\pi_v)/2} \in J_v$ for all $v \in S$.*

The so-called *convexity bound* on $L_{\mathrm{fin}}(1/2,\pi)$ is

$$|L_{\mathrm{fin}}(1/2,\pi)| \ll_\epsilon \Big\{\mathrm{N}(\mathfrak{n}) \prod_{v\in\Sigma_\infty} l_v^2\Big\}^{1/4+\epsilon}, \qquad \pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n}),$$

for any $\epsilon > 0$. When $F = \mathbb{Q}$ (so the weight l is a number) and $\mathfrak{n} = \mathbb{Z}$, the bound $|L_{\mathrm{fin}}(1/2,\pi)| \ll_\epsilon l^{1/3+\epsilon}$, which breaks the convexity bound in the weight aspect, has long been known [15], [9]. Thanks to a recent result by Michel and Venkatesh [12], existence of a subconvexity bound for $L_{\mathrm{fin}}(1/2,\pi)$ in any aspect in the general setting is now known; however, beyond its existence, an explicit form of the subconvex exponent is not obvious from their work. As a second application of our formula, we deduce a bound with an explicit subconvex exponent in the *weight aspect* for the L-function $L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta)$ with η an idele class character of $F^\times$ which is odd at all archimedean places, where F is a general totally real number field.

THEOREM 1.3. *Let $l = (l_v)_{v\in\Sigma_\infty}$ be a weight such that $l_v \geq 6$ for all $v \in \Sigma_\infty$. Let $\mathfrak{n}$ be an arbitrary ideal of $\mathfrak{o}$, and η a real valued idele class character of $F^\times$ such that $\eta_v(-1) = -1$ for all $v \in \Sigma_\infty$. Suppose that the conductor $\mathfrak{f}$ of η is relatively prime to $\mathfrak{n}$. Then, for any $\epsilon > 0$,*

$$|L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta)| \ll_\epsilon \mathrm{N}(\mathfrak{f})^{3/4+\epsilon}\,\mathrm{N}(\mathfrak{n})^{1+\epsilon}\Big\{\prod_{v\in\Sigma_\infty} l_v\Big\}^{7/8+\epsilon}$$

with the implied constant independent of $l = (l_v)_{v\in\Sigma_\infty}$, $\mathfrak{n}$, η and $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$.

Theorems 1.1 and 1.3 are deduced from the relative trace formula stated in §9 together with the explicit formulas for local terms given in §10 and §11. Technically speaking, there are substantial differences between [16], [4] and our paper, explained below. In [4], by using the Jacquet–Langlands transfer and some refinements of Waldspurger's period formula, when the character η is odd at all archimedean places, the equidistribution theorem is deduced from the relative trace formula developed in [8], and specified by the authors of [4] for an anisotropic inner form of GL(2) which is certainly an easier place to do analysis than GL(2). By contrast, as in [16], we perform an explicit computation of the relative trace formula on GL(2) for the period along the split torus, which is slightly harder analytically due to the non-compactness of the spaces but much easier algebraically because we only have to consider the Hecke zeta integral on the spectral side. The analytical difficulty can be resolved by the technique developed in [21] (see §6.3 and §6.6; for a different approach, we refer to [16]). For the algebraic aspect, we have an advantage thanks to [18] which completed the computation of local Hecke zeta integrals for local old forms. Due to the direct nature of the method, we can rather easily drop several local and global constraints on automorphic representations and the character η, which is essential to moving to an anisotropic group by the Jacquet–Langlands transfer. For example, the character η is allowed to be trivial in our work. Moreover, the use of "Shintani functions" (see §3.2) in place of the matrix coefficients of discrete series at archimedean places simplifies some computation of archimedean orbital integrals compared with [16]. Similarly, the use of "Green functions" (see §4) at finite places makes it possible for us to compute non-archimedean orbital integrals directly; our result is exhaustive in the sense that it covers not only the unit element of the spherical Hecke algebra but also all of its elements.

This article is organized as follows. After a preliminary section, in §3, we recall the definition of Shintani functions studied in [6] for the symmetric pair $(\mathrm{GL}(2,\mathbb{R}), T)$ with T being the diagonal maximal torus, and prove several of their properties that are necessary later. In §4, we briefly review the Green function on GL(2) over a non-archimedean local field, which was introduced in [21, §5]. Combining these, in §5, we define a left $H_{\mathbb{A}}$-equivariant smooth function on the adele group $\mathrm{GL}(2,\mathbb{A})$ with the "reproducing property" (Lemma 5.5), calling it the adelic Green function. Here H denotes the diagonal maximal split torus of GL(2).

In §6, after reviewing the explicit formulas for the toral period integrals of GL(2) cusp forms with arbitrary level given in [18], we compute the spectral expansion of the automorphic renormalized kernel (5.5), which is constructed by forming the sum of the adelic Green function translated by $H_F \backslash \mathrm{GL}(2,F)$ after a regularization to compensate for $\mathrm{vol}(Z_{\mathbb{A}} H_F \backslash H_{\mathbb{A}}) = \infty$, where Z is the center of GL(2). Although such a regularization is not needed

on the spectral side since (5.5) is cuspidal, the regularization plays a role on the geometric side in §8.

In §7 and §8, closely following [21, §12], we compute the geometric expression of the period integral of the automorphic renormalized kernel. Up to §8, most of the estimates and computations are obtained from the corresponding ones in [21] and [19] by modification at archimedean places; we make the proofs as brief as possible, referring for detailed arguments to our previous works.

In the final formula (Theorem 9.1), two linear functionals $\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l, \mathfrak{n}|\alpha)$ and $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)$ in the test function α arise. We can deduce Theorem 1.1 easily from Theorem 9.1 as explained in §9; the point is to show that the term $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)$ amounts to at most $\mathrm{N}(\mathfrak{n})^{-\delta}$, giving an error term.

The new and essential contribution of this paper to the relative trace formula is probably §10 and §11, devoted to computing the functionals $\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l, \mathfrak{n}|\alpha)$ and $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)$ explicitly for particular but sufficiently general test functions α. For the results, we refer to Theorems 9.2 and 9.3. In the final section §12, we prove Theorem 1.3 by applying the relative trace formula (Theorem 9.1) to a specially chosen test function (see §12.1) originally due to Iwaniec. In the proof, our explicit formula for orbital integrals for arbitrary Hecke functions plays an essential role.

Finally, we mention our work [20] where we obtain an analogue of the results of [17] for the central (derivative of) L-values of Hilbert modular forms; in [20], the explicit relative trace formula to be developed in this article is also indispensable.

Basic notation and convention. Let $\mathbb{N}$ be the set of all positive integers, and we write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. For any condition P, we set $\delta(P) = 1$ if P is true, and $\delta(P) = 0$ if P is false. For any $z \in \mathbb{C}^{\times}$ and $\alpha \in \mathbb{C}$, we define

$$\log z = \log r + i\theta, \qquad z^{\alpha} = \exp(\alpha \log z)$$

for $z = re^{i\theta}$ $(r > 0, \theta \in (-\pi, \pi])$. For a complex function $f(z)$ of $z \in \mathbb{C}$ and for $\sigma \in \mathbb{R}$, the contour integral $\int_{\sigma-i\infty}^{\sigma+i\infty} f(z)\, dz$ along the vertical line $\mathrm{Re}(z) = \sigma$ is sometimes denoted by $\int_{L_\sigma} f(z)\, dz$. We set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Set $1_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, the identity matrix. All the fractional ideals appearing in this paper are supposed to be non-zero.

2. Preliminaries. We now introduce basic objects and notation, which are used throughout this article.

2.1. Let F, $\mathbb{A}$, $\mathfrak{o}$, Σ_{∞}, Σ_{fin} and $\mathfrak{o}_v$, q_v for $v \in \Sigma_{\mathrm{fin}}$ be as in the introduction. Set $\Sigma_F = \Sigma_{\infty} \cup \Sigma_{\mathrm{fin}}$ and $d_F = [F : \mathbb{Q}]$. The finite adele ring of F is denoted by $\mathbb{A}_{\mathrm{fin}}$. For $v \in \Sigma_{\mathrm{fin}}$, F_v denotes the completion of F at v, and we fix a prime element ϖ_v of F_v and set $\mathfrak{p}_v = \varpi_v \mathfrak{o}_v$; the modulus of $F_v^{\times}$ is

denoted by $|\ |_v$ and the associated order function is $\mathrm{ord}_v = -\log_{q_v}|\ |_v$. Let d_v be the local differential exponent of F_v over $\mathbb{Q}_p$, where p is the characteristic of $\mathfrak{o}_v/\mathfrak{p}_v$. The discriminant D_F of $F/\mathbb{Q}$ is defined to be the absolute norm $\mathrm{N}(\mathfrak{d}_{F/\mathbb{Q}})$, where $\mathfrak{d}_{F/\mathbb{Q}}$ is the global different of $F/\mathbb{Q}$. The completed Dedekind zeta function of F is denoted by $\zeta_F(s)$. For any ideal $\mathfrak{a}$ of $\mathfrak{o}$, let $S(\mathfrak{a})$ denote the set of all $v \in \Sigma_{\mathrm{fin}}$ such that $\mathrm{ord}_v(\mathfrak{a}) \geq 1$.

2.2. Let G be the F-algebraic group $\mathrm{GL}(2)$. For any F-subgroup M of G, we set $M_{\mathbb{A}} = M(\mathbb{A})$, $M_F = M(F)$, $M_{\mathrm{fin}} = M(\mathbb{A}_{\mathrm{fin}})$, $M_\infty = M(F \otimes_{\mathbb{Q}} \mathbb{R})$ and $M_v = M(F_v)$ for any $v \in \Sigma_F$. The set G_{fin} of points of finite adeles of G is realized as a restricted direct product of the local groups G_v with respect to the maximal compact subgroups $\mathbf{K}_v = \mathrm{GL}(2, \mathfrak{o}_v)$ over all $v \in \Sigma_{\mathrm{fin}}$. For an ideal $\mathfrak{n}$ of $\mathfrak{o}$, let $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$ be as in the introduction and set $\mathbf{K}_0(\mathfrak{n}) = \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$, which is an open compact subgroup of $\mathbf{K}_{\mathrm{fin}} = \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_v$. The Lie group G_∞ is isomorphic to $\prod_{v\in\Sigma_\infty} G_v$. For each $v \in \Sigma_\infty$, let $\mathbf{K}_v$ be the image of $\mathrm{O}(2,\mathbb{R})$ by the isomorphism $\mathrm{GL}(2,\mathbb{R}) \cong G_v$. Note that $\mathbf{K}_v^0$ is isomorphic to the rotation group $\mathrm{SO}(2,\mathbb{R})$. Set $\mathbf{K}_\infty = \prod_{v\in\Sigma_\infty} \mathbf{K}_v$ and $\mathbf{K} = \mathbf{K}_{\mathrm{fin}}\mathbf{K}_\infty$. Let Z be the center of G, H the F-split torus of G consisting of all the diagonal matrices, and N the F-subgroup of G consisting of all the upper triangular unipotent matrices. Set $B = HN$.

2.3. Haar measures. For $v \in \Sigma_F$, let dx_v be the additive Haar measure on F_v such that $\mathrm{vol}(\mathfrak{o}_v) = q_v^{-d_v/2}$ if $v \in \Sigma_{\mathrm{fin}}$, and $\mathrm{vol}(\{x \in F_v \mid |x|_v < 1\}) = 2$ if $v \in \Sigma_\infty$. Fix a multiplicative Haar measure $d^\times x_v$ on $F_v^\times$ by $d^\times x_v = c_v\, dx_v/|x_v|_v$, where $c_v = 1$ if $v \in \Sigma_\infty$, and $c_v = (1 - q_v^{-1})^{-1}$ if $v \in \Sigma_{\mathrm{fin}}$. We fix a Haar measure on the idele group $\mathbb{A}^\times$ by $d^\times x = \prod_{v\in\Sigma_F} d^\times x_v$. For $y > 0$, let $\underline{y} \in \mathbb{A}^\times$ be the idele such that $\underline{y}_\iota = y^{1/d_F}$ for all $\iota \in \Sigma_\infty$, and $\underline{y}_v = 1$ for all $v \in \Sigma_{\mathrm{fin}}$. Then $y \mapsto \underline{y}$ is a section of the idele norm $|\ |_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_+^\times$, which allows us to identify $\mathbb{A}^\times$ with the direct product of $\{\underline{y} \mid y > 0\}$ and the norm one subgroup $\mathbb{A}^1 = \{x \in \mathbb{A}^\times \mid |x|_{\mathbb{A}} = 1\}$. We fix a Haar measure d^1u on $\mathbb{A}^1$ so that $d^\times x = d^1u\, d^\times y$ when $x = u\underline{y}$ with $x \in \mathbb{A}^\times$, $u \in \mathbb{A}^1$ and $y > 0$.

We fix Haar measures dh_v, dn_v and dk_v on the groups H_v, N_v and $\mathbf{K}_v$ respectively by setting $dh_v = d^\times t_{1,v}\, d^\times t_{2,v}$ if $h_v = \left[\begin{smallmatrix} t_{1,v} & 0 \\ 0 & t_{2,v} \end{smallmatrix}\right]$, $dn_v = dx_v$ if $n_v = \left[\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}\right]$, and by requiring $\mathrm{vol}(\mathbf{K}_v, dk_v) = 1$. Then we normalize the Haar measure $dg_v = dh_v\, dn_v\, dk_v$ on G_v by using the Iwasawa decomposition $G_v = H_vN_v\mathbf{K}_v$. We note that $\mathrm{vol}(\mathbf{K}_v, dg_v) = q_v^{-3d_v/2}$. By taking the tensor product of the measures dg_v on G_v, we fix a Haar measure dg on $G_{\mathbb{A}}$.

Let φ be a smooth function on $G_{\mathbb{A}}$. The right translation of φ by $g \in G_{\mathbb{A}}$ is denoted by $R(g)\varphi$, i.e., $[R(g)\varphi](h) = \varphi(hg)$. The derived action of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)_{\mathbb{C}}$ on smooth functions on $G_{\mathbb{A}}$ is also denoted by R. Let W and

$\overline{W}$ be the element $\frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right]$ of $\mathfrak{sl}_2(\mathbb{C})$ and its complex conjugate, respectively. For any $v \in \Sigma_\infty$, the elements of $\mathrm{Lie}(G_v)_\mathbb{C}$ corresponding to W and $\overline{W}$ are denoted by W_v and $\overline{W}_v$, respectively. For any compactly supported smooth function f on the direct product G_S of $\{G_v\}_{v\in S}$ with a finite subset $S \subset \Sigma_F$, the right translation of φ by f is defined by the convolution $R(f)\varphi(x) = \int_{G_S} \varphi(xg_S)f(g_S)\,dg_S$ for $x \in G_\mathbb{A}$ with respect to the product measure $dg_S = \bigotimes_{v\in S} dg_v$.

2.4. Given a real valued idele class character η of $F^\times$ with conductor $\mathfrak{f}$, we set $f(\eta_v) = \mathrm{ord}_v(\mathfrak{f})$ for $v \in \Sigma_{\mathrm{fin}}$. For any $v \in \Sigma_\infty$, there exists $\epsilon_v \in \{0,1\}$ such that $\eta_v(x) = (x/|x|_v)^{\epsilon_v}$; we call ϵ_v the *sign* of η at v, and set $\epsilon(\eta) = \sum_{v\in\Sigma_\infty} \epsilon_v$. Let $I(\mathfrak{f})$ be the group of fractional ideals relatively prime to $\mathfrak{f}$; then we define a character $\tilde{\eta} : I(\mathfrak{f}) \to \{\pm 1\}$ by setting $\tilde{\eta}(\mathfrak{p}_v \cap \mathfrak{o}) = \eta_v(\varpi_v)$ for any $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f})$. The Gauss sum $\mathcal{G}(\eta)$ for η is defined to be the product of $\mathcal{G}(\eta_v) = \int_{\mathfrak{o}_v^\times} \eta_v(u\varpi_v^{-d_v-f(\eta_v)})\psi_{F,v}(u\varpi_v^{-d_v-f(\eta_v)})\,d^\times u$ over all $v \in \Sigma_{\mathrm{fin}}$, where $\psi_F = \psi_\mathbb{Q} \circ \mathrm{tr}_{F/\mathbb{Q}}$ with $\psi_\mathbb{Q}$ being the character of $\mathbb{Q}\backslash\mathbb{A}$ such that $\psi_\mathbb{Q}(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$.

2.5. Fix a relatively compact subset ω_B of $B^1_\mathbb{A} = \left\{\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mid a, d \in \mathbb{A}^1, b \in \mathbb{A}\right\}$ such that $B^1_\mathbb{A} = B_F\omega_B$. Let $\mathfrak{S}^1 = \omega_B\left\{\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] \mid t > 0, t^2 > c\right\}\mathbf{K}$ with some $c > 0$ be a Siegel domain such that $G_\mathbb{A} = Z_\mathbb{A}G_F\mathfrak{S}^1$. Define $y : G_\mathbb{A} \to \mathbb{R}^\times_+$ by setting $y\left(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]k\right) = |a/d|_\mathbb{A}$ for any $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in B_\mathbb{A}$ and $k \in \mathbf{K}$. Set $G^1_\mathbb{A} = \{g \in G_\mathbb{A} \mid \det g \in \mathbb{A}^1\}$.

3. Holomorphic Shintani functions on $\mathrm{GL}(2,\mathbb{R})$.

Consider the following one-parameter subgroups in $\mathrm{GL}(2,\mathbb{R})$:

$$k_\theta = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right], \qquad a_r = \left[\begin{smallmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{smallmatrix}\right],$$

where $\theta, r \in \mathbb{R}$. We have $\mathrm{SO}(2,\mathbb{R}) = \{k_\theta \mid \theta \in \mathbb{R}\}$.

3.1. Discrete series of $\mathrm{PGL}(2,\mathbb{R})$. For $n \in \mathbb{Z}$, let τ_n be the character of $\mathrm{SO}(2,\mathbb{R})$ defined by

$$\tau_n(k_\theta) = e^{in\theta}, \qquad \theta \in \mathbb{R}.$$

Let $l \geq 2$ be an even integer. There are discrete series representations D^+_l and D^-_l of $\mathrm{SL}(2,\mathbb{R})$ such that $D^\pm_l|\mathrm{SO}(2,\mathbb{R})$ is the direct sum of the characters τ_n for all $n \in \pm(l + 2\mathbb{N}_0)$. We have a unitary representation D_l of $\mathrm{GL}(2,\mathbb{R})$ such that (a) D_l has the trivial central character and (b) $D_l|\mathrm{SL}(2,\mathbb{R}) = D^+_l \oplus D^-_l$. We call D_l the *discrete series representation of* $\mathrm{PGL}(2,\mathbb{R})$ *of minimal* $\mathrm{SO}(2,\mathbb{R})$*-type* l.

3.2. Shintani functions. Let $f(\tau)$ be a cusp form on the upper half-plane satisfying the modularity condition $f((a\tau+b)/(c\tau+d)) = (c\tau+d)^l f(\tau)$

for any matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in a fixed congruence subgroup Γ of $\mathrm{PSL}(2,\mathbb{Z})$. It is lifted to a left Γ-invariant function $\tilde{f}$ on the group $\mathrm{GL}(2,\mathbb{R})$ by setting

$$\tilde{f}(g) = (\det g)^{l/2}(ci+d)^{-l} f\left(\frac{ai+b}{ci+d}\right) \times \delta(\det g > 0), \quad g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R}).$$

Let $\tilde{f}_c$ be the complex conjugate of $\tilde{f}$. Then $\tilde{f}_c$ satisfies the conditions

$$\tilde{f}_c(gk_\theta) = \tau_l(k_\theta)\, \tilde{f}_c(g) \quad (\forall k_\theta \in \mathrm{SO}(2,\mathbb{R})), \quad [R(\overline{W})\tilde{f}_c](g) = 0.$$

Since $\mathrm{Ad}(k_\theta)\overline{W} = e^{-2i\theta}\,\overline{W}$ in any $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2,\mathbb{R}))$-module (π, V), we have $\pi(\overline{W})V[\tau_l] \subset V[\tau_{l-2}]$, where

$$V[\tau_l] = \{v \in V \mid \pi(k_\theta)v = e^{il\theta}v \ (\forall k_\theta \in \mathrm{SO}(2,\mathbb{R}))\}.$$

Let V be the $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2,\mathbb{R}))$-submodule of the regular representation $L^2(\Gamma\backslash \mathrm{GL}(2,\mathbb{R}))$ generated by $\tilde{f}_c$. Then the condition above, or equivalently $\tilde{f}_c \in V[\tau_l]$ and $R(\overline{W})\tilde{f}_c = 0$, tells us that inside the module V (which is a finite sum of discrete series D_l) the vector $\tilde{f}_c$ is extremal. For $z \in \mathbb{C}$, let χ_z be the quasi-character of the diagonal split torus T defined by $\chi_z\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]\right) = |t_1/t_2|^z$. The integral

$$\phi(g) = \int\limits_{\Gamma\cap T\backslash T} \tilde{f}_c(hg)\chi_{-z}(h)\, dh, \quad g \in \mathrm{GL}(2,\mathbb{R}),$$

often called the (T,χ_z)-*period integral* of $\tilde{f}_c$, satisfies the following two conditions:

- $\phi\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g k_\theta\right) = |t_1/t_2|^z \tau_l(k_\theta)\phi(g)$ for all $\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in T$ and $\theta \in \mathbb{R}$,
- $R(\overline{W})\phi = 0$.

A function having these properties is called a *holomorphic Shintani function* of weight l. The next proposition tells us that these conditions determine the function $\phi(g)$ uniquely up to a constant multiple.

Proposition 3.1 ([6, Proposition 5.3]). *Let $z \in \mathbb{C}$. For each even integer $l \geq 2$, there exists a unique $\mathbb{C}$-valued C^∞-function $\Psi^{(z)}(l;-)$ on $\mathrm{GL}(2,\mathbb{R})$ with the properties:*

(S-i) *It satisfies the equivariance condition*

$$\Psi^{(z)}\left(l; \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g k_\theta\right) = |t_1/t_2|^z \tau_l(k_\theta)\Psi^{(z)}(l;g) \quad \text{for all } \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in T \text{ and } \theta \in \mathbb{R}.$$

(S-ii) *It satisfies the differential equation*

$$R(\overline{W})\Psi^{(z)}(l;-) = 0.$$

(S-iii) $\Psi^{(z)}(l;1_2) = 1$.

We have the explicit formula

$$\Psi^{(z)}(l;a_r) = 2^{-l/2}(-y)^{(2z-l)/4}(1-y)^{l/2} \quad \textit{with} \quad y = \left(\frac{e^{2r}-i}{e^{2r}+i}\right)^2.$$

We remark that the function $\Psi^{(z)}(l;-)$ is characterized by its restriction to the group $A = \{a_r \mid r \in \mathbb{R}\}$ due to property (S-i) and the decomposition $\mathrm{GL}(2,\mathbb{R}) = TA\,\mathrm{SO}(2,\mathbb{R})$ (cf. [6, Lemma 3.1]).

LEMMA 3.2. *Let $\Psi^{(z)}(l;-)$ be as in Proposition* 3.1. *Then*

$$\Psi^{(z)}\big(l;\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]\big) = (1+ix)^{z-l/2}, \qquad x \in \mathbb{R}.$$

Proof. By a direct computation, $\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] = \left[\begin{smallmatrix}t & 0\\ 0 & t^{-1}\end{smallmatrix}\right] a_r k_\theta$ with

$$t = (1+x^2)^{1/4}, \qquad \cosh 2r = (1+x^2)^{1/2}, \qquad \sinh 2r = x,$$

$$e^{i\theta} = \frac{(\sqrt{1+x^2}+1)^{1/2}}{\sqrt{2}\,(1+x^2)^{1/4}}\left(1 - \frac{ix}{\sqrt{1+x^2}+1}\right),$$

and $y = \frac{x-i}{x+i}$, $1-y = \frac{2i}{x+i}$. Using these, we obtain the formula by a direct computation. ∎

LEMMA 3.3. *We have the estimate*

$$\big|\Psi^{(z)}\big(l;\left[\begin{smallmatrix}t_1 & 0\\ 0 & t_2\end{smallmatrix}\right] a_r k\big)\big| \le 2^{-l/2}|t_1/t_2|^{\mathrm{Re}(z)} e^{\pi|\mathrm{Im}(z)|/2}(\cosh 2r)^{-l/2}$$

for any $t_1, t_2 \in \mathbb{R}^\times$, $r \in \mathbb{R}$ and $k \in \mathrm{SO}(2,\mathbb{R})$.

Proof. Set $y = \big(\frac{e^{2r}-i}{e^{2r}+i}\big)^2$. Then

$$y = \left(\tanh 2r - \frac{i}{\cosh 2r}\right)^2 = 1 - \frac{2}{(\cosh 2r)^2} - \frac{2i\tanh 2r}{\cosh 2r}.$$

By a direct computation, we obtain $|1-y| = (\cosh 2r)^{-1}$. Furthermore, since $|y| = 1$, we have $|(-y)^{(2z-l)/4}| \le e^{\pi|\mathrm{Im}(z)|/2}$. This completes the proof. ∎

3.3. An inner-product formula for Shintani functions. For an even integer $l \ge 2$ and $z \in \mathbb{C}$, let us consider the integral

$$C_l(z) = \int_1^\infty \left\{\left(-\left(\frac{u-i}{u+i}\right)^2\right)^z + \left(-\left(\frac{u+i}{u-i}\right)^2\right)^z\right\}(1+u^2)^{1-l}u^{l-2}\,du.$$

LEMMA 3.4. *The integral $C_l(z)$ converges absolutely. It has the following properties:*

(i) *The function $z \mapsto C_l(z)$ is entire and satisfies the functional equation*

$$C_l(-z) = C_l(z).$$

(ii) *The value at $z = 0$ is given by*

$$C_l(0) = 2^{-1}\Gamma((l-1)/2)^2\Gamma(l-1)^{-1} = 2^{3-2l}\pi\Gamma(l-1)\Gamma(l/2)^{-2}.$$

(iii) *We have*

$$|C_l(z)| \le C_l(0)\exp(\pi|\mathrm{Im}(z)|), \qquad z \in \mathbb{C}.$$

Proof. By the change of variable $v^{-1} = 1 + u^2$, we have

$$\begin{aligned} C_l(0) &= 2\int_1^\infty (1+u^2)^{1-l}u^{l-2}\,du = 2^{-1}\int_0^1 (1-v)^{(l-3)/2}v^{(l-3)/2}\,dv \\ &= 2^{-1}\Gamma((l-1)/2)^2\Gamma(l-1)^{-1}, \end{aligned}$$

as desired in (ii). Note that the second equality in (ii) is obtained by the duplication formula. Since $w = -((u-i)/(u+i))^2$ satisfies $|w| = 1$, by definition, we have $w^z = \exp(i\theta z)$ with $\theta \in (-\pi, \pi]$. Thus, $|w^z| = \exp(-\mathrm{Im}(z)\theta)$ $\le \exp(\pi|\mathrm{Im}\, z|)$, from which (iii) is immediate. By definition, we have $w^{-z} = (w^{-1})^z$, which yields the functional equation in (i). ∎

The inner product of the Shintani functions $\Psi^{(z)}(l;-)$ and $\Psi^{(-\bar{z})}(l;-)$ is as follows.

Proposition 3.5. *We have*

$$\int_{T\backslash \mathrm{GL}(2,\mathbb{R})} \Psi^{(z)}(l;g)\,\overline{\Psi^{(-\bar{z})}(l;g)}\,dg = 2^{l-1}C_l(z).$$

Proof. Set $f(g) = \Psi^{(z)}(l;g)\,\overline{\Psi^{(-\bar{z})}(l;g)}$. We have

$$\int_{T\backslash \mathrm{GL}(2,\mathbb{R})} f(g)\,dg = 2\int_{\mathbb{R}} f(a_r)\cosh 2r\,dr$$

by [21, (3.3)], which is checked by computing the Jacobian of the coordinate transform from the Iwasawa decomposition to the decomposition $\mathrm{GL}(2,\mathbb{R}) = TA\,\mathrm{SO}(2)$. From Proposition 3.1, $f(a_r) = 2^{-l}(-y)^{-l/2+z}(1-y)^l$ with $y = \left(\frac{e^{2r}-i}{e^{2r}+i}\right)^2$. Using this, we compute

$$\begin{aligned} 2\int_0^\infty f(a_r)\cosh 2r\,dr &= 2^{1-l}\int_0^\infty (-y)^{-l/2+z}(1-y)^l\cosh 2r\,dr \\ &= 2^{l-1}\int_1^\infty \left\{-\left(\frac{u-i}{u+i}\right)^2\right\}^z (1+u^2)^{1-l}u^{l-2}\,du, \end{aligned}$$

by setting $u = e^{2r}$. In the same way, we obtain

$$2\int_{-\infty}^0 f(a_r)\cosh 2r\,dr = 2^{l-1}\int_1^\infty \left\{-\left(\frac{u+i}{u-i}\right)^2\right\}^z (1+u^2)^{1-l}u^{l-2}\,du. \ ∎$$

3.4. Orbital integrals of Shintani functions. Set $w_0 = k_{\pi/2} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

LEMMA 3.6. *If* $0 < \mathrm{Re}(z) < l/2$, *then, for* $\epsilon, \epsilon' \in \{0, 1\}$,

$$\int_{\mathbb{R}^\times} \Psi^{(0)}\big(l; \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] w_0^{\epsilon'}\big)|x|^z \,\mathrm{sgn}^\epsilon(x)\, d^\times x = 2i^{l\epsilon'}\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1} i^\epsilon \cos\big(\tfrac{\pi}{2}(z+\epsilon)\big),$$

$$\int_{\mathbb{R}^\times} \Psi^{(0)}\big(l; \left[\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right] w_0^{\epsilon'}\big)|x|^z \,\mathrm{sgn}^\epsilon(x)\, d^\times x = 2i^{l\epsilon'}\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1} (-i)^\epsilon \cos\big(\tfrac{\pi}{2}(z+\epsilon)\big).$$

Proof. Let $J_{l,\epsilon}(z)$ denote the first integral with $\epsilon' = 0$. From Lemma 3.2, we have $J_{l,\epsilon}(z) = J_l^+(z) + (-1)^\epsilon J_l^-(z)$ with $J_l^\pm(z) = \int_0^\infty (1 \pm ix)^{-l/2} x^z \, d^\times x$. By [5, 3.194.3],

$$\begin{aligned} J_l^\pm(z) &= (\pm i)^{-z} B(z, l/2-z) \\ &= (\pm i)^{-z}\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1} \quad (l/2 > \mathrm{Re}(z) > 0). \end{aligned}$$

Hence, using the relation $i^{-z} + (-1)^\epsilon(-i)^{-z} = 2i^\epsilon \cos(\pi(z+\epsilon)/2)$, we get

$$\begin{aligned} J_{l,\epsilon}(z) &= \Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1}\{i^{-z} + (-1)^\epsilon(-i)^{-z}\} \\ &= 2i^\epsilon \cos(\pi(z+\epsilon)/2)\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1}. \end{aligned}$$

We have the Iwasawa decomposition $\left[\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1/\sqrt{1+x^2} & 0 \\ 0 & \sqrt{1+x^2} \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] k_\theta$ with $e^{i\theta} = \frac{1+ix}{\sqrt{1+x^2}}$. Hence, by Lemma 3.2,

$$\Psi^{(z)}\big(l; \left[\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right]\big) = \left(\frac{1}{1+x^2}\right)^z \times \left(\frac{1+ix}{\sqrt{1+x^2}}\right)^l \times (1+ix)^{z-l/2} = (1-ix)^{-z-l/2}.$$

Using this formula, in the same way as above, we can prove the second formula with $\epsilon' = 0$. The remaining two formulas follow immediately from the proved ones and the relation $\Psi^{(0)}(l; gw_0) = i^l \Psi^{(0)}(l; g)$. ∎

4. Green's functions on $\mathrm{GL}(2)$ **over non-archimedean local fields.** This section is a review of results in [21, §5]. We fix a place $v \in \Sigma_{\mathrm{fin}}$. For $z \in \mathbb{C}$, there exists a unique function $\Phi_{0,v}^{(z)} : G_v \to \mathbb{C}$ such that

$$\Phi_{0,v}^{(z)}\big(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] k\big) = |t_1/t_2|_v^z \delta(x \in \mathfrak{o}_v), \quad \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in H_v,\ \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \in N_v,\ k \in \mathbf{K}_v. \tag{4.1}$$

Given $z \in \mathbb{C}$ and $s \in \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$, we consider the inhomogeneous equation

$$R\big(\mathbb{T}_v - (q_v^{(1-s)/2} + q_v^{(1+s)/2}) 1_{\mathbf{K}_v}\big)\Psi = \Phi_{0,v}^{(z)} \tag{4.2}$$

with the unknown function $\Psi : G_v \to \mathbb{C}$ possessing the $(H_v, \mathbf{K}_v)$-equivariance

$$\Psi\big(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] gk\big) = |t_1/t_2|_v^z \Psi(g), \qquad \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in H_v,\ k \in \mathbf{K}_v. \tag{4.3}$$

Here $\mathbb{T}_v$ and $1_{\mathbf{K}_v}$ are elements of the spherical Hecke algebra $\mathcal{H}(G_v, \mathbf{K}_v)$ defined by

$$\mathbb{T}_v = \mathrm{vol}(\mathbf{K}_v, dg_v)^{-1} \,\mathrm{ch}_{\mathbf{K}_v \left[\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right] \mathbf{K}_v}, \qquad 1_{\mathbf{K}_v} = \mathrm{vol}(\mathbf{K}_v, dg_v)^{-1} \,\mathrm{ch}_{\mathbf{K}_v}.$$

We note that $\mathrm{vol}(\mathbf{K}_v, dg_v) = q_v^{-3d_v/2}$ (see §2.3).

Lemma 4.1 ([21, Lemma 5.2]). *Suppose* $\mathrm{Re}(s) > |2\,\mathrm{Re}(z) - 1|$. *Then there exists a unique bounded function* $\Psi_v^{(z)}(s; -) : G_v \to \mathbb{C}$ *satisfying* (4.2) *and* (4.3), *whose values on* N_v *are given by*

$$\Psi_v^{(z)}\big(s; \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\big) = -q_v^{-(s+1)/2}(1 - q_v^{-(s-2z+1)/2})^{-1}(1 - q_v^{-(s+2z+1)/2})^{-1} \\ \times \max(1, |x|_v)^{-(s-2z+1)/2}, \qquad x \in F_v.$$

Proof. We review the proof of [21, Lemmas 5.1 and 5.2]. By the decomposition $G_v = \coprod_{m \geq 0} H_v \mathrm{n}_m \mathbf{K}_v$ with $\mathrm{n}_m = \left[\begin{smallmatrix} 1 & \varpi_v^{-m} \\ 0 & 1 \end{smallmatrix}\right]$, the condition (4.3) implies that a function Ψ satisfying (4.2) and (4.3) is determined by the system of numbers $a(m) = \Psi(\mathrm{n}_m)$, $m \geq 0$. The equation (4.2) yields a recurrence relation among $a(m-1)$, $a(m)$ and $a(m+1)$. By solving it, we are done. ∎

The following lemma is necessary in the proof of Proposition 5.3.

Lemma 4.2 ([21, Lemma 5.4]). *Let* $f : G_v \to \mathbb{C}$ *be a smooth function such that* $f\big(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] gk\big) = |t_1/t_2|_v^{-z} f(g)$ *for any* $t_1, t_2 \in F_v^\times$ *and* $k \in \mathbf{K}_v$. *Then*

$$\int_{H_v \backslash G_v} \Psi_v^{(z)}(s; g)\big[R\big(\mathbb{T}_v - (q_v^{(1+s)/2} + q_v^{(1-s)/2}) 1_{\mathbf{K}_v}\big) f\big](g)\, dg \\ = \mathrm{vol}(H_v \backslash H_v \mathbf{K}_v) f(1_2) \tag{4.4}$$

as long as the integral on the left-hand side converges absolutely.

Proof. We review the proof of [21, Lemma 5.4]. On the left-hand side of (4.4), we move the operator $R(\mathbb{T}_v - (q_v^{(1+s)/2} + q_v^{(1-s)/2}) 1_{\mathbf{K}_v})$ applied to f in front of $\Psi_v^{(z)}$ by a simple change of variable; then due to (4.2), we have

$$\int_{H_v \backslash G_v} \big[R\big(\mathbb{T}_v - (q_v^{(1+s)/2} + q_v^{(1-s)/2}) 1_{\mathbf{K}_v}\big) \Psi_v^{(z)}(s)\big](g) f(g)\, dg \\ = \int_{H_v \backslash G_v} \Phi_{0,v}^{(z)}(g) f(g)\, dg,$$

where the right-hand side equals $\mathrm{vol}(H_v \backslash H_v \mathbf{K}_v) f(1_2)$ by (4.1). ∎

5. Automorphic Green functions. Let $S \subset \Sigma_{\mathrm{fin}}$ be a finite subset. Set

$$\mathfrak{X}_S = \prod_{v \in S} (\mathbb{C}/4\pi i (\log q_v)^{-1}\mathbb{Z}),$$

which we regard as a complex manifold in the obvious way. Note that for any $\mathbf{c} \in \mathbb{R}^S$, the slice $\mathbb{L}_S(\mathbf{c}) = \{\mathbf{s} \in \mathfrak{X}_S \mid \mathrm{Re}(\mathbf{s}) = \mathbf{c}\}$ is a compact set homeomorphic to the torus $(\mathbf{S}^1)^S$.

Given $\mathbf{s} \in \mathfrak{X}_S$, $z \in \mathbb{C}$, an ideal $\mathfrak{n} \subset \mathfrak{o}$ such that $S(\mathfrak{n}) \cap S = \emptyset$, and a family $l = (l_v)_{v \in \Sigma_\infty} \in (2\mathbb{Z}_{\geq 2})^{\Sigma_\infty}$, the *adelic Green function* $\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}, -)$ is defined by

$$\begin{aligned}\Psi_l^{(z)}&(\mathfrak{n}|\mathbf{s}; g)\\ &:= \prod_{v \in \Sigma_\infty} \Psi_v^{(z)}(l_v; g_v) \prod_{v \in S} \Psi_v^{(z)}(s_v; g_v) \prod_{v \in S(\mathfrak{n})} \Phi_{\mathfrak{n},v}^{(z)}(g_v) \prod_{v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))} \Phi_{0,v}^{(z)}(g_v)\end{aligned}$$

for any $g = (g_v)_{v \in \Sigma_F} \in G_{\mathbb{A}}$, where $\Psi_v^{(z)}(l_v; -)$ for $v \in \Sigma_\infty$ is the holomorphic Shintani function on $G_v \cong \mathrm{GL}(2, \mathbb{R})$ defined in Proposition 3.1, $\Psi_v^{(z)}(s_v; -)$ for $v \in S$ is the Green function recalled in §4, and for any $v \in \Sigma_{\mathrm{fin}}$, we set

$$\Phi_{\mathfrak{n},v}^{(z)}\big(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]k\big) = |t_1/t_2|_v^z \delta(x \in \mathfrak{o}_v)\delta(k \in \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)),$$
$$t_1, t_2 \in F_v^\times,\ x \in F_v,\ k \in \mathbf{K}_v.$$

We remark that $\Phi_{\mathfrak{n},v}^{(z)} = \Phi_{0,v}^{(z)}$ if $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})$. It is easy to check that the adelic Green function $\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; -)$ is a smooth function on $G_{\mathbb{A}}$ having the equivariance

$$\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; hgk_\infty k_{\mathrm{fin}}) = \Big\{ \prod_{v \in \Sigma_\infty} \tau_{l_v}(k_v) \Big\} \chi_z(h) \Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; g), \quad g \in G_{\mathbb{A}},$$

for any $h \in H_{\mathbb{A}}$, $k_\infty = (k_v)_{v \in \Sigma_\infty} \in \mathbf{K}_\infty^0$ and $k_{\mathrm{fin}} \in \mathbf{K}_0(\mathfrak{n})$, where $\chi_z : H_F \backslash H_{\mathbb{A}} \to \mathbb{C}^\times$ is the quasi-character defined by

$$\chi_z\big(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]\big) = |t_1/t_2|_{\mathbb{A}}^z, \quad t_1, t_2 \in \mathbb{A}^\times.$$

To state the most important property of the adelic Green functions, we introduce the (H, χ_z)-period integral of a cusp form $\varphi \in C^\infty(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})$ by setting

$$\varphi^{H,(z)}(g) = \int_{Z_{\mathbb{A}} H_F \backslash H_{\mathbb{A}}} \varphi(hg) \chi_z(h)\, dh, \quad g \in G_{\mathbb{A}},\ z \in \mathbb{C}.$$

Let $\|g\|_{\mathbb{A}}$ denote the height of $g \in G_{\mathbb{A}}$ (see [21, §3.2]).

Lemma 5.1. *The integral $\varphi^{H,(z)}(g)$ converges absolutely and satisfies*

$$\varphi^{H,(z)}(hg) = \chi_z(h)^{-1} \varphi^{H,(z)}(g), \quad h \in H_{\mathbb{A}}.$$

For any $\epsilon > 0$, $|\varphi^{H,(z)}(g)| \ll_{\varphi,\epsilon} \|g\|_{\mathbb{A}}^{2|\mathrm{Re}(z)|+\epsilon}$ on $G_{\mathbb{A}}^1$.

Proof. From [2, Theorem 2.14], there exists $f \in C_{\mathrm{c}}^{\infty}(G_{\mathbb{A}})$ such that $\varphi = R(f)\varphi$. Hence

$$\varphi(g) = \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \varphi(x)K_x(g)\,dx \quad \text{with} \quad K_x(g) = \sum_{\gamma\in Z_F\backslash G_F} \int_{Z_{\mathbb{A}}} f(zg^{-1}\gamma x)\,dz.$$

Let $\epsilon > 0$. We shall prove the estimate

$$\int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}} |K_x(hg)\chi_z(h)|\,dh \ll_{\varphi,\epsilon} \|g\|_{\mathbb{A}}^{2|\mathrm{Re}(z)|+\epsilon}\|x\|_{\mathbb{A}}^{2|\mathrm{Re}(z)|+\epsilon}, \qquad g, x \in G_{\mathbb{A}}^1.$$

Take a compact set $\omega \subset G_{\mathbb{A}}^1$ such that $\mathrm{supp}(f) \subset Z_{\mathbb{A}}\omega$. Suppose $K_x(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] g) \neq 0$ for $t > 0$ and $g, x \in G_{\mathbb{A}}^1$. Since $\mathrm{supp}(K_x) \subset G_F x Z_{\mathbb{A}}(\mathrm{supp}\, f)^{-1}$, there exist $\gamma \in G_F$, $z \in Z_{\mathbb{A}}$ and $u \in \omega$ such that $\left[\begin{smallmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{smallmatrix}\right] g = z\gamma x u$. Let $\|\xi\|_{\mathbb{A}}$ denote the height of a primitive element $\xi \in \mathbb{A}^2$ introduced in [21, p. 27]. By noting $z \in G_{\mathbb{A}}^1$, we have

$$\begin{aligned} t^{-1/2}\big\|g^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\big\|_{\mathbb{A}} &= \big\|u^{-1}x^{-1}\gamma^{-1}z^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\big\|_{\mathbb{A}} \asymp_f \big\|x^{-1}\gamma^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\big\|_{\mathbb{A}} \\ &\geq \|x\|_{\mathbb{A}}^{-1}\big\|\gamma^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\big\|_{\mathbb{A}} \gg \|x\|_{\mathbb{A}}^{-1}, \end{aligned}$$

and hence $t^{1/2} \ll_f \big\|g^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\big\|_{\mathbb{A}}\|x\|_{\mathbb{A}} \ll \|g\|_{\mathbb{A}}\|x\|_{\mathbb{A}}$. Thus we are done by replacing $\|g\|_{\mathbb{A}}$ with $\|g\|_{\mathbb{A}}\|x\|_{\mathbb{A}}$ in the proof of [21, Lemma 6.2]. Consequently,

$$\begin{aligned} \int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}} |\varphi(hg)\chi_z(h)|\,dh &\leq \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} |\varphi(x)|\Big(\int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}} |K_x(hg)\chi_z(h)|\,dh\Big)\,dx \\ &\ll_{f,\epsilon} \|g\|_{\mathbb{A}}^{2|\mathrm{Re}(z)|+\epsilon} \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} |\varphi(x)|\,\|x\|_{\mathbb{A}}^{2|\mathrm{Re}(z)|+\epsilon}\,dx. \end{aligned}$$

The last integral is convergent by the cuspidality of φ. ∎

Let $\mathcal{A}_{\mathrm{cus}}(G;\tau_l)$ be the space of cusp forms $\varphi \in C^{\infty}(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ such that for all $v \in \Sigma_{\infty}$, $R(\overline{W}_v)\varphi = 0$ and $R(k_v)\varphi = \tau_{l_v}(k_v)\varphi$ for all $k_v \in \mathbf{K}_v^0$.

Lemma 5.2. *Suppose $\varphi \in \mathcal{A}_{\mathrm{cus}}(G;\tau_l)$. Then*

$$\overline{\varphi}^{H,(z)}(g_{\mathrm{fin}}g_{\infty}) = \Big\{\prod_{v\in\Sigma_{\infty}} \overline{\Psi_v^{(-\bar{z})}(l_v, g_v)}\Big\}\overline{\varphi}^{H,(z)}(g_{\mathrm{fin}})$$

for $g_{\infty} = (g_v)_{v\in\Sigma_{\infty}} \in G_{\infty}$ and $g_{\mathrm{fin}} \in G_{\mathrm{fin}}$.

Proof. We first note that $\overline{\varphi}^{H,(z)} = \overline{\varphi^{H,(\bar{z})}}$. Let $g_{\mathrm{fin}} \in G_{\mathrm{fin}}$. For any $v \in \Sigma_{\infty}$, we easily see $\varphi^{H,(\bar{z})}\big(g_{\mathrm{fin}}\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g_{\infty}k\big) = |t_1/t_2|_v^{-\bar{z}}\tau_{l_v}(k)\varphi^{H,(\bar{z})}(g_{\mathrm{fin}}g_{\infty})$ for all $t_1, t_2 \in F_v^{\times}$, $k \in \mathbf{K}_v^0$ and $g_{\infty} \in G_{\infty}$. Moreover, $R(\overline{W}_v)(\varphi^{H,(\bar{z})}) = (R(\overline{W}_v)\varphi)^{H,(\bar{z})} = 0$. Thus the uniqueness of Shintani functions (Proposition 3.1) yields a constant C such that $\varphi^{H,(\bar{z})}(g_{\mathrm{fin}}g_{\infty}) = C\prod_{v\in\Sigma_{\infty}} \Psi_v^{(-\bar{z})}(l_v; g_v)$ for all $g_{\infty} \in G_{\infty}$. By setting $g_{\infty} = 1_2$ and using property (S-iii) of Proposition 3.1, we get $C = \varphi^{H,(\bar{z})}(g_{\mathrm{fin}})$. ∎

For $\mathbf{s} \in \mathfrak{X}_S$, we consider the element

$$\mathbf{T}_S(\mathbf{s}) = \bigotimes_{v \in S} \{\mathbb{T}_v - (q_v^{(1-s_v)/2} + q_v^{(1+s_v)/2}) 1_{\mathbf{K}_v}\}$$

of the Hecke algebra $\bigotimes_{v \in S} \mathcal{H}(G_v, \mathbf{K}_v)$. We also set

$$q(\mathbf{s}) = \inf\{(\mathrm{Re}(s_v) + 1)/4 \mid v \in S\}.$$

Proposition 5.3. *Suppose $q(\mathbf{s}) > 2|\mathrm{Re}(z)|+1$. For $\varphi \in \mathcal{A}_{\mathrm{cus}}(G; \tau_l)^{\mathbf{K}_0(\mathfrak{n})}$, the function $g \mapsto \Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; g)\overline{\varphi}^{H,(z)}(g)$ is integrable on $H_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Moreover,*

$$\int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; g)[R(\mathbf{T}_S(\mathbf{s}))\overline{\varphi}^{H,(z)}](g)\, dg = \Big\{ \prod_{v \in \Sigma_\infty} 2^{l_v - 1} C_{l_v}(z) \Big\} \mathrm{vol}(H_{\mathrm{fin}} \backslash H_{\mathrm{fin}} \mathbf{K}_0(\mathfrak{n})) \overline{\varphi}^{H,(z)}(1_2).$$

Proof. We follow the argument in [21, proof of Lemma 6.3]; to show the absolute convergence of the integral, we use Lemma 5.1. By Lemma 5.2, the integral on the left-hand side is the product of

$$\prod_{v \in \Sigma_\infty} \int_{H_v \backslash G_v} \Psi_v^{(z)}(l_v; g_v) \overline{\Psi_v^{(-\overline{z})}(l_v; g_v)}\, dg_v,$$

which is evaluated by Proposition 3.5, and

$$\int_{H_{\mathrm{fin}} \backslash G_{\mathrm{fin}}} \Big\{ \prod_{v \in S} \Psi_v^{(z)}(s_v; g_v) \prod_{v \in S(\mathfrak{n})} \Phi_{\mathfrak{n},v}^{(z)}(g_v) \prod_{v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))} \Phi_{0,v}^{(z)}(g_v) \Big\} \times [R(\mathbf{T}_S(\mathbf{s}))\overline{\varphi}^{H,(z)}](g_{\mathrm{fin}})\, dg_{\mathrm{fin}},$$

which yields the factor $\mathrm{vol}(H_{\mathrm{fin}} \backslash H_{\mathrm{fin}} \mathbf{K}_0(\mathfrak{n})) \overline{\varphi}^{H,(z)}(1_2)$ by Lemma 4.2. ∎

5.1. Regularization of periods and automorphic smoothed kernels. For a weight $l = (l_v)_{v \in \Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$, set $\underline{l} = \inf_{v \in \Sigma_\infty} l_v$. In this subsection, we introduce the automorphic renormalized smoothed kernel function $\hat{\mathbf{\Psi}}_{\beta,\lambda}^l(\mathfrak{n}|\alpha; g)$ depending on a complex parameter λ. We show that $\hat{\mathbf{\Psi}}_{\beta,\lambda}^l(\mathfrak{n}|\alpha; g)$, originally defined by the Poincaré series (5.5) convergent for $\mathrm{Re}(\lambda) > 0$, becomes square-integrable (even cuspidal) when $\underline{l} \geq 4$ and $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$.

5.1.1. Let $\mathcal{B}$ denote the space of all entire functions $\beta(z)$ on $\mathbb{C}$ such that $\beta(z) = \beta(-z)$ satisfying the following condition: For any interval $[a, b] \subset \mathbb{R}$, there exist $A > 0$ and $B \in \mathbb{R}$ such that

$$|\beta(\sigma + it)| \ll e^{-A(|t|+B)^2}, \quad \sigma \in [a, b],\ t \in \mathbb{R}.$$

We impose a stronger condition than [21, (6.1)] to have the inclusion $C_l \mathcal{B} \subset \mathcal{B}$, which is seen from Lemma 3.4(iii). For $\beta \in \mathcal{B}$ and $(\mathbf{s}, \lambda) \in \mathfrak{X}_S \times \mathbb{C}$ such that

$q(\mathbf{s}) > 1$ and $\mathrm{Re}(\lambda) > 1 - q(\mathbf{s})$, we define the renormalized Green function by

$$\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g) = \frac{1}{2\pi i} \int\limits_{L_\sigma} \frac{\beta(z)}{z+\lambda} \{\Psi^{(z)}_l(\mathfrak{n}|\mathbf{s}; g) + \Psi^{(-z)}_l(\mathfrak{n}|\mathbf{s}; g)\}\, dz$$

for the contour L_σ defined at the end of the introduction, where σ is taken so that $-\min(q(\mathbf{s}) - 1, \mathrm{Re}(\lambda)) < \sigma < q(\mathbf{s}) - 1$. The defining integral is absolutely convergent and is independent of the choice of the contour; the function $\lambda \mapsto \Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ is holomorphic in the region $\mathrm{Re}(\lambda) > 1 - q(\mathbf{s})$ which contains $\lambda = 0$.

5.1.2. Our main interest is the central L-value $L(1/2, \pi)$ which is essentially the $(H, \mathbf{1})$-period of cusp forms belonging to π by Hecke's zeta integral. Proposition 5.3 strongly suggests that the automorphic object

$$\sum_{\gamma \in H_F \backslash G_F} \Psi^{(0)}_l(\mathfrak{n}|\mathbf{s}; \gamma g), \tag{5.1}$$

if well defined, might have the spectral resolution describable by the $(H, \mathbf{1})$-period integral of cusp forms φ through the following formal computation:

$$\begin{aligned} &\int\limits_{Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}}} \Big\{ \sum_{\gamma \in H_F \backslash G_F} \Psi^{(0)}_l(\mathfrak{n}|\mathbf{s}; \gamma g) \Big\} [R(\mathbf{T}_S(\mathbf{s}))\varphi](g)\, dg \\ &\quad = \int\limits_{Z_{\mathbb{A}} H_F \backslash G_{\mathbb{A}}} \Psi^{(0)}_l(\mathfrak{n}|\mathbf{s}; g) [R(\mathbf{T}_S(\mathbf{s}))\varphi](g)\, dg \\ &\quad = \int\limits_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Psi^{(0)}_l(\mathfrak{n}|\mathbf{s}; g) [R(\mathbf{T}_S(\mathbf{s}))\varphi^{H,(0)}](g) dg = \mathrm{const} \cdot \varphi^{H,(0)}(1_2). \end{aligned} \tag{5.2}$$

Unfortunately, this is not attained for free due to the divergence of the series $\sum_\gamma |\Psi^{(0)}_l(\mathfrak{n}|\mathbf{s}; \gamma g)|$ for almost all g caused by the fact that $H_F Z_{\mathbb{A}} \backslash H_{\mathbb{A}} \cong F^\times \backslash \mathbb{A}^\times$ is of infinite volume. Since $\mathrm{CT}_{\lambda=0} \Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g) = \Psi^{(0)}_l(\mathfrak{n}|\mathbf{s}; g)\beta(0)$ by [21, Lemma 6.5 or 6.9], and since $\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ with large $\mathrm{Re}(\lambda)$ behaves on $G_{\mathbb{A}}$ well enough to ensure the absolute convergence of the Poincaré series

$$\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g) = \sum_{\gamma \in H_F \backslash G_F} \Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; \gamma g), \quad g \in G_{\mathbb{A}}, \tag{5.3}$$

we expect that some substitute for (5.1) could be gained as the constant term at $\lambda = 0$ of the analytic continuation in λ of the series (5.3). This circle of ideas motivates our study of the series (5.3).

Lemma 5.4. *Suppose* $\underline{l} \geq 4$.

(1) *The series* $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ *converges absolutely and locally uniformly in* $(\lambda, \mathbf{s}, g) \in \{\mathrm{Re}(\lambda) > 0\} \times \{q(\mathbf{s}) > 1\} \times G_{\mathbb{A}}$. *For a fixed* $(\lambda, \mathbf{s})$ *in this region,* $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ *is a continuous function of* $g \in G_{\mathbb{A}}$, *which is left*

$Z_{\mathbb{A}}G_F$-invariant and right $\mathbf{K}_0(\mathfrak{n})$-invariant, and satisfies

$$\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};gk_v) = \tau_{l_v}(k_v)\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)$$

for all $v \in \Sigma_\infty$ and $k_v \in \mathbf{K}^0_v$.

(2) *Let $(\lambda,\mathbf{s}) \in \mathbb{C}\times\mathfrak{X}_S$ be such that $2\,\mathrm{Re}(\lambda) > 1$, $q(\mathbf{s}) > 2\,\mathrm{Re}(\lambda)+1$ and $\underline{l}/2 > \mathrm{Re}(\lambda)+1$. Then, for any $\sigma \in (1/2, \mathrm{Re}(\lambda))$,*

$$|\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)| \ll y(g)^{1-\sigma}, \quad g \in \mathfrak{S}^1.$$

Proof. The same proof of [21, Proposition 8.1] goes through with a minor modification. The outline is as follows. For $p > 0$ and $q > 1$, set

$$\begin{aligned}\Xi_{l,p,q,S}\Big(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2\end{smallmatrix}\right](a_{r_v})_{v\in\Sigma_\infty}\Big(\left[\begin{smallmatrix} 1 & x_v \\ 0 & 1\end{smallmatrix}\right]\Big)_{v\in\Sigma_{\mathrm{fin}}}k\Big)&\\ = \min\{|t_1/t_2|^p_{\mathbb{A}}, |t_1/t_2|^{-p}_{\mathbb{A}}\}\prod_{v\in\Sigma_\infty}(\cosh 2r_v)^{-\underline{l}/2}&\\ \times\prod_{v\in S}\max(1,|x_v|_v)^{-q}\prod_{v\in\Sigma_{\mathrm{fin}}-S}\delta(x_v\in\mathfrak{o}_v)&\end{aligned}$$

for $t_1, t_2 \in \mathbb{A}^\times$, $(r_v)_{v\in\Sigma_\infty} \in \mathbb{R}^{\Sigma_\infty}$ and $(x_v)_{v\in\Sigma_{\mathrm{fin}}} \in \mathbb{A}_{\mathrm{fin}}$, and set

$$\mathbf{\Xi}_{l,p,q,S}(g) = \sum_{\gamma\in H_F\backslash G_F}\Xi_{l,p,q,S}(\gamma g), \quad g \in G_{\mathbb{A}}.$$

By Lemma 3.3, $\Xi_{l,\sigma,q(\mathbf{s}),S}$ with $0 < \sigma < \min(\mathrm{Re}(\lambda), q(\mathbf{s})-1)$ gives a majorant of $\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$ in the same way as in [21, Lemma 6.7]. Thus to prove the convergence and the estimation for $\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$, it is enough to establish that $\mathbf{\Xi}_{l,p,q,S}$ is locally uniformly convergent in $G_{\mathbb{A}}$, and that

$$\mathbf{\Xi}_{l,p,q,S}(g) \ll y(g)^{1-p}, \quad g \in \mathfrak{S}^1,$$

if $1+2p < q$ and $1+p < \underline{l}/2$. To achieve these, we modify the proof of [21, Lemma 3.5] by replacing q in the archimedean factors of $\Xi_{p,q,S}$ used there with $\underline{l}/2 > 1$. We also note that the condition $1+p < \underline{l}/2$ is necessary to guarantee $\int_{\mathbb{R}}(\cosh 2r_v)^{p-\underline{l}/2+1}\,dr_v < \infty$. ∎

For a fixed $(\lambda,\mathbf{s})$ such that $2\,\mathrm{Re}(\lambda) > 1$, $q(\mathbf{s}) > 2\,\mathrm{Re}(\lambda)+1$ and $\underline{l}/2 > \mathrm{Re}(\lambda)+1$, the function $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$ defines a distribution on $Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}$ by

$$\langle\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}),\varphi\rangle = \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}}\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)\varphi(g)\,dg, \quad \varphi \in C^\infty_c(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}).$$

We remark that the integral is absolutely convergent for any $\varphi \in \mathcal{A}_{\mathrm{cus}}(G;\tau_l)$ by Lemma 5.4(2).

5.1.3. We need to make the computation (5.2) rigorous using $\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)$. For this, the notion of periods should be modified properly. Let us recall the regularization of period integrals along H, which was introduced in [21, §7].

For $t > 0$, set

$$\hat{\beta}_\lambda(t) = \frac{1}{2\pi i} \int\limits_{L_\sigma} \frac{\beta(z)}{z+\lambda} t^z \, dz \quad (\sigma > -\operatorname{Re}(\lambda)).$$

We have $|\hat{\beta}_\lambda(t)| \ll \min\{t^\sigma, t^{-\operatorname{Re}(\lambda)}\}$, $t > 0$ [21, Lemma 7.1]. Given a real valued idele class character η of $F^\times$, let $x_\eta = (x_{\eta,v})_{v \in \Sigma_F}$ be the adele such that $x_{\eta,v} = 0$ for $v \in \Sigma_\infty$, and $x_{\eta,v} = \varpi_v^{-f(\eta_v)}$ for $v \in \Sigma_{\mathrm{fin}}$, and let x_η^* be the idele such that the finite component of x_η^* coincides with the projection of x_η to $\mathbb{A}_{\mathrm{fin}}^\times$, and all archimedean components of x_η^* are 1. A continuous function φ on $Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}}$ is said to have the *regularized* (H, η)*-period* $P^\eta_{\mathrm{reg}}(\varphi) \in \mathbb{C}$ if for any $\beta \in \mathcal{B}$, the integral

$$P^\eta_{\beta,\lambda}(\varphi) = \int\limits_{F^\times \backslash \mathbb{A}^\times} \varphi\big(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big) \eta(t x_\eta^*) \{\hat{\beta}_\lambda(|t|_{\mathbb{A}}) + \hat{\beta}_\lambda(|t|_{\mathbb{A}}^{-1})\} \, d^\times t \tag{5.4}$$

converges absolutely when $\operatorname{Re}(\lambda) \gg 0$ and can be continued meromorphically in a neighborhood of $\lambda = 0$ with the constant term $\operatorname{CT}_{\lambda=0} P^\eta_{\beta,\lambda}(\varphi) = P^\eta_{\mathrm{reg}}(\varphi)\beta(0)$ in its Laurent expansion at 0. We note that for $\varphi \in \mathcal{A}_{\mathrm{cus}}(G; \tau_l)$, the regularized period $P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)$ coincides with the $(H, \mathbf{1})$-period [21, Lemma 7.3].

The following lemma is shown along the lines of the formal computation (5.2).

LEMMA 5.5. *Assume* $\underline{l} \geq 4$. *Let* $(\lambda, \mathbf{s})$ *be an element of* $\mathbb{C} \times \mathfrak{X}_S$ *such that* $2\operatorname{Re}(\lambda) > 1$, $q(\mathbf{s}) > 2\operatorname{Re}(\lambda) + 1$ *and* $\underline{l}/2 > \operatorname{Re}(\lambda) + 1$. *Then, for any* $\varphi \in \mathcal{A}_{\mathrm{cus}}(G; \tau_l)^{\mathbf{K}_0(\mathfrak{n})}$,

$$\langle \mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}), R(\mathbf{T}_S(\mathbf{s}))\overline{\varphi} \rangle = \Big\{ \prod_{v \in \Sigma_\infty} 2^{l_v - 1} \Big\} \operatorname{vol}(H_{\mathrm{fin}} \backslash H_{\mathrm{fin}} \mathbf{K}_0(\mathfrak{n})) P^{\mathbf{1}}_{\beta C_l, \lambda}(\overline{\varphi}),$$

where $C_l(z) = \prod_{v \in \Sigma_\infty} C_{l_v}(z)$.

Proof. The proof is as in [21, Lemma 8.2] with the aid of Lemma 3.3 and Proposition 5.3. We note that $P^{\mathbf{1}}_{\beta C_l, \lambda}(\overline{\varphi})$ is well defined because βC_l belongs to $\mathcal{B}$. ∎

5.1.4. Assume $\underline{l} \geq 4$. Given a holomorphic function $\alpha(\mathbf{s})$ on $\mathfrak{X}_S$ such that $\alpha(\varepsilon \mathbf{s}) = \alpha(\mathbf{s})$ for all $\varepsilon \in \{\pm 1\}^S$, we define the renormalized smoothed kernel

$$\hat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \left(\frac{1}{2\pi i}\right)^{\#S} \int\limits_{\mathbb{L}_S(\mathbf{c})} \Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g) \alpha(\mathbf{s}) \, d\mu_S(\mathbf{s})$$

for $\operatorname{Re}(\lambda) > 0$ with $\mathbf{c} \in \mathbb{R}^S$ such that $q(\mathbf{c}) > \max(\operatorname{Re}(\lambda) + 1, 2)$, where $\int_{\mathbb{L}_S(\mathbf{c})} f(\mathbf{s}) \, d\mu_S(\mathbf{s})$ means the multidimensional contour integral along the slice $\mathbb{L}_S(\mathbf{c}) = \{\mathbf{s} \in \mathfrak{X}_S \mid \operatorname{Re}(\mathbf{s}) = \mathbf{c}\}$ oriented naturally with respect to the

form $d\mu_S(\mathbf{s}) = \prod_{v\in S} d\mu_v(s_v)$ with

$$d\mu_v(s_v) = 2^{-1}(\log q_v)(q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2})\, ds_v.$$

For $\mathrm{Re}(\lambda) > 0$, let us consider the Poincaré series

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \sum_{\gamma\in H_F\backslash G_F} \hat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; \gamma g), \quad g \in G_{\mathbb{A}}, \tag{5.5}$$

which is a central object in this paper. We introduce (5.5) because it has a much nicer spectral expansion than $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$ (see Lemma 6.5). In the same way as in [21], we analyze this series and obtain the following.

Lemma 5.6.

(1) *The series* $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ *converges absolutely and locally uniformly in* $(\lambda, g) \in \{\mathrm{Re}(\lambda) > 0\} \times G_{\mathbb{A}}$. *The function* $g \mapsto \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ *is continuous on* $G_{\mathbb{A}}$, *left* $Z_{\mathbb{A}}G_F$*-invariant, and right* $\mathbf{K}_0(\mathfrak{n})$*-invariant; moreover, for all* $v \in \Sigma_\infty$,

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; gk_v) = \tau_{l_v}(k_v)\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) \quad \text{for all } k_v \in \mathbf{K}^0_v. \tag{5.6}$$

(2) *For* $0 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$, *the function* $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ *belongs to* $L^m(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ *for any* $m > 0$ *such that* $m(1 - \mathrm{Re}(\lambda)) < 1$.

Proof. The argument in [21, proof of Proposition 9.1] works with a minor modification: we use $\Xi_{l,p,q,S}$ and $\mathbf{\Xi}_{l,p,q,S}$ given in the proof of Lemma 5.4. ∎

Proposition 5.7. *For* $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$, *the function* $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ *is cuspidal.*

Proof. From Proposition 3.1 and Lemma 5.6(1), the function $g \mapsto \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ on $G_{\mathbb{A}}$ is smooth and satisfies

$$R(\overline{W}_v)\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = 0, \quad g \in G_{\mathbb{A}}, \tag{5.7}$$

for all $v \in \Sigma_\infty$. From this equation together with the $\mathbf{K}^0_v$-equivariance (5.6) of $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$, the Casimir operator for G_v for each $v \in \Sigma_\infty$ acts on $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ as multiplication by a scalar. Hence there exists $f \in C^\infty_c(G_{\mathbb{A}})$ such that $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha) * f = \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)$ by [2, Theorem 2.14]. From Lemma 5.6(2), $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)$ belongs to $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})}$. Thus, for any $X \in \mathfrak{g}_\infty$, the derivative $R(X)\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha) = \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha) * R(-X)f$ also belongs to the same L^2-space. Let V be the $(\mathfrak{g}_\infty, \mathbf{K}_\infty)$-submodule of $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})}$ generated by $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$; from (5.6) and (5.7), V is decomposed into a finite sum $\boxtimes_{v\in\Sigma_\infty} D_{l_v}$ of discrete series representations of $\mathrm{PGL}(2, F \otimes_{\mathbb{Q}} \mathbb{R})$ of weight $(l_v)_{v\in\Sigma_\infty}$. By Wallach's criterion [22, Theorem 4.3], the space V is contained in the cuspidal part of $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$. ∎

By Proposition 5.7, for $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$, the function $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ has the spectral expansion

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \sum_{\pi \in \mathit{\Pi}_{\mathrm{cus}}(l,\mathfrak{n})} \sum_{\varphi \in \mathcal{B}(\pi; l, \mathfrak{n})} \langle \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha) | \varphi \rangle_{L^2} \varphi(g) \tag{5.8}$$

for almost all $g \in G_{\mathbb{A}}$. Here $\langle \cdot | \cdot \rangle_{L^2}$ is the L^2-inner product on $L^2(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})$, and $\mathcal{B}(\pi; l, \mathfrak{n})$ is an orthonormal basis of $\{\varphi \in L^2(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})[\tau_l]^{\mathbf{K}_0(\mathfrak{n})} \mid R(\overline{W}_v)\varphi = 0 \ (\forall v \in \Sigma_\infty)\}$ which coincides with $\mathcal{A}_{\mathrm{cus}}(G; \tau_l)^{\mathbf{K}_0(\mathfrak{n})}$, as seen from the proof of Proposition 5.7. From the finite dimensionality of $\mathcal{A}_{\mathrm{cus}}(G; \tau_l)^{\mathbf{K}_0(\mathfrak{n})}$, the sum in (5.8) is finite and the equality holds pointwise for all $g \in G_{\mathbb{A}}$.

6. Spectral side. From this section until §11, we fix an even weight $l = (l_v)_{v \in \Sigma_\infty}$, an ideal $\mathfrak{n} \subset \mathfrak{o}$, an idele class character η of $F^\times$ such that $\eta^2 = \mathbf{1}$ whose conductor $\mathfrak{f}$ is relatively prime to $\mathfrak{n}$, a finite subset $S \subset \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f})$, and a holomorphic function $\alpha(\mathbf{s})$ on $\mathfrak{X}_S$ such that $\alpha(\varepsilon \mathbf{s}) = \alpha(\mathbf{s})$ for all $\varepsilon \in \{\pm 1\}^S$. Using the spectral expansion (5.8), we show that $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ has an entire extension to the whole λ-plane. We define the regularized kernel $\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha; g)$ to be the value at $\lambda = 0$ of the entire extension, which is our desired substitute for the divergent series (5.1), and obtain its spectral expression. The upshot of this section is Proposition 6.6, which gives the period integral of the regularized kernel.

6.1. Extremal Whittaker vectors of discrete series. For $v \in \Sigma_\infty$, let π_v be the discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$ of minimal $\mathbf{K}^0_v$-type l_v. Let V_{π_v} denote the Whittaker model of π_v with respect to the character $\psi_{F,v}$ (see §2.4). It is known that $V_{\pi_v}[\tau_{l_v}]$ contains a unique vector $\phi_{0,v}$ characterized by

$$\phi_{0,v}\left(\left[\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) = 2|y|_v^{l_v/2} e^{2\pi y} \delta(y < 0), \qquad y \in \mathbb{R}^\times. \tag{6.1}$$

We remark that $\phi_{0,v}$ is extremal, i.e., $\pi_v(\overline{W})\phi_{0,v} = 0$, and $V_{\pi_v}[\tau_{l_v}] = \mathbb{C}\phi_{0,v}$. We should also note that the local epsilon factor of π_v is given as $\epsilon(s, \pi_v \otimes \mathrm{sgn}^m, \psi_{F,v}) = i^{l_v}$ for $m \in \{0, 1\}$.

6.2. Construction of basis. Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_{\mathbb{A}}$ with trivial central character such that $V_\pi \subset L^2(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})$. We fix a family $\{(\pi_v, V_{\pi_v})\}_{v \in \Sigma_F}$ of unitarizable irreducible admissible representations of G_v with V_{π_v} being contained in the $\psi_{F,v}$-Whittaker functions on G_v such that $\pi \cong \bigotimes_{v \in \Sigma_F} \pi_v$. The conductor of π is defined to be the ideal $\mathfrak{f}_\pi$ determined by the condition $\mathfrak{f}_\pi \mathfrak{o}_v = \mathfrak{p}_v^{c(\pi_v)}$ for all $v \in \Sigma_{\mathrm{fin}}$, where $c(\pi_v)$ is the minimal non-negative integer among those $c \in \mathbb{N}_0$ such that $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{p}_v^c)} \neq \{0\}$. Let $\mathit{\Pi}_{\mathrm{cus}}(l, \mathfrak{n})$ denote the set of all those cuspidal representations π such that $\pi_v \cong D_{l_v}$ for all $v \in \Sigma_\infty$ and $\mathfrak{n} \subset \mathfrak{f}_\pi$.

For $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$, let $\Lambda_\pi(\mathfrak{n})$ be the set of all maps $\rho : \Sigma_{\mathrm{fin}} \to \mathbb{N}_0$ such that $\rho(v) \in \{0, \dots, \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})\}$ for all $v \in \Sigma_{\mathrm{fin}}$. Corresponding to each $\rho \in \Lambda_\pi(\mathfrak{n})$, we have a cusp form $\varphi_{\pi,\rho} \in V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ as the image of the decomposable tensor

$$\bigotimes_{v\in\Sigma_\infty} \phi_{0,v} \otimes \bigotimes_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \phi_{\rho(v),v} \otimes \bigotimes_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \phi_{0,v}$$

under the isomorphism $V_\pi \cong \bigotimes_{v\in\Sigma_F} V_{\pi_v}$, where for each $v \in \Sigma_{\mathrm{fin}}$, the system $\{\phi_{k,v} \mid 0 \le k \le \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})\}$ is the basis of $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)}$ constructed in [18]. We remark that $\phi_{0,v}$ is the local new vector of π_v. In this way, we have an orthogonal basis $\{\varphi_{\pi,\rho} \mid \rho \in \Lambda_\pi(\mathfrak{n})\}$ of the finite-dimensional space $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ equipped with the L^2-inner product on $Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}$ [18, Proposition 17]. The vector φ_{π,ρ_0} with $\rho_0(v) = 0$ for all $v \in \Sigma_{\mathrm{fin}}$ is denoted by $\varphi_\pi^{\mathrm{new}}$.

Remark. Let $S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ be the set of $v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ such that $\mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1}) = k$, and n the maximal non-negative integer k such that $S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}) \neq \emptyset$. For $\rho \in \Lambda_\pi(\mathfrak{n})$, by writing $\rho_k = \rho|_{S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}$ for each $0 \le k \le n$, we can identify ρ with the family $(\rho_k)_{0\le k\le n}$ of maps as done in [18].

6.3. Regularized periods and standard L-values. (For details, see [21, §7] and [18].) In this subsection, π denotes an element of $\Pi_{\mathrm{cus}}(l, \mathfrak{n})$. We note that for any cusp form $\varphi \in V_\pi$, the regularized period $P^\eta_{\mathrm{reg}}(\varphi)$ defined by (5.4) coincides with the global zeta integral

$$Z^*(1/2, \eta, \varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi\Big(\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x_\eta \\ 0 & 1 \end{bmatrix}\Big)\eta(tx_\eta^*)\, d^\times t,$$

which is absolutely convergent.

Proposition 6.1. *For any $\rho \in \Lambda_\pi(\mathfrak{n})$, $\varphi_{\pi,\rho}$ has the regularized (H, η)-period given by*

$$\begin{aligned} P^\eta_{\mathrm{reg}}(\varphi_{\pi,\rho}) &= Z^*(1/2, \eta, \varphi_{\pi,\rho}) \\ &= (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)\Big\{\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q^{\pi_v}_{\rho(v),v}(\eta_v, 1)\Big\}L(1/2, \pi\otimes\eta). \end{aligned}$$

Here $Q^{\pi_v}_{k,v}(\eta_v, 1)$ with $v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and $k \in \{1, \dots, \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})\}$ is the constant appearing in [18, Main Theorem A]*, and $\mathcal{G}(\eta)$ is the Gauss sum defined in* §2.4.

Proof. The first identity is obtained in [21, Lemma 7.3]. The second identity follows basically from [18, Main Theorem A]. Although the hypothesis $\eta_v(-1) = 1$ for all $v \in \Sigma_\infty$ in [18, Main Theorem A] is not satisfied in our setting, it is easy to modify the proof at archimedean places by means of (6.1). ∎

Set

$$\mathbb{P}^{\eta}(\pi; l, \mathfrak{n}) = \sum_{\varphi \in \mathcal{B}(\pi; l, \mathfrak{n})} \overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\, P^{\eta}_{\mathrm{reg}}(\varphi),$$

where $\mathcal{B}(\pi; l, \mathfrak{n})$ is an orthonormal basis of $V_{\pi}[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$.

LEMMA 6.2. *The sum $\mathbb{P}^{\eta}(\pi; l, \mathfrak{n})$ is independent of the choice of $\mathcal{B}(\pi; l, \mathfrak{n})$. We have*

$$\mathbb{P}^{\eta}(\pi; l, \mathfrak{n}) = D_F^{-1/2}(-1)^{\epsilon(\eta)}\mathcal{G}(\eta) w^{\eta}_{\mathfrak{n}}(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\|\varphi^{\mathrm{new}}_{\pi}\|^2},$$

and the value $(-1)^{-\epsilon(\eta)}\mathcal{G}(\eta)^{-1}\mathbb{P}^{\eta}(\pi; l, \mathfrak{n})$ is non-negative. Here $w^{\eta}_{\mathfrak{n}}(\pi)$ is the explicit non-negative constant given by

$$w^{\eta}_{\mathfrak{n}}(\pi) = \prod_{v \in S(\mathfrak{n}\mathfrak{f}_{\pi}^{-1})} r(\pi_v, \eta_v)$$

with $r(\pi_v, \eta_v)$ defined as follows. Set $k_v = \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_{\pi}^{-1})$. If $\eta_v(\varpi_v) = -1$, then

$$r(\pi_v, \eta_v) = \frac{1 + (-1)^{k_v}}{2} \begin{cases} (q_v + 1)(q_v - 1)^{-1} & (c(\pi_v) = 0), \\ 1 & (c(\pi_v) \geq 1). \end{cases}$$

If $\eta_v(\varpi_v) = 1$, then

$$r(\pi_v, \eta_v) = \begin{cases} \dfrac{q_v + 1}{(1 + q_v^{1/2}\alpha_v)(1 + q_v^{1/2}\alpha_v^{-1})} \left\{ 2 + \dfrac{k_v - 1}{q_v - 1}(1 - \alpha_v q_v^{1/2})(1 - \alpha_v^{-1} q_v^{1/2}) \right\} & (c(\pi_v) = 0), \\ 1 + k_v \dfrac{1 - q_v^{-1}\chi_v(\varpi_v)}{1 + q_v^{-1}\chi_v(\varpi_v)} & (c(\pi_v) = 1), \\ k_v + 1 & (c(\pi_v) \geq 2), \end{cases}$$

where $(\alpha_v, \alpha_v^{-1})$ is the Satake parameter of π_v if $c(\pi_v) = 0$, and χ_v is the unramified character of $F_v^{\times}$ such that $\pi_v \cong \sigma(\chi_v |\ |_v^{1/2}, \chi_v |\ |_v^{-1/2})$ if $c(\pi_v) = 1$.

If $\eta_v(\varpi_v) = -1$ for all $v \in S(\mathfrak{n})$, then $w^{\eta}_{\mathfrak{n}}(\pi) = 0$ unless $\mathfrak{n}\mathfrak{f}_{\pi}^{-1}$ is the square of an integral ideal.

Proof. With the aid of Proposition 6.1, we obtain the assertion as in [19, Lemma 12]. The non-negativity of $(-1)^{-\epsilon(\eta)}\mathcal{G}(\eta)^{-1}\mathbb{P}^{\eta}(\pi; l, \mathfrak{n})$ follows from $w^{\eta}_{\mathfrak{n}}(\pi) \geq 0$ combined with the non-negativity of $L(1/2, \pi)L(1/2, \pi \otimes \eta)$ proved in [8]. ∎

The sign of the functional equation of $L(s, \pi)L(s, \pi \otimes \eta)$ is given as follows.

LEMMA 6.3. *We have $\epsilon(1/2, \pi)\epsilon(1/2, \pi \otimes \eta) = (-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{f}_{\pi})$. In particular, $L(1/2, \pi)L(1/2, \pi \otimes \eta) = 0$ unless $(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{f}_{\pi}) = 1$.*

Proof. Since l_v is even for all $v \in \Sigma_\infty$, by [19, Lemma 13], we have

$$\begin{aligned}\epsilon(1/2,\pi)\epsilon(1/2,\pi\otimes\eta)&\\ &= \prod_{v\in\Sigma_\infty} i^{2l_v} \prod_{v\in S(\mathfrak{f})} \eta_v(-1) \prod_{v\in S(\mathfrak{f}_\pi)} \eta_v(\varpi_v^{c(\pi_v)}) = \eta_{\mathrm{fin}}(-1)\tilde{\eta}(\mathfrak{f}_\pi)\\ &= (-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{f}_\pi).\end{aligned}$$

By the functional equation, we are done. ∎

6.4. Adjoint L-functions. Let $E(\nu,g) = \sum_{\gamma\in B_F\backslash G_F} y(\gamma g)^{(\nu+1)/2}$ ($\mathrm{Re}(\nu) > 1$) be the $\mathbf{K}$-spherical Eisenstein series on $G_{\mathbb{A}}$ (for the definition of $y(g)$, see §2.5).

LEMMA 6.4. *For any $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$,*

$$\begin{aligned}(6.2)\qquad \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} &\varphi_\pi^{\mathrm{new}}(g)\,\overline{\varphi_\pi^{\mathrm{new}}}(g)E(2s-1,g)\,dg\\ &= \Big\{\prod_{v\in\Sigma_\infty} 2^{1-l_v}\Big\}\frac{\mathrm{N}(\mathfrak{f}_\pi)^s D_F^{s-3/2}}{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{f}_\pi)]}\,\frac{\zeta_F(s)L(s,\pi;\mathrm{Ad})}{\zeta_F(2s)}\\ &\quad\times\prod_{v\in S_\pi}\frac{q_v^{d_v(3/2-s)}Z_v(s)}{q_v^{c(\pi_v)(s-1)}L(s,\pi_v;\mathrm{Ad})}\,\frac{1+q_v^{-1}}{1+q_v^{-s}}\end{aligned}$$

for $\mathrm{Re}(s) \gg 0$ *and*

$$\|\varphi_\pi^{\mathrm{new}}\|^2 = 2\Big\{\prod_{v\in\Sigma_\infty} 2^{1-l_v}\Big\}\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{f}_\pi)]^{-1}L^{S_\pi}(1,\pi;\mathrm{Ad}).$$

Here we have set $S_\pi := \{v\in\Sigma_{\mathrm{fin}} \mid \mathrm{ord}_v(\mathfrak{f}_\pi) \ge 2\}$ *and*

$$Z_v(s) := \int_{\mathbf{K}_v}\int_{F_v^\times} \phi_{0,v}\big(\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]k\big)\overline{\phi_{0,v}\big(\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]k\big)}|t|_v^{s-1}\,d^\times t\,dk$$

for $v\in\Sigma_F$.

Proof. By the standard procedure, we see that the left-hand side of (6.2) is a product of the integrals $Z_v(s)$ over all $v\in\Sigma_F$. If $v\in\Sigma_\infty$, using (6.1), we easily obtain $Z_v(s) = 2^{1-l_v}\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(2s)^{-1}L(s,\pi_v;\mathrm{Ad})$. Together with the computations at finite places (cf. [21, Lemma 2.14 and Corollary 2.15] and [19, Lemma 14]), this completes the proof. ∎

REMARK. Nelson, Pitale and Saha [14] also considered the integrals $Z_v(s)$ and gave explicit formulas for $Z_v(s)$. However, as was already remarked in [14, 1.3], it seems difficult to give a simple formula for $Z_v(s)$ for $v\in S_\pi$.

6.5. Spectral parameters. Let $\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})$. For any $v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{f}_\pi)$, the v-component π_v of π is isomorphic to the $\mathbf{K}_v$-spherical principal series

representation $I_v(\nu_v) = \mathrm{Ind}_{B_v}^{G_v}(|\ |_v^{\nu_v/2} \boxtimes |\ |_v^{-\nu_v/2})$ with $\nu_v \in \mathbb{C}$ such that ν_v or $\nu_v + \frac{2\pi i}{\log q_v}$ belongs to $(0,1) \cup i\mathbb{R}_{\geq 0}$. The point $\nu_S(\pi) = \{\nu_v\}_{v\in S}$ of $\mathfrak{X}_S$ is called the *spectral parameter* of π at S. We have the Hecke eigenequation

$$R(\mathbb{T}_v)\varphi = (q_v^{(1+\nu_v)/2} + q_v^{(1-\nu_v)/2})\varphi, \quad \varphi \in V_\pi^{\mathbf{K}_v},\ v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi). \tag{6.3}$$

Since the Hecke operator $R(\mathbb{T}_v)$ acting on the space $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ is self-adjoint, the eigenvalue $q_v^{(1+\nu_v)/2} + q_v^{(1-\nu_v)/2}$ is a real number.

6.6. The spectral side. By means of Lemma 5.5, we can explicitly describe the coefficients of $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)$ in the L^2-expansion in terms of $(H,\mathbf{1})$-period integrals and the spectral parameters of cuspidal representations.

LEMMA 6.5. *Let $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$ and $\nu_S(\pi) = (\nu(\pi_v))_{v\in S}$ the spectral parameter of π at S. Then, for $\varphi \in V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ and $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$,*

$$\langle \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2} = (-1)^{\#S}\Big\{\prod_{v\in\Sigma_\infty} 2^{l_v-1}\Big\} D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1}\alpha(\nu_S(\pi))P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi}).$$

Proof. We note the inclusion $V_\pi[\tau_l] \subset \mathcal{A}_{\mathrm{cus}}(G;\tau_l)$. In the same way as in [21, Lemma 9.2] with the aid of the majorant $\Xi_{l,\mathrm{Re}(\lambda)-\epsilon,q(\mathbf{c}),S}$ for any sufficiently small $\epsilon > 0$, we obtain

$$\langle \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2} = \Big(\frac{1}{2\pi i}\Big)^{\#S} \int\limits_{\mathbb{L}_S(\mathbf{c})} \langle \mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}), \bar{\varphi}\rangle \alpha(\mathbf{s})\, d\mu_S(\mathbf{s})$$

for any $\varphi \in \mathcal{A}_{\mathrm{cus}}(G;\tau_l)^{\mathbf{K}_0(\mathfrak{n})}$, where $q(\mathbf{c})$ is sufficiently large ([1]). In contrast to [21, Lemma 9.2], the condition $\mathrm{Re}(\lambda) > 1$ is not needed. Indeed, in the proof of [21, Lemma 9.2], the estimate $|\varphi(g)| \ll \|g\|_{\mathbb{A}}^{1+\epsilon}$ is replaced with $|\varphi(g)| \ll \|g\|_{\mathbb{A}}^{-m}$ for any $m > 0$, and moreover $\int_1^\infty y^{-\mathrm{Re}(\lambda)+1+2\epsilon}\, d^\times y$ is replaced with $\int_1^\infty y^{-\mathrm{Re}(\lambda)-m+\epsilon}\, d^\times y$ ([2]). Thus, by Lemma 5.5 and (6.3), $\langle \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2}$ is equal to $\{\prod_{v\in\Sigma_\infty} 2^{l_v-1}\}\,\mathrm{vol}(H_{\mathrm{fin}}\backslash H_{\mathrm{fin}}\mathbf{K}_0(\mathfrak{n}))P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})$ multiplied by the integral

$$\Big(\frac{1}{2\pi i}\Big)^{\#S} \int\limits_{\mathbb{L}_S(\mathbf{c})} \Big\{\prod_{v\in S}(q_v^{(1+\nu(\pi_v))/2} + q_v^{(1-\nu(\pi_v))/2} - q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2})\Big\}^{-1} \times \alpha(\mathbf{s})\, d\mu_S(\mathbf{s}).$$

We have $q_v^{(1+\nu(\pi_v))/2} + q_v^{(1-\nu(\pi_v))/2} \in \mathbb{R}$ (see §6.5) and $\mathrm{vol}(H_{\mathrm{fin}}\backslash H_{\mathrm{fin}}\mathbf{K}_0(\mathfrak{n})) = D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]$ from [21, Lemma 8.3]. The integral is computed as

([1]) In the proof of [21, Lemma 9.2], the majorant of the integral (9.3) should be $\Xi_{\mathrm{Re}(\lambda)-\epsilon,q(\mathbf{c}),S_{\mathrm{fin}}}$.

([2]) In the proof of [21, Lemma 9.2], the first factor of the last integral should be $\int_1^\infty y^{-\mathrm{Re}(\lambda)+1+\epsilon}\, d^\times y$.

$(-1)^{\#S}\alpha(\mathbf{s})$ by invoking the formula

$$\frac{1}{2\pi i}\int_{c_v-2\pi i(\log q_v)^{-1}}^{c_v+2\pi i(\log q_v)^{-1}}\{q_v^{(1+\nu(\pi_v))/2}+q_v^{(1-\nu(\pi_v))/2}-q_v^{(1+s_v)/2}-q_v^{(1-s_v)/2})\}^{-1}\\ \times\alpha_v(s_v)\,d\mu_v(s_v)=-\alpha_v(s_v),$$

proved in [21, Lemma 9.5] for any $v\in S$ and any even holomorphic function $\alpha_v(s_v)$ on $\mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$. ∎

By this lemma and (5.8), we have

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\frac{(-1)^{\#S}\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}D_F^{-1/2}}{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]}\\ \times\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})}\alpha(\nu_S(\pi))P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})\varphi(g),\qquad g\in G_{\mathbb{A}}.$$

The integral $P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})$ is continued to an entire function in λ for any $\varphi\in\mathcal{A}_{\mathrm{cus}}(G;\tau_l)$ by [21, Lemma 7.3]. As a finite linear combination of such, the function $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ has a holomorphic analytic continuation to the whole λ-plane. Since $\mathrm{CT}_{\lambda=0}P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})=C_l(0)P^{\mathbf{1}}_{\mathrm{reg}}(\overline{\varphi})\beta(0)$, we can define the regularized automorphic smoothed kernel $\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)$ by

$$\mathrm{CT}_{\lambda=0}\,\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)\beta(0)$$

for any $\beta\in\mathcal{B}$. Indeed, we have the pointwise expression

$$\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)=\frac{(-1)^{\#S}\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}C_l(0)D_F^{-1/2}}{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]}\\ \times\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})}\alpha(\nu_S(\pi))\overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\varphi(g),$$

with the summation being finite. By computing the period integral $P^{\eta}_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ in terms of this expansion, we obtain one side of the relative trace formula, the *spectral side*.

Proposition 6.6. *Suppose $l\geq 4$. The function $\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ has the regularized (H,η)-period given by*

$$P^{\eta}_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))\\ =(-1)^{\#S}\Big\{\prod_{v\in\Sigma_\infty}2\pi\frac{\Gamma(l_v-1)}{\Gamma(l_v/2)^2}\Big\}D_F^{-1}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\times(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)\\ \times\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}w^{\eta}_{\mathfrak{n}}(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{2\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{f}_\pi)]^{-1}L^{S_\pi}(1,\pi;\mathrm{Ad})}\alpha(\nu_S(\pi)).$$

Proof. As was remarked in §6.3, for a cusp form φ, the regularized period $P^{\eta}_{\mathrm{reg}}(\varphi)$ becomes the usual zeta integral $Z^*(1/2,\eta,\varphi)$, which is absolutely convergent. Thus, by termwise integration,

$$P^{\eta}_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha)) = \frac{(-1)^{\#S}\{\prod_{v\in\Sigma_\infty} 2^{l_v-1}\}C_l(0)D_F^{-1/2}}{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]} \times \sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})} \overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\, P^{\eta}_{\mathrm{reg}}(\varphi)\alpha(\nu_S(\pi)).$$

Then we obtain the assertion by Lemma 3.4(ii), Proposition 6.1, and Lemmas 6.2 and 6.4. ∎

7. Geometric expansions. The reader might wonder why we take the regularized period in Proposition 6.6: this seems unnecessary because the function $\hat{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ is cuspidal and the usual period integral makes sense. The reason should become evident from our computation of the other side of the relative trace formula, the *geometric side*, to be performed in this section and the next. Suppose $\underline{l} = \inf_{v\in\Sigma_\infty} l_v \geq 4$. We compute $P^{\eta}_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ by using the series expression (5.5). The first step is to break the sum in (5.5) over $H_F\backslash G_F$ into a sum of subseries according to double cosets $H_F\delta H_F$. For $\delta\in G_F$, we set $\mathrm{St}(\delta) := H_F\cap\delta^{-1}H_F\delta$. Then the following elements of G_F form a complete set of representatives for the double coset space $H_F\backslash G_F/H_F$:

$$e=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\quad w_0=\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right],$$
$$u=\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right],\quad \overline{u}=\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right],\quad uw_0=\left[\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right],\quad \overline{u}w_0=\left[\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right],$$
$$\delta_b=\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right],\quad b\in F^\times-\{-1\}.$$

Moreover, $\mathrm{St}(e)=\mathrm{St}(w_0)=H_F$ and $\mathrm{St}(\delta)=Z_F$ for any $\delta\in\{u,\overline{u},uw_0,\overline{u}w_0\}\cup\{\delta_b\mid b\in F^\times-\{-1\}\}$ (see [16, Lemma 1] and [21, Lemma 73]). Thus we obtain the following expression for $\mathrm{Re}(\lambda)>0$:

$$\hat{\mathbf{\Psi}}^{l}_{\beta,\lambda}\big(\mathfrak{n}|\alpha;\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)=\sum_{\delta} J_\delta(\beta,\lambda,\alpha;t),$$

where δ runs through the double coset representatives listed above, and for each such δ, $J_\delta(\beta,\lambda,\alpha;t)$ is the sum of $\hat{\Psi}^{l}_{\beta,\lambda}\big(\mathfrak{n}|\alpha;\delta\gamma\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)$ for γ in $\mathrm{St}(\delta)\backslash H_F$.

LEMMA 7.1. *The functions $\lambda\mapsto J_e(\beta,\lambda,\alpha;t)$ and $\lambda\mapsto J_{w_0}(\beta,\lambda,\alpha;t)$ are entire on $\mathbb{C}$. Moreover, their values at $\lambda=0$ are $J_{\mathrm{id}}(\alpha;t)\beta(0)$ and $i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})J_{\mathrm{id}}(\alpha;t)\beta(0)$, respectively, where*

$$J_{\mathrm{id}}(\alpha;t)=\delta(\mathfrak{f}=\mathfrak{o})\left(\frac{1}{2\pi i}\right)^{\#S}\int\limits_{\mathbb{L}_S(\mathbf{c})}\Upsilon^{\mathbf{1}}_S(\mathbf{s})\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

with $\tilde{l} = \sum_{v\in\Sigma_\infty} l_v$ *and*

$$\Upsilon_S^{\mathbf{1}}(\mathbf{s}) = \prod_{v\in S}(1-q_v^{-(s_v+1)/2})^{-1}(1-q_v^{(s_v+1)/2})^{-1}.$$

Proof. Since $\Psi_v^{(0)}(l_v;1_2)=1$ for all $v\in\Sigma_\infty$ (Proposition 3.1(S-iii)), the assertion is proved in the same way as [21, Lemma 11.2]. ∎

We set

$$J_{\mathrm{u}}(\beta,\lambda,\alpha;t) = J_u(\beta,\lambda,\alpha,t) + J_{\bar{u}w_0}(\beta,\lambda,\alpha,t),$$
$$J_{\bar{\mathrm{u}}}(\beta,\lambda,\alpha;t) = J_{uw_0}(\beta,\lambda,\alpha,t) + J_{\bar{u}}(\beta,\lambda,\alpha,t).$$

LEMMA 7.2. *For* $*\in\{\mathrm{u},\bar{\mathrm{u}}\}$, *the function* $\lambda\mapsto J_*(\beta,\lambda,\alpha;t)$ *on* $\mathrm{Re}(\lambda)>0$ *has a holomorphic continuation to* $\mathbb{C}$ *whose value at* $\lambda=0$ *is* $J_*(\alpha,t)\beta(0)$, *where*

$$J_{\mathrm{u}}(\alpha;t) = \left(\frac{1}{2\pi i}\right)^{\#S}\sum_{a\in F^\times}\int_{\mathbb{L}_S(\mathbf{c})}\Big\{\Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big) + \Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\at^{-1}&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\big)\Big\}\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

and

$$J_{\bar{\mathrm{u}}}(\alpha;t) = \left(\frac{1}{2\pi i}\right)^{\#S}\sum_{a\in F^\times}\int_{\mathbb{L}_S(\mathbf{c})}\Big\{\Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\at&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big) + \Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\big)\Big\}\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}).$$

Proof. We follow the proof of [21, Lemma 11.3]. Take $\sigma>0$ such that $\underline{l}/2>\sigma+1$. Let us examine $J_{\mathrm{u}}(\beta,\lambda,\alpha;t)$. First we consider the sum of the functions

$$\hat{\Psi}^l_{\beta,\lambda}\big(\mathfrak{n}|\alpha;u\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)$$
$$= \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\left\{\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z+\lambda}\big\{|t|_{\mathbb{A}}^{z}\Psi^{(z)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big) + |t|_{\mathbb{A}}^{-z}\Psi^{(-z)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)\big\}\,dz\right\}\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

over all $a\in F^\times$. Here $\mathbf{c}$ is taken so that $q(\mathbf{c})$ is sufficiently large. By Lemmas 3.2 and 4.1, there exists an ideal $\mathfrak{a}$ of F depending on t such that

$$\big|\Psi_l^{(\pm z)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)\big| \ll_{\sigma,\mathbf{c}} f(a)e^{d_F\pi|\mathrm{Im}(z)|/2},\qquad a\in F^\times,\ (\mathbf{s},z)\in\mathbb{L}_S(\mathbf{c})\times L_\sigma,$$

where we set

$$f(a) = \prod_{v\in\Sigma_\infty} |1+ia_vt_v^{-1}|_v^{\sigma-l_v/2} \prod_{v\in S} \max(1,|a_vt_v^{-1}|_v)^{-(2q(\mathbf{c})-\sigma)} \times \prod_{\Sigma_{\mathrm{fin}}-S} \delta(a_v\in\mathfrak{a}\mathfrak{d}_v), \qquad a\in\mathbb{A},$$

and use $|(1+ia_vt_v^{-1})^{z-l_v/2}|_v \le |1+ia_vt_v^{-1}|_v^{\mathrm{Re}(z)-l_v/2} e^{|\mathrm{Im}(z)|\pi/2}$. Thus to establish the absolute convergence of the sum of $\hat{\Psi}^l_{\beta,\lambda}\big(\mathfrak{n}|\alpha; u\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)$ over $a\in F^\times$, it is enough to show $\sum_{a\in F^\times} f(a)<\infty$. The convergence of the latter sum in turn follows from the convergence of the integral $\int_{\mathbb{A}} f(a)\,da$, which is a product of the archimedean integrals for all $v\in\Sigma_\infty$ convergent when $l_v/2-\sigma>1$, and the non-archimedean ones convergent for sufficiently large $q(\mathbf{c})$.

The sum of the functions $\hat{\Psi}^l_{\beta,\lambda}\big(\mathfrak{n}|\alpha; \bar{u}w_0\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)$ over $a\in F^\times$ is analyzed similarly. By the estimate

$$\big|\Psi_l^{(\pm z)}\big(\mathfrak{n}|\mathbf{s}; \left[\begin{smallmatrix}1&0\\at^{-1}&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\big)\big| \ll_{\sigma,\mathbf{c}} f(a)e^{d_F\pi|\mathrm{Im}(z)|/2}, \qquad a\in F^\times,\ (\mathbf{s},z)\in\mathbb{L}_S(\mathbf{c})\times L_\sigma,$$

the problem is reduced to the convergence of the same series $\sum_{a\in F^\times} f(a)$ as above. Hence the assertion on $J_{\mathrm{u}}(\beta,\lambda,\alpha;t)$ is obtained. The integral $J_{\bar{\mathrm{u}}}(\beta,\lambda,\alpha;t)$ is examined in the same way. ∎

7.1. Hyperbolic terms. We consider the convergence of

$$J_{\mathrm{hyp}}(\beta,\lambda,\alpha;t) = \sum_{b\in F^\times-\{-1\}} J_{\delta_b}(\beta,\lambda,\alpha;t).$$

Let $v\in\Sigma_\infty$. For $t\in F_v^\times$, $b\in F_v^\times-\{-1\}$ and $\sigma,\rho\in\mathbb{R}$, set

$$f^{(\sigma)}(l_v;t,b) = \{(b+1)^2t^2+b^2\}^{\sigma/2-l_v/4}(1+t^{-2})^{-\sigma/2-l_v/4}|t|_v^{-2\sigma},$$
$$M_v(\sigma,\rho,l_v;b) = |b+1|_v^{-(\sigma-\rho)_-}|b|_v^{l_v/4-\sigma/2} \int_{F_v^\times} f^{(\sigma)}(l_v;t,b)|t|_v^{\sigma+\rho}\,d^\times t,$$

where $q_- = \min(0,q)$ for $q\in\mathbb{R}$.

Lemma 7.3. *Let $v\in\Sigma_\infty$. Then, for any $\sigma\in\mathbb{R}$,*

$$\big|\Psi_v^{(z)}\big(l_v;\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\big)\big| \le |b|_v^{-\sigma}|t|_v^\sigma e^{\pi|\mathrm{Im}(z)|/2} f_v^{(\sigma)}(l_v;t,b), \qquad t\in F_v^\times,\ b\in F_v^\times-\{-1\},\ z\in L_\sigma.$$

Proof. This follows from Proposition 3.1, Lemma 3.2 and $\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right] = \left[\begin{smallmatrix}b^{-1}tr^{-1}&0\\0&r\end{smallmatrix}\right]\left[\begin{smallmatrix}1&(b+1)t+bt^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1/r&-t/r\\t/r&1/r\end{smallmatrix}\right]$ with $r=(t^2+1)^{1/2}$. Note that $e^{\pi|\mathrm{Im}(z)|/2}$ arises as in the proof of Lemma 7.2. ∎

LEMMA 7.4. *Let $v \in \Sigma_\infty$ and $l_v \in 2\mathbb{Z}_{\geq 2}$. Let $\sigma, \rho \in \mathbb{R}$. Then*

$$M_v(\sigma,\rho,l_v;b) \ll |b+1|_v^{-l_v/4+\sigma/2-(\sigma-\rho)_-}, \quad b \in F_v^\times - \{-1\}, \tag{7.1}$$

if $l_v/4 > |\rho| - \sigma/2$ and $l_v/4 > \sigma/2$. Moreover, for $\epsilon > 0$ and $c \in \mathbb{R}$, the function $|b(b+1)|_v^\epsilon |b|_v^{-l_v/4+(c+1)/4} M_v(\sigma,\rho,l_v;b)$ in $b \in F_v$ is locally bounded if

$$\begin{aligned} &\big||\rho|-\sigma\big| + (\sigma-\rho)_- < \epsilon/3 < 1, \quad l_v/4 > \sigma/2 - (\sigma-\rho)_- + 1, \\ &(c+1)/4 > \sigma/2 - (\sigma-\rho)_-. \end{aligned} \tag{7.2}$$

Proof. The assertion is proved in a similar way to [21, Lemma 11.14]. From $b^2 + t^2(b+1)^2 \geq 2|b|\,|b+1|\,|t|$ and $\sigma/2 - l_v/4 < 0$, we estimate

$$\begin{aligned} M_v(\sigma,\rho,l_v;b) &\ll |b+1|^{-(\sigma-\rho)_-}|b|^{l_v/4-\sigma/2} \\ &\quad \times \int_0^\infty \{|b|\,|b+1|\,|t|\}^{\sigma/2-l_v/4}(1+t^{-2})^{-\sigma/2-l_v/4}|t|^{-\sigma+\rho}\, d^\times t \\ &= |b+1|^{-l_v/4+\sigma/2-(\sigma-\rho)_-} \int_0^\infty |t|^{\rho+l_v/4+\sigma/2}(1+t^2)^{-\sigma/2-l_v/4}\, d^\times t. \end{aligned}$$

The integral converges absolutely if $l_v/4 > |\rho| - \sigma/2$. As in [21, proof of Lemma 11.14], we have

$$\begin{aligned} &|b(b+1)|^\epsilon |b|^{-l_v/4+(c+1)/4} M_v(\sigma,\rho,l_v;b) \\ &\quad \ll |b+1|^{\sigma-|\rho|-(\sigma-\rho)_-+\epsilon/3}|b|^{(c+1)/4+\sigma/2-|\rho|+\epsilon/3}|b(b+1)|^{\epsilon/3}\mathfrak{m}(r;b(b+1)), \end{aligned}$$

where $r = l_v + 2\sigma - 4|\rho| - 4\epsilon/3$ and

$$\mathfrak{m}(r;b(b+1)) = \int_0^\infty [(1+t^{-2})(b^2+t^2(b+1)^2\}]^{-r/4}\, d^\times t.$$

By [21, Lemma 15.5], the function $|b(b+1)|^\epsilon \mathfrak{m}(r;b(b+1))$ (with $r > 0$) is locally bounded on F_v. From this, $|b(b+1)|^\epsilon |b|^{-l_v/4+(c+1)/4} M_v(\sigma,\rho,l_v;b)$ is also locally bounded on F_v if

$$\begin{aligned} &\sigma - |\rho| - (\sigma-\rho)_- + \epsilon/3 \geq 0, \quad r = l_v + 2\sigma - 4|\rho| - 4\epsilon/3 > 0, \\ &(c+1)/4 + \sigma/2 - |\rho| + \epsilon/3 \geq 0, \end{aligned}$$

which in turn follow from (7.2). Thus, under (7.2), the estimate (7.1) is extendable to F_v; from this, the last assertion is obvious. ∎

Let $\mathbf{c} = (c_v)_{v\in S} \in \mathbb{R}^S$, $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{Z}_{\geq 2})^{\Sigma_\infty}$, $t \in \mathbb{A}^\times$, $b \in F^\times - \{-1\}$ and $\sigma, \rho \in \mathbb{R}$. For $v \in S$, we set

$$f_v^{(\sigma)}(c_v;t_v,b) = \min(1,|t_v|_v^{-2})^\sigma \begin{cases} \max(1,|t_v|_v^{-1}|b|_v)^{-(c_v+1)/2+\sigma} & (|t_v|_v \leq 1), \\ \max(1,|t_v|_v|b+1|_v)^{-(c_v+1)/2+\sigma} & (|t_v|_v > 1), \end{cases}$$

$$M_v(\sigma,\rho,c;b) = \max(1,|b+1|_v)^{-(c+1)/4+\sigma/2+|\sigma-\rho|},$$

and for $v \in \Sigma_{\mathrm{fin}} - S$, we set

$$f_v^{(\sigma)}(t_v, b) = \min(1, |t_v|^{-2})^{\sigma} \delta\big(b \in \mathfrak{p}_v^{-f(\eta_v)}, q_v^{-2f(\eta_v)} |b|_v \leq |t_v|_v \leq |b+1|_v^{-1}\big).$$

Then, define

$$\begin{aligned} N(\mathfrak{n}|\sigma, l, \mathbf{c}; t, b) = |t|_{\mathbb{A}}^{\sigma} & \prod_{v \in \Sigma_\infty} f_v^{(\sigma)}(l_v; t_v, b) \prod_{v \in S} f_v^{(\sigma)}(c_v; t_v, b) \\ & \times \prod_{v \in S(\mathfrak{n})} \delta(t_v \in \mathfrak{n}\mathfrak{o}_v) f_v^{(\sigma)}(t_v, b) \prod_{v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}))} f_v^{(\sigma)}(t_v, b), \\ M(\mathfrak{n}|\sigma, \rho, l, \mathbf{c}; b) = & \prod_{v \in \Sigma_\infty} |b|_v^{-l_v/4 + \sigma/2} M_v(\sigma, \rho, l_v; b) \\ & \times \prod_{v \in S} |b|_v^{-(c_v+1)/4 + \sigma/2} M_v(\sigma, \rho; c_v, b) \\ & \times \prod_{v \in \Sigma_{\mathrm{fin}}} \max(1, |b|_v^{\sigma+\rho}) \prod_{v \in \Sigma_{\mathrm{fin}} - S} \delta(b \in \mathfrak{f}^{-1}\mathfrak{n}\mathfrak{o}_v) \end{aligned}$$

and $M_\epsilon(\mathfrak{n}|\sigma, \rho, l, \mathbf{c}; b) = \{\prod_{v \in \Sigma_\infty} |b(b+1)|_v^{\epsilon}\} M(\mathfrak{n}|\sigma, \rho, l, \mathbf{c}; b)$ for $\epsilon \geq 0$. By closely following [21, §11.4], we obtain the following series of lemmas.

LEMMA 7.5. *If $q(\mathbf{c}) > |\sigma| + 1$, then*

$$\big|\Psi_l^{(z)}\big(\mathfrak{n}|\mathbf{s}; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big)\big| \ll N(\mathfrak{n}|\sigma, l, \mathbf{c}; t, b) e^{d_F \pi |\mathrm{Im}(z)|/2},$$

uniformly in $(z, \mathbf{s}) \in L_\sigma \times \mathbb{L}_S(\mathbf{c})$, $b \in F^\times - \{-1\}$ and $t \in \mathbb{A}^\times$.

Proof. This follows from Lemma 7.3 and [21, Corollary 11.6 and Lemma 11.10]. ∎

LEMMA 7.6. *If $q(\mathbf{c}) > |\sigma| + |\rho| + 1$, $\underline{l}/4 > \max(\sigma/2, |\rho| - \sigma/2)$ and $\sigma \neq \pm\rho$, then*

$$\int_{\mathbb{A}^\times} N(\mathfrak{n}|\sigma, l, \mathbf{c}; t, b) |t|_{\mathbb{A}}^{\rho} \, d^\times t \ll_\epsilon M_\epsilon(\mathfrak{n}|\sigma, \rho, l, \mathbf{c}; b) \, \mathrm{N}(\mathfrak{n})^\epsilon, \qquad b \in F^\times - \{-1\},$$

for any $\epsilon > 0$, with the implied constant independent of the ideal $\mathfrak{n}$.

Proof. We can apply the same argument as in [21, Lemma 11.16] by using l_ι in place of $c_\iota + 1$ for all $\iota \in \Sigma_\infty$. ∎

LEMMA 7.7. *Let U be a compact subset of $\mathbb{A}^\times$. If $q(\mathbf{c}) > |\sigma| + |\rho| + 1$, $\underline{l}/4 > \max(\sigma/2, |\rho| - \sigma/2)$ and $\sigma \neq \pm\rho$, then*

$$\sum_{a \in F^\times} N(\mathfrak{n}|\sigma, l, \mathbf{c}; at, b) \ll_\epsilon M_\epsilon(\mathfrak{n}|\sigma, \rho, l, \mathbf{c}; b) \, \mathrm{N}(\mathfrak{n})^\epsilon, \qquad b \in F^\times, \, t \in U,$$

for any $\epsilon > 0$, with the implied constant independent of the ideal $\mathfrak{n}$.

Proof. This follows from Lemma 7.6 and the argument in [21, Corollary 11.17]. ∎

LEMMA 7.8. *If* $\sigma+\rho>-1$, $\sigma\neq\pm\rho$, $(c+1)/4>5|\sigma|/2+2|\rho|+1$, $\underline{l}/4>|\sigma|+|\rho|+1$ *and* $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1$, *then*

$$\sum_{b\in F^\times-\{-1\}} M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b)\ll \mathrm{N}(\mathfrak{n})^{-(c+1)/4+\sigma/2+|\sigma+\rho|}$$

for any $\epsilon>0$ *such that* $\big||\rho|-\sigma\big|+(\sigma-\rho)_-<\epsilon/3<1$ *and* $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1+2\epsilon$, *with the implied constant independent of* $\mathfrak{n}$. *Here* $\mathbf{c}=(c_v)_{v\in S}$ *with* $c_v=c$ *(for all* $v\in S$*).*

Proof. The proof follows that of [21, Lemma 11.19], with $c_\iota+1$ replaced by l_ι for all $\iota\in\Sigma_\infty$. Under the present assumption on l,σ,ρ,c, the series

$$\sum_{b\in\mathfrak{o}(S)-\{-1\}}\Big\{\prod_{v\in\Sigma_\infty}|b|_v^{-l_v/4+(c+1)/4}M_v(\sigma,\rho,l_v;b)\Big\}\times\Big\{\prod_{v\in S}\max(1,|b|_v^{\sigma+\rho})M_v(\sigma,\rho,c_v;b)\Big\}|\mathrm{N}(b(b+1))|^\epsilon,$$

which is denoted by $A_S(\sigma,\rho,l,c)$, converges for any $\epsilon>0$ such that $\big||\rho|-\sigma\big|+(\sigma-\rho)_-<\epsilon/3<1$ and $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1+2\epsilon$. Here $\mathfrak{o}(S)$ denotes the S-integer ring of F. Indeed, this follows from Lemma 7.4 and [21, Lemma 11.18]. By the Artin product formula $|b|_{\mathbb{A}}=1$ for $b\in F^\times$, we have

$$\begin{aligned}
&\sum_{b\in F^\times-\{-1\}} M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b)\\
&=\sum_{b\in\mathfrak{f}^{-1}\mathfrak{n}\mathfrak{o}(S)-\{0,-1\}}\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}}\max(1,|b|_v^{\sigma+\rho})\Big\}\Big\{\prod_{v\in S}|b|_v^{-(c+1)/4+\sigma/2}M_v(\sigma,\rho,c;b)\Big\}\\
&\quad\times\Big\{\prod_{v\in\Sigma_\infty}|b|^{-l_v/4+\sigma/2}M_v(\sigma,\rho,l_v;b)\Big\}|\mathrm{N}(b(b+1))|^\epsilon\\
&=\sum_{\substack{b\in\mathfrak{f}^{-1}\mathfrak{n}\mathfrak{o}(S)/\mathfrak{o}(S)^\times\\ b\neq0,-1}}\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}-S}\max(1,|b|_v^{\sigma+\rho})|b|_v^{(c+1)/4-\sigma/2}\Big\}\\
&\quad\times\sum_{u\in\mathfrak{o}(S)^\times}\Big\{\prod_{v\in\Sigma_\infty}|ub|^{-l_v/4+(c+1)/4}M_v(\sigma,\rho,l_v;ub)\Big\}\\
&\quad\times\Big\{\prod_{v\in S}\max(1,|ub|_v^{\sigma+\rho})M_v(\sigma,\rho,c;ub)\Big\}|\mathrm{N}(ub(ub+1))|^\epsilon\\
&\ll\sum_{\substack{b\in\mathfrak{f}^{-1}\mathfrak{n}\mathfrak{o}(S)/\mathfrak{o}(S)^\times\\ b\neq0,-1}}\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}-S}\max(1,|b|_v^{\sigma+\rho})|b|_v^{(c+1)/4-\sigma/2}\Big\}\times A_S(\sigma,\rho,l,c).
\end{aligned}$$

We note that the series in the last line is majorized by $\mathrm{N}(\mathfrak{n})^{-(c+1)/4+\sigma/2+|\sigma+\rho|}$ as in the proof of [21, Lemma 11.19]. ∎

For $c \in \mathbb{R}$, set $\mathbb{L}_S(c) = \{\mathbf{s} \in \mathfrak{X}_S \mid \mathrm{Re}(s_v) = c \ (\forall v \in S)\}$.

LEMMA 7.9. *Let* $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{Z}_{\geq 2})^{\Sigma_\infty}$ *and* $c, \sigma \in \mathbb{R}$. *Assume that*

$$\underline{l} \geq 6, \quad \sigma > -1, \quad (c+1)/4 > 9|\sigma|/2 + 1, \quad \underline{l}/2 > (c+1)/4 + 5|\sigma|/2 + 1.$$

Then, for any compact subset U *of* $\mathbb{A}^\times$, *the series*

$$\sum_{b\in F^\times - \{-1\}} \sum_{a\in F^\times} \left|\Psi_l^{(z)}\big(\mathfrak{n}|\mathbf{s}; \delta_b \left[\begin{smallmatrix} at & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big)\right|$$

converges uniformly in $(t, z, \mathbf{s}) \in U \times L_\sigma \times \mathbb{L}_S(c)$, *and there exists* $\epsilon > 0$ *such that, for any* $\rho \in \mathbb{R}$ *satisfying* $0 < \big||\rho| - \sigma\big| < \epsilon$ *and* $\sigma + \rho > -1$, *the integral*

$$\sum_{b\in F^\times - \{-1\}} \int_{t\in\mathbb{A}^\times} \left|\Psi_l^{(z)}\big(\mathfrak{n}|\mathbf{s}; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big)\right| |t|_{\mathbb{A}}^{\rho} \, d^\times t$$

converges uniformly in $(z, \mathbf{s}) \in L_\sigma \times \mathbb{L}_S(c)$.

Proof. By assumption, we can take $\rho \in \mathbb{R}$ such that $(c+1)/4 > 5|\sigma|/2 + 2|\rho| + 1$, $\sigma + \rho > -1$, $\underline{l}/4 > |\sigma| + |\rho| + 1$ and $\underline{l}/2 > (c+1)/4 + 3|\sigma|/2 + |\rho| + 1$ (we can take $\rho = 0$ if $\sigma > -1$ and $\sigma \neq 0$). Thus the assertion follows from Lemmas 7.5–7.8. We remark that the condition $\underline{l} \geq 6$ is forced by the third and the fourth inequalities in Lemma 7.9; indeed, they imply $\underline{l}/2 > 7|\sigma| + 2$, and hence $\underline{l} > 4$. ∎

LEMMA 7.10. *Suppose* $\underline{l} \geq 6$. *The function* $J_{\mathrm{hyp}}(\beta, \lambda, \alpha; t)$ *on* $\mathrm{Re}(\lambda) > 1$ *has a holomorphic continuation to* $\mathbb{C}$ *whose value at* $\lambda = 0$ *equals* $J_{\mathrm{hyp}}(\alpha; t)\beta(0)$, *where*

$$J_{\mathrm{hyp}}(\alpha; t) = \sum_{b\in F^\times - \{-1\}} \sum_{a\in F^\times} \hat{\Psi}_l^{(0)}\big(\mathfrak{n}|\alpha; \delta_b \left[\begin{smallmatrix} at & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big).$$

The series converges absolutely and uniformly in $t \in \mathbb{A}^\times$. *Here we have set*

$$\hat{\Psi}_l^{(0)}(\mathfrak{n}|\alpha; g) = \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(c)} \Psi_l^{(0)}(\mathfrak{n}|\mathbf{s}; g)\alpha(\mathbf{s}) \, d\mu_S(\mathbf{s})$$

with c *being a real number such that* $(c+1)/4 > 1$ (cf. [21, §6.3]).

Proof. This follows from Lemma 7.9 in the same way as [21, Lemma 11.21]. ∎

From Lemmas 7.1, 7.2 and 7.10, we have

$$\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}\big(\mathfrak{n}|\alpha; \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big) = (1 + i^{\tilde{l}}\delta(\mathfrak{n} = \mathfrak{o}))J_{\mathrm{id}}(\alpha; t) + J_{\mathrm{u}}(\alpha; t) + J_{\bar{\mathrm{u}}}(\alpha; t) + J_{\mathrm{hyp}}(\alpha; t) \tag{7.3}$$

for any $t \in \mathbb{A}^\times$. Some terms on the right-hand side, viewed as functions on $Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}$ individually, have divergent (H, η)-period integrals; to proceed further, we need to regularize them.

8. Geometric side. Suppose $\underline{l} = \inf_{v\in\Sigma_\infty} l_v \geq 6$. Let $\beta \in \mathcal{B}$. For $\natural \in \{\mathrm{id}, \mathrm{u}, \bar{\mathrm{u}}, \mathrm{hyp}\}$, we set

$$\mathbb{J}^\eta_\natural(\beta,\lambda;\alpha) = \int\limits_{F^\times\backslash\mathbb{A}^\times} J_\natural(\alpha;t)\{\hat\beta_\lambda(|t|_\mathbb{A}) + \hat\beta_\lambda(|t|_\mathbb{A}^{-1})\}\eta(tx^*_\eta)\,d^\times t.$$

In this section, we shall show that this integral converges absolutely when $\mathrm{Re}(\lambda) \gg 0$ and has a meromorphic continuation to a neighborhood of $\lambda = 0$; at the same time, we determine the constant term in its Laurent expansion at $\lambda = 0$. As a result, by the identity

$$\begin{aligned} P^\eta_{\beta,\lambda}(\hat{\boldsymbol{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)) &= \mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha) + i^{\tilde l}\delta(\mathfrak{n}=\mathfrak{o})\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha) \\ &\quad + \mathbb{J}^\eta_{\mathrm{u}}(\beta,\lambda;\alpha) + \mathbb{J}^\eta_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha) + \mathbb{J}^\eta_{\mathrm{hyp}}(\beta,\lambda;\alpha) \end{aligned}$$

obtained from (7.3), we have another expression of $P^\eta_{\mathrm{reg}}(\hat{\boldsymbol{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ already computed in Proposition 6.6 by means of the spectral expansion.

LEMMA 8.1. *For* $\mathrm{Re}(\lambda) > 0$, *the* $\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha)$ *converges absolutely and*

$$\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha) = \delta_{\eta,\mathbf{1}}\,\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\left(\frac{1}{2\pi i}\right)^{\#S}\int\limits_{\mathbb{L}_S(\mathbf{c})} \Upsilon^{\mathbf{1}}_S(\mathbf{s})\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})\,\frac{2\beta(0)}{\lambda},$$

where $\delta_{\eta,\mathbf{1}} = \delta(\eta = \mathbf{1})$. *We have a meromorphic continuation of* $\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha)$ $(\mathrm{Re}(\lambda) > 0)$ *to* $\mathbb{C}$ *with* $\mathrm{CT}_{\lambda=0}\,\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha) = 0$.

Proof. The first claim is shown in the same way as [21, Lemma 12.2]. From the expression, the function $\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha)$ $(\mathrm{Re}(\lambda) > 0)$, which is just a constant multiple of λ^{-1}, obviously has a meromorphic continuation to $\mathbb{C}$ with $\mathrm{CT}_{\lambda=0}\,\mathbb{J}^\eta_{\mathrm{id}}(\beta,\lambda;\alpha) = 0$. ∎

Let us examine the terms $\mathbb{J}^\eta_{\mathrm{u}}(\beta,\lambda;\alpha)$ and $\mathbb{J}^\eta_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$. Assume that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma > 1$ and $1 < \sigma < \underline{l}/2$, and set

$$U^\pm_{0,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int\limits_{L_{\mp\sigma}}\frac{\beta(z)}{z+\lambda}\int\limits_{\mathbb{A}^\times}\Psi^{(0)}_l\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1 & t^{-1}\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\big)\eta(tx^*_\eta)|t|^{\pm z}_\mathbb{A}\,d^\times t\,dz,$$

$$U^\pm_{1,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int\limits_{L_{\mp\sigma}}\frac{\beta(z)}{z+\lambda}\int\limits_{\mathbb{A}^\times}\Psi^{(0)}_l\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1 & 0\\ t^{-1} & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ -x_\eta & 1\end{smallmatrix}\right]w_0\big)\eta(tx^*_\eta)|t|^{\pm z}_\mathbb{A}\,d^\times t\,dz$$

with $w_0 = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, and

$$\Upsilon^\eta_S(z;\mathbf{s}) = \prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(z+(s_v+1)/2)})^{-1}(1-q_v^{(s_v+1)/2})^{-1},$$

$$\begin{aligned}\Upsilon^\eta_{S,l}(z;\mathbf{s}) &= D_F^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f})^\times\}^{-1} \\ &\quad\times\left\{\prod_{v\in\Sigma_\infty}\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_\mathbb{R}(-z+\epsilon_v)\Gamma(l_v/2)}i^{\epsilon_v}\cos\left(\frac{\pi}{2}(-z+\epsilon_v)\right)\right\}\Upsilon^\eta_S(z;\mathbf{s}).\end{aligned}$$

Here $\epsilon_v \in \{0,1\}$ is the sign of η_v for $v \in \Sigma_\infty$ (see §2.4).

LEMMA 8.2. *The double integrals* $U^{\pm}_{0,\eta}(\lambda;\mathbf{s})$ *and* $U^{\pm}_{1,\eta}(\lambda;\mathbf{s})$ *converge absolutely, and*

$$U^{\pm}_{0,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int\limits_{L_{\mp\sigma}} \frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f})^{\mp z}L(\mp z,\eta)(-1)^{\epsilon(\eta)}\Upsilon^{\eta}_{S,l}(\pm z;\mathbf{s})\,dz,$$

$$U^{\pm}_{1,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i} \times \int\limits_{L_{\mp\sigma}} \frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f})^{\mp z}\mathrm{N}(\mathfrak{n})^{\pm z}\tilde{\eta}(\mathfrak{n})\delta(\mathfrak{n}=\mathfrak{o})L(\mp z,\eta)i^{\tilde{l}}\Upsilon^{\eta}_{S,l}(\pm z;\mathbf{s})\,dz,$$

where $\tilde{l} = \sum_{v\in\Sigma_\infty} l_v$ *and* $\epsilon(\eta) = \sum_{v\in\Sigma_\infty}\epsilon_v$.

Proof. This is proved in the same way as [21, Lemma 12.3]; to compute the archimedean integral, we use Lemma 3.6. ∎

Since $1 < \sigma < \underline{l}/2$, the possible poles of the integrand of $U^{+}_{0,\eta}(\lambda;\mathbf{s})$ in the region $-\sigma < \mathrm{Re}(z) < \sigma$ are $z = 0, -1$. In fact, we observe that the integrand is holomorphic at $z = -1$. We shift the contour $L_{-\sigma}$ to L_σ; by the residue theorem,

$$U^{+}_{0,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int\limits_{L_\sigma} \frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f})^{-z}L(-z,\eta)(-1)^{\epsilon(\eta)}\Upsilon^{\eta}_{S,l}(z;\mathbf{s})\,dz - \frac{\beta(0)}{\lambda}\delta_{\eta,\mathbf{1}}R_F\Upsilon^{\mathbf{1}}_S(\mathbf{s}),$$

where R_F is the residue of $\zeta_F(s)$ at $s = 1$. In a similar manner,

$$U^{+}_{1,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int\limits_{L_\sigma} \frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f})^{-z}L(-z,\eta)\delta(\mathfrak{n}=\mathfrak{o})i^{\tilde{l}}\Upsilon^{\eta}_{S,l}(z;\mathbf{s})\,dz - \frac{\beta(0)}{\lambda}\delta_{\eta,\mathbf{1}}R_F\delta(\mathfrak{n}=\mathfrak{o})i^{\tilde{l}}\Upsilon^{\mathbf{1}}_S(\mathbf{s}).$$

Define $C_0(\eta)$ and $R(\eta)$ by

$$L(s,\eta) = R(\eta)(s-1)^{-1} + C_0(\eta) + \mathcal{O}(s-1) \quad (s\to 1).$$

We remark that $R_F = R(\eta)$ if η is trivial. Let C_{Euler} be Euler's constant.

LEMMA 8.3. *The function* $\lambda \mapsto \mathbb{J}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha)$ *on* $\mathrm{Re}(\lambda) > 1$ *has a meromorphic continuation to the region* $\mathrm{Re}(\lambda) > -\underline{l}/2$. *The constant term of* $\mathbb{J}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha)$ *at* $\lambda = 0$ *equals* $\mathbb{J}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha)\beta(0)$. *Here we have set*

$$\mathbb{J}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\big(1+(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big) \times \left(\frac{1}{2\pi i}\right)^{\#S}\int\limits_{\mathbb{L}_S(\mathbf{c})}\Upsilon^{\eta}_S(\mathbf{s})\mathfrak{C}^{\eta}_{S,\mathrm{u}}(\mathbf{s})\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

with

$$\mathfrak{C}^{\eta}_{S,\mathrm{u}}(\mathbf{s}) = \pi^{\epsilon(\eta)}C_0(\eta) + R(\eta)\Bigg\{-\frac{d_F}{2}(C_{\mathrm{Euler}} + \log\pi) + \sum_{v\in\Sigma_\infty}\sum_{k=1}^{l_v/2-1}\frac{1}{k} + \sum_{v\in S}\frac{\log q_v}{1-q_v^{(s_v+1)/2}} + \log D_F\Bigg\}.$$

In particular, $\mathfrak{C}^{\eta}_{S,\mathrm{u}}(\mathbf{s}) = L_{\mathrm{fin}}(1,\eta)$ *if* η *is non-trivial.*

Proof. By definition,

$$\mathbb{J}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int\limits_{\mathbb{L}_S(\mathbf{c})}\big(U^{+}_{0,\eta}(\lambda;\mathbf{s})+U^{-}_{0,\eta}(\lambda;\mathbf{s})+U^{+}_{1,\eta}(\lambda;\mathbf{s})+U^{-}_{1,\eta}(\lambda;\mathbf{s})\big)\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}).$$

From Lemma 8.2 and the computation after it,

$$\begin{aligned}\mathbb{J}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}&\int\limits_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int\limits_{L_\sigma}\frac{\beta(z)}{z+\lambda}((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\\ &\{\mathrm{N}(\mathfrak{f})^{-z}L(-z,\eta)\Upsilon^{\eta}_{S,l}(z;\mathbf{s})+\mathrm{N}(\mathfrak{f})^{z}L(z,\eta)\Upsilon^{\eta}_{S,l}(-z;\mathbf{s})\}\,dz\,\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})\\ &-\frac{(-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})}{2}\mathbb{J}^{\eta}_{\mathrm{id}}(\beta,\lambda;\alpha),\end{aligned}$$

with $1<\sigma<\underline{l}/2$ and $\mathrm{Re}(\lambda)>-\sigma$. Since σ is arbitrary, this gives a meromorphic continuation of $\mathbb{J}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha)$ to $\mathrm{Re}(\lambda)>-\underline{l}/2$. By the above expression,

$$\begin{aligned}&\mathrm{CT}_{\lambda=0}\,\mathbb{J}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha)\\ &\quad=\left(\frac{1}{2\pi i}\right)^{\#S}\int\limits_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\times\int\limits_{L_\sigma}\frac{\beta(z)}{z}((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\\ &\qquad\times\{\mathrm{N}(\mathfrak{f})^{-z}L(-z,\eta)\Upsilon^{\eta}_{S,l}(z;\mathbf{s})+\mathrm{N}(\mathfrak{f})^{z}L(z,\eta)\Upsilon^{\eta}_{S,l}(-z;\mathbf{s})\}\,dz\,\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})\\ &\quad=((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\frac{1}{2\pi i}\int\limits_{L_\sigma}\frac{\beta(z)}{z}\{f_{\mathrm{u}}(z)+f_{\mathrm{u}}(-z)\}\,dz\\ &\quad=((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\,\mathrm{Res}_{z=0}\left(\frac{\beta(z)}{z}f_{\mathrm{u}}(z)\right)\\ &\quad=((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\beta(0)\,\mathrm{CT}_{z=0}\,f_{\mathrm{u}}(z).\end{aligned}$$

Here we have set $f_{\mathrm{u}}(z)=\mathrm{N}(\mathfrak{f})^{-z}L(-z,\eta)\Upsilon^{\eta}_{S,l}(z;\mathbf{s})$. By setting

$$\tilde{\Upsilon}^{\eta}_{S,l}(z;\mathbf{s}) = D_F^{1/2}\{\#(\mathfrak{o}/\mathfrak{f})^{\times}\}\Upsilon^{\eta}_{S,l}(z;\mathbf{s}),$$

the constant term is computed as follows:

$$\begin{aligned}
\mathrm{CT}_{z=0}\, f_{\mathrm{u}}(z) &= \frac{d}{dz}\mathrm{N}(\mathfrak{f})^{-z} z L(-z,\eta)\Upsilon^{\eta}_{S,l}(z;\mathbf{s})\Big|_{z=0} \\
&= \frac{d}{dz}\{\mathrm{N}(\mathfrak{f})^{-z} \times z i^{\epsilon(\eta)} D_F^{1/2}\mathrm{N}(\mathfrak{f})^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f})^{\times}\} \\
&\qquad \times \mathcal{G}(\eta) D_F^{1/2+z}\mathrm{N}(\mathfrak{f})^{1/2+z} L(z+1,\eta) D_F^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f})^{\times}\}^{-1}\tilde{\Upsilon}^{\eta}_{S,l}(z;\mathbf{s})\}\Big|_{z=0} \\
&= i^{\epsilon(\eta)}\mathcal{G}(\eta) D_F^{1/2} \times \frac{d}{dz}\big\{D_F^{z} z L(z+1,\eta) \times \tilde{\Upsilon}^{\eta}_{S,l}(z;\mathbf{s})\big\}\Big|_{z=0} \\
&= \mathcal{G}(\eta) D_F^{1/2}\pi^{\epsilon(\eta)}\tilde{\Upsilon}^{\eta}_{S}(\mathbf{s})\Big\{(\log D_F)R(\eta) + C_0(\eta) + R(\eta)\frac{\frac{d}{dz}\tilde{\Upsilon}^{\mathbf{1}}_{S,l}(z;\mathbf{s})\big|_{z=0}}{\tilde{\Upsilon}^{\mathbf{1}}_{S,l}(0;\mathbf{s})}\Big\}.
\end{aligned}$$

Here $\epsilon(\eta) = \sum_{v\in\Sigma_\infty}\epsilon_v$. We note that $\tilde{\Upsilon}^{\eta}_{S,l}(0;\mathbf{s}) = (-i\pi)^{\epsilon(\eta)}\Upsilon^{\eta}_{S}(0;\mathbf{s})$ since

$$\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z+\epsilon_v)\Gamma(l_v/2)} i^{\epsilon_v}\cos\Big(\frac{\pi}{2}(-z+\epsilon_v)\Big)\Big|_{z=0} = (-i\pi)^{\epsilon_v}$$

for $v\in\Sigma_\infty$. The logarithmic derivative of $\tilde{\Upsilon}^{\mathbf{1}}_{S,l}(z;\mathbf{s})$ at $z=0$ is computed as

$$\begin{aligned}
&\sum_{v\in\Sigma_\infty}\frac{d}{dz}\log\Big\{\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z)\Gamma(l_v/2)}\cos\Big(\frac{\pi z}{2}\Big)\Big\}\Big|_{z=0} \\
&\qquad\qquad + \sum_{v\in S}\frac{d}{dz}\log(1-q_v^{-(z+(s_v+1)/2)})^{-1}(1-q_v^{(s_v+1)/2})^{-1}\Big|_{z=0} \\
&= \sum_{v\in\Sigma_\infty}\Big\{\psi(l_v/2) - \frac{1}{2}\log\pi + \Big(\frac{1}{2}\psi\Big(\frac{-z}{2}\Big) - \psi(-z)\Big)\Big|_{z=0}\Big\} + \sum_{v\in S}\frac{\log q_v}{1-q_v^{(s_v+1)/2}}.
\end{aligned}$$

By the formulas

$$\psi(l_v/2) = -C_{\mathrm{Euler}} + \sum_{k=1}^{l_v/2-1}\frac{1}{k}, \qquad \Big(\frac{1}{2}\psi\Big(\frac{-z}{2}\Big) - \psi(-z)\Big)\Big|_{z=0} = \frac{1}{2}C_{\mathrm{Euler}},$$

we are done. ∎

Assume that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma$ and $1 < \sigma < \underline{l}/2$. Analyzing the integrals

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}\,d^\times t\,dz, \\
&\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}-1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\big)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}\,d^\times t\,dz
\end{aligned}$$

in the same way as $U^{\pm}_{\epsilon,\eta}(\lambda;\mathbf{s})$, we obtain the following lemma.

LEMMA 8.4. *The function* $\lambda \mapsto \mathbb{J}^{\eta}_{\mathfrak{u}}(\beta, \lambda; \alpha)$ *on* $\mathrm{Re}(\lambda) > 1$ *has a meromorphic continuation to the region* $\mathrm{Re}(\lambda) > -\underline{l}/2$. *The constant term of* $\mathbb{J}^{\eta}_{\mathfrak{u}}(\beta, \lambda; \alpha)$ *at* $\lambda = 0$ *equals* $\mathbb{J}^{\eta}_{\bar{\mathfrak{u}}}(l, \mathfrak{n}|\alpha)\beta(0)$. *Here we have set*

$$\mathbb{J}^{\eta}_{\bar{\mathfrak{u}}}(l, \mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\big((-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}) + i^{\tilde{l}}\delta(\mathfrak{n} = \mathfrak{o})\big) \times \left(\frac{1}{2\pi i}\right)^{\#S} \int\limits_{\mathbb{L}_S(\mathbf{c})} \Upsilon^{\eta}_S(\mathbf{s})\mathfrak{C}^{\eta}_{S,\bar{\mathfrak{u}}}(\mathbf{s})\alpha(\mathbf{s})\, d\mu_S(\mathbf{s})$$

with $\mathfrak{C}^{\eta}_{S,\bar{\mathfrak{u}}}(\mathbf{s}) = \mathfrak{C}^{\eta}_{S,\mathrm{u}}(\mathbf{s}) + R(\eta)\log \mathrm{N}(\mathfrak{n})$.

Let us consider the term $\mathbb{J}^{\eta}_{\mathrm{hyp}}(\beta, \lambda; \alpha)$, which is, by definition, equal to

$$\int\limits_{\mathbb{A}^\times} \sum_{b \in F^\times - \{-1\}} \hat{\Psi}^{(0)}_l\big(\mathfrak{n}|\alpha; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big)\{\hat{\beta}_\lambda(|t|_{\mathbb{A}}) + \hat{\beta}_\lambda(|t|^{-1}_{\mathbb{A}})\}\eta(tx^*_\eta)\, d^\times t.$$

LEMMA 8.5. *The integral* $\mathbb{J}^{\eta}_{\mathrm{hyp}}(\beta, \lambda; \alpha)$ *converges absolutely and has an analytic continuation to the region* $\mathrm{Re}(\lambda) > -\epsilon$ *for some* $\epsilon > 0$. *Moreover,* $\mathrm{CT}_{\lambda=0}\, \mathbb{J}^{\eta}_{\mathrm{hyp}}(\beta, \lambda; \alpha) = \mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)\beta(0)$. *Here* $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)$ *is defined by*

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S} \int\limits_{\mathbb{L}_S(c)} \mathfrak{K}^{\eta}_S(l, \mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})\, d\mu_S(\mathbf{s})$$

with

$$\mathfrak{K}^{\eta}_S(l, \mathfrak{n}|\mathbf{s}) = \sum_{b \in F^\times - \{-1\}} \int\limits_{\mathbb{A}^\times} \Psi^{(0)}_l\big(\mathfrak{n}|\mathbf{s}; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big)\eta(tx^*_\eta)\, d^\times t.$$

Proof. We choose $c \in \mathbb{R}$ such that $\underline{l}/2 - 1 > (c+1)/4 > 1$. Set $\mathbf{c} = (c)_{v \in S}$. Then, from Lemmas 7.5, 7.6 and 7.8, there exists $\epsilon > 0$ such that for $0 < |\rho| < \epsilon$ the integral

$$\int\limits_{\mathbb{L}_S(\mathbf{c})} |\alpha(\mathbf{s})|\,|\, d\mu_S(\mathbf{s})| \times \int\limits_{L_\rho} \frac{|\beta(z)|}{|z+\lambda|} \sum_{b \in F^\times - \{-1\}} \int\limits_{\mathbb{A}^\times} \big|\Psi^{(0)}_l\big(\mathfrak{n}|\mathbf{s}; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big)\big|\{|t|^{\rho}_{\mathbb{A}} + |t|^{-\rho}_{\mathbb{A}}\}\, d^\times t\, |dz|, \tag{8.1}$$

which is majorized by

$$\int\limits_{\mathbb{L}_S(\mathbf{c})} |\alpha(\mathbf{s})|\,|\, d\mu_S(\mathbf{s})| \int\limits_{L_\rho} \frac{|\beta(z)|}{|z+\lambda|}\, |dz| \times \sum_{b \in F^\times - \{-1\}} \{M_\epsilon(\mathfrak{n}|0, \rho, l, \mathbf{c}; b) + M_\epsilon(\mathfrak{n}|0, -\rho, l, \mathbf{c}; b)\}\, \mathrm{N}(\mathfrak{n})^\epsilon,$$

is convergent. Since $|t|^{\rho}_{\mathbb{A}} + |t|^{-\rho}_{\mathbb{A}} \geq 2$ $(t \in \mathbb{A}^\times)$, the integral (8.1) is finite even

for $\rho = 0$. Hence, we obtain an analytic continuation of the function

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(\beta, \lambda; \alpha) = \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \left\{ \frac{1}{2\pi i} \int_{L_\rho} \frac{\beta(z)}{z+\lambda} \Big(\sum_{b \in F^\times - \{-1\}} \int_{\mathbb{A}^\times} \Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s}; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\big) \times (|t|_{\mathbb{A}}^{z} + |t|_{\mathbb{A}}^{-z}) \eta(t x_\eta^*)\, d^\times t \Big)\, dz \right\} \alpha(\mathbf{s})\, d\mu_S(\mathbf{s})$$

in the variable λ to the region $\mathrm{Re}(\lambda) > -\epsilon$. ∎

9. The relative trace formula. Let $\mathfrak{n}$ be an integral ideal of F, $l = (l_v)_{v \in \Sigma_\infty}$ an even weight with $l_v \geq 6$ for all $v \in \Sigma_\infty$, and η a real valued idele class character of $F^\times$ unramified at all $v \in S(\mathfrak{n})$. Let $\mathfrak{f}$ denote the conductor of η. We assume $(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}) = 1$ (for the definition of $\epsilon(\eta)$, see §2.4). Set $\tilde{l} = \sum_{v \in \Sigma_\infty} l_v$. Let S be a finite subset of Σ_{fin} disjoint from $S(\mathfrak{n}) \cup S(\mathfrak{f})$. For $v \in S$, let $\mathcal{A}_v$ be the space of all holomorphic functions $\alpha_v(s_v)$ in $s_v \in \mathbb{C}$ satisfying $\alpha_v(s_v) = \alpha_v(-s_v)$ and $\alpha_v\big(s_v + \frac{4\pi i}{\log q_v}\big) = \alpha_v(s_v)$. Let $\mathcal{A}_S$ denote the space of holomorphic functions on $\mathfrak{X}_S$ such that $\alpha(\epsilon\mathbf{s}) = \alpha(\mathbf{s})$ for all $\epsilon \in \{\pm 1\}^S$. Then $\bigotimes_{v \in S} \mathcal{A}_v \subset \mathcal{A}_S$.

THEOREM 9.1. *For any $\alpha \in \mathcal{A}_S$,*

$$C(l, \mathfrak{n}, S) \sum_{\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})} \mathbb{I}^{\eta}_{\mathrm{cus}}(\pi; l, \mathfrak{n}) \alpha(\nu_S(\pi)) = \tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l, \mathfrak{n}|\alpha) + \mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha). \tag{9.1}$$

Here $\nu_S(\pi) = \{\nu_v(\pi)\}_{v \in S}$ is the spectral parameter of π at S (see §6.5),

$$C(l, \mathfrak{n}, S) = (-1)^{\#S} \left\{ \prod_{v \in \Sigma_\infty} \frac{2\pi \Gamma(l_v - 1)}{\Gamma(l_v/2)^2} \right\} \frac{D_F^{-1}}{2} [\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1},$$

$$\mathbb{I}^{\eta}_{\mathrm{cus}}(\pi; l, \mathfrak{n}) = (-1)^{\epsilon(\eta)} \mathcal{G}(\eta) w^{\eta}_{\mathfrak{n}}(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1} L^{S_\pi}(1, \pi; \mathrm{Ad})}$$

with $w^{\eta}_{\mathfrak{n}}(\pi)$ given in Lemma 6.2, and

$$\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l, \mathfrak{n}|\alpha) = 2(-1)^{\epsilon(\eta)} \mathcal{G}(\eta) D_F^{1/2} (1 + i^{\tilde{l}} \delta(\mathfrak{n} = \mathfrak{o})) \times \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \mathfrak{U}^{\eta}_S(l, \mathfrak{n}|\mathbf{s}) \alpha(\mathbf{s})\, d\mu_S(\mathbf{s}),$$

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S} \int_{L_S(c)} \mathfrak{K}^{\eta}_S(l, \mathfrak{n}|\mathbf{s}) \alpha(\mathbf{s})\, d\mu_S(\mathbf{s})$$

with

$$\mathfrak{U}_S^{\eta}(l,\mathfrak{n}|\mathbf{s}) = \Big\{\prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(s_v+1)/2})^{-1}(1-q_v^{(s_v+1)/2})^{-1}\Big\} \times \Big\{\mathbf{C}_F^{\eta}(l,\mathfrak{n}) + R(\eta)\sum_{v\in S}\frac{\log q_v}{1-q_v^{(s_v+1)/2}}\Big\},$$

$$\mathfrak{K}_S^{\eta}(l,\mathfrak{n}|\mathbf{s}) = \sum_{b\in F^\times-\{-1\}}\int_{\mathbb{A}^\times} \Psi_l^{(0)}\big(\mathfrak{n}|\mathbf{s};\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\big)\eta(tx_\eta^*)\,d^\times t$$

and

$$\mathbf{C}_F^{\eta}(l,\mathfrak{n}) = \pi^{\epsilon(\eta)}C_0(\eta) + R(\eta)\Big\{-\frac{d_F}{2}(C_{\mathrm{Euler}}+\log\pi)+\log(D_F\mathrm{N}(\mathfrak{n})^{1/2})+\sum_{v\in\Sigma_\infty}\sum_{k=1}^{l_v/2-1}\frac{1}{k}\Big\}.$$

We remark that $\mathbf{C}_F^{\eta}(l,\mathfrak{n}) = L_{\mathrm{fin}}(1,\eta)$ *if* η *is non-trivial.*

Proof. From Lemmas 8.1, 8.3, 8.4 and 8.5, $P_{\mathrm{reg}}^{\eta}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha))$ is given by the right-hand side of (9.1); the left-hand side is provided by Proposition 6.6. ∎

We restrict our attention to the test functions of the form $\alpha(\mathbf{s}) = \prod_{v\in S}\alpha_v^{(m_v)}(s_v)$ with

$$\alpha_v^{(m)}(s_v) = q_v^{ms_v/2}+q_v^{-ms_v/2}, \qquad v\in S,\ m\in\mathbb{N}_0. \tag{9.2}$$

As is well known, these functions form a $\mathbb{C}$-basis of the image of the spherical Hecke algebra $\mathcal{H}(G_v,\mathbf{K}_v)$ under the spherical Fourier transform. Thus, by restricting our consideration to these functions, no generality is lost practically. The following two theorems are proved in §10 and §11.

THEOREM 9.2. *For* $\alpha=\bigotimes_{v\in S}\alpha_v$, *we have*

$$\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha) = \sum_{b\in F^\times-\{-1\}}\Big\{\prod_{v\in S}J_v^{\eta_v}(b;\alpha_v)\Big\}\Big\{\prod_{v\in\Sigma_\infty}J_v^{\eta_v}(l_v;b)\Big\}\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}-S}J_v^{\eta_v}(b)\Big\}.$$

Here $J_v^{\eta_v}(b;\alpha_v)$ *is given by Lemma* 10.2, $J_v^{\eta_v}(b)$ *is given by Lemmas* 10.4, 10.5 *and* 10.9, *and* $J_v^{\eta_v}(l_v;b)$ *is given by Lemma* 10.15.

THEOREM 9.3. *For* $\alpha=\bigotimes_{v\in S}\alpha_v$, *we have*

$$\tilde{\mathbb{J}}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha) = 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})) \times\Big\{\mathbf{C}_F^{\eta}(l,\mathfrak{n})\prod_{v\in S}U_v^{\eta_v}(\alpha_v)+R(\eta)\sum_{v\in S}U_v'(\alpha_v)\prod_{w\in S-\{v\}}U_w^{\eta_w}(\alpha_w)\Big\}.$$

Here $U_v^{\eta_v}(\alpha_v)$ *and* $U_v'(\alpha_v)$ *are explicitly given in Proposition* 11.1.

The proof of Theorem 1.1. By the same procedure as in [19, §7.1], the estimation is reduced to that for a similar average over $\Pi_{\mathrm{cus}}(l,\mathfrak{n})$ (in

place of $\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$). From [21, Lemma 13.15], we have

$$\begin{aligned}&\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha)\\ &\quad= 2(-1)^{\epsilon(\eta)+\#S}(1+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))D_F^{1/2}\mathcal{G}(\eta)L_{\mathrm{fin}}(1,\eta)\int\limits_{\mathfrak{X}^0_S}\alpha(\mathbf{s})\,d\mu^{\eta}_S(\mathbf{s}).\end{aligned}$$

By Theorem 9.1, it suffices to show

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)=\mathcal{O}_{\epsilon,l,\eta,\alpha}(\mathrm{N}(\mathfrak{n})^{-\inf_{v\in\Sigma_\infty} l_v/2+1+\epsilon})$$

for any sufficiently small $\epsilon>0$. This follows from the proof of Lemma 8.5 and Lemma 7.8 by taking $c\in\mathbb{R}$ and $\rho\neq 0$ such that $\inf_{v\in\Sigma_\infty} l_v/2-1>(c+1)/4>(c+1)/4-|\rho|>\inf_{v\in\Sigma_\infty} l_v/2-1-\epsilon>1$ and $|\rho|$ is sufficiently small. When $\mathfrak{n}$ is square-free, by noting $w^{\eta}_{\mathfrak{n}}(\pi)=\delta(\mathfrak{n}=\mathfrak{f}_\pi)$, we need no procedure as in [19, §7.1]. Thus the exponent of the error term is not affected and remains $-\inf_{v\in\Sigma_\infty} l_v/2+1+\epsilon$ in the final result. ∎

Corollary 9.4. *Suppose $\eta_v(-1)=-1$ for all $v\in\Sigma_\infty$. Let $\mathfrak{a}$ be an integral ideal. Then $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)=0$ for any ideal $\mathfrak{n}$ such that $\mathrm{N}(\mathfrak{n})\geq\mathrm{N}(\mathfrak{f}\mathfrak{a})$ and for any $\alpha\in\mathcal{A}_S$ of the form $\bigotimes_{v\in S}\alpha_v$ with $\alpha_v(s_v)$ being a linear combination of $\alpha_v^{(m)}(s_v)$ $(0\leq m\leq\mathrm{ord}_v(\mathfrak{a}))$.*

Proof. From Theorem 9.2, the condition on α implies that the hyperbolic term is a sum of terms indexed by $b\in F^\times-\{-1\}$ such that $b\in\mathfrak{n}\mathfrak{f}^{-1}\mathfrak{a}^{-1}$ (Lemmas 10.3–10.5 and 10.9) and $0<|\mathrm{N}(b)|<1$ (Lemma 10.15). Thus, the summation becomes empty if $\mathrm{N}(\mathfrak{n})\geq\mathrm{N}(\mathfrak{f}\mathfrak{a})$. ∎

Remark. The vanishing of the term $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$ for $(\mathfrak{n},\alpha)$ with both $\mathrm{N}(\mathfrak{n})$ and $\deg(\alpha)$ large is called *stability*, and was already observed in [11], [4] and [13] at least when $\mathfrak{n}$ is square-free. Actually, even when η admits a place $v\in\Sigma_\infty$ such that $\eta_v(-1)=1$, our relative trace formula (Theorems 9.1–9.3 combined) gives an exact formula for the spectral average, although the expression involves an infinite sum.

10. Explicit formula for the hyperbolic term. In this section, we compute $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$ further for particular test functions $\alpha=\bigotimes_{v\in S}\alpha_v$. By changing the order of integrals, we have

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)=\sum_{b\in F^\times-\{-1\}}\Big\{\prod_{v\in S}J^{\eta_v}_v(b;\alpha_v)\Big\}\Big\{\prod_{v\in\Sigma_\infty}J^{\eta_v}_v(l_v;b)\Big\}\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}-S}J^{\eta_v}_v(b)\Big\},$$

where

$$J^{\eta_v}_v(b;\alpha_v)=\frac{1}{2\pi i}\int\limits_{L_v(c)}\Big\{\int\limits_{F_v^\times}\Psi^{(0)}_v\big(s_v;\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\big)\eta_v(t)\,d^\times t\Big\}\alpha_v(s_v)\,d\mu_v(s_v)$$

for $v \in S$ with $\Psi_v^{(0)}(s_v;-)$ being the Green function on G_v (Lemma 4.1),

$$J_v^{\eta_v}(b) = \int_{F_v^\times} \Phi_{v,\mathfrak{n}}^{(0)}\big(\delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & \varpi_v^{-f(\eta_v)} \\ 0 & 1 \end{smallmatrix}\right]\big)\eta_v(t\varpi_v^{-f(\eta_v)})\, d^\times t \quad \text{if } v \in \Sigma_{\mathrm{fin}} - S,$$

$$J_v^{\eta_v}(l_v;b) = \int_{\mathbb{R}^\times} \Psi_v^{(0)}\big(l_v; \delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\big)\eta_v(t)\, d^\times t \quad \text{if } v \in \Sigma_\infty,$$

with $\Psi_v^{(0)}(l_v;-)$ being the Shintani function (Proposition 3.1), and $L_v(c)$ denotes the vertical contour directed from $c - \frac{2\pi i}{\log q_v}$ to $c + \frac{2\pi i}{\log q_v}$.

10.1. An evaluation of non-archimedean integrals (for unramified η_v). In this subsection, we explicitly compute the integrals $J_v^{\eta_v}(b;\alpha_v^{(m)})$ at $v \in S$ and the integrals $J_v^{\eta_v}(b)$ at $v \in \Sigma_{\mathrm{fin}} - S \cup S(\mathfrak{f})$.

Lemma 10.1. *Let $v \in S$. Let $\alpha_v^{(m)}(s_v) = q_v^{ms_v/2} + q_v^{-ms_v/2}$ with $m \in \mathbb{N}_0$. Set*

$$\widehat{\Phi_{vm}}(g_v) = \frac{1}{2\pi i} \int_{L_v(c)} \Psi_v^{(0)}(s_v;g_v)\alpha_v^{(m)}(s_v)\, d\mu_v(s_v).$$

If $m > 0$, then for any $x \in F_v$ with $\max(|x|_v, 1) = q_v^l$ with $l \in \mathbb{N}_0$, we have

$$\widehat{\Phi_{vm}}\big(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\big) = \begin{cases} 0 & (l \geq m+1), \\ -q_v^{-m/2} & (l = m), \\ (m-l-1)q_v^{1-m/2} - (m-l+1)q_v^{-m/2} & (0 \leq l < m). \end{cases}$$

If $m = 0$, then for any $x \in F_v$ with $\max(|x|_v, 1) = q_v^l$ with $l \in \mathbb{N}_0$, we have

$$\widehat{\Phi_{v0}}\big(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\big) = -2\delta(l = 0).$$

Proof. From Lemma 4.1 and the formula $d\mu_v(s) = 2^{-1}(\log q_v)(q_v^{(1+s)/2} - q_v^{(1-s)/2})\, ds$, we have

$$\widehat{\Phi_{vm}}\big(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\big) = \frac{1}{2\pi i} \int_{L_v(c)} q_v^{-(s+1)l/2}(1 - q_v^{-(s+1)/2})^{-1}(1 - q_v^{(s+1)/2})^{-1} \times (q_v^{-ms/2} + q_v^{ms/2})2^{-1}(\log q_v)(q_v^{(1+s)/2} - q_v^{(1-s)/2})\, ds.$$

By the change of variable $z = q_v^{s/2}$, this becomes

$$\frac{q_v^{(1-l)/2}}{2\pi i} \oint_{|z|=R} z^{-l}(1 - q_v^{-1/2}z^{-1})^{-1}(1 - q_v^{1/2}z)^{-1}(z^m + z^{-m})(z - z^{-1})\, \frac{dz}{z}$$

with $R = q_v^{c/2}$ (> 1). Thus, by the residue theorem,

$$\widehat{\Phi_{vm}}\big(\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\big) = q_v^{(1-l)/2}\big(\mathrm{Res}_{z=q_v^{-1/2}}\phi(z) + \mathrm{Res}_{z=0}\phi(z)\big) \tag{10.1}$$

with $\phi(z)=\frac{(1-z^2)(z^m+z^{-m})}{(1-q_v^{1/2}z)^2}\,\frac{q_v^{1/2}}{z^{1+l}}$. By a direct computation, we obtain

$$\text{(10.2)}\qquad \mathrm{Res}_{z=q_v^{-1/2}}\phi(z)=-q_v^{(1+l)/2}$$
$$\times\big(\{(m+l+1)(1-q_v^{-1})+2q_v^{-1}\}q_v^{m/2}+\{(-m+l+1)(1-q_v^{-1})+2q_v^{-1}\}q_v^{-m/2}\big),$$

$$\text{(10.3)}\qquad \mathrm{Res}_{z=0}\phi(z)$$
$$=\delta(l\ge m+1)\{(l-m+1)q_v^{(l-m+1)/2}-(l-m-1)q_v^{(l-m-1)/2}\}$$
$$+\delta(l\ge 1-m)\{(l+m+1)q_v^{(l+m+1)/2}-(l+m-1)q_v^{(l+m-1)/2}\}$$
$$+\{\delta(m=l)+\delta(m=-l)\}q_v^{1/2}.$$

From (10.1)–(10.3), we easily obtain the desired formula. ∎

Lemma 10.2. *Let $v\in S$. Let $\alpha_v^{(m)}(s_v)=q_v^{ms_v/2}+q_v^{-ms_v/2}$ with $m\in\mathbb{N}_0$. Then, for any $b\in F_v^\times-\{-1\}$,*

$$J_v^{\eta_v}(b;\alpha_v^{(m)})=I_v^+(m;b)+\eta_v(\varpi_v)I_v^+(m;\varpi_v^{-1}(b+1))$$

with

$$I_v^+(m;b)=\mathrm{vol}(\mathfrak{o}_v^\times)2^{\delta(m=0)}$$
$$\times\Big(-q_v^{-m/2}\delta_m^{\eta_v}(b)+\sum_{l=\max(0,-\,\mathrm{ord}_v(b))}^{m-1}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\delta_l^{\eta_v}(b)\Big),$$

where for $n\in\mathbb{N}_0$,

$$\delta_n^{\eta_v}(b)=\delta(|b|_v\le q_v^n)\eta_v(\varpi_v^n)\begin{cases}(\mathrm{ord}_v(b)+1)^{\delta(n=0)} & (\eta_v(\varpi_v)=1),\\ (2^{-1}(\eta_v(b)+1))^{\delta(n=0)} & (\eta_v(\varpi_v)=-1).\end{cases}$$

Proof. Let $m>0$. By definition, $J_v^{\eta_v}(b;\alpha_v)=I_v^+(m;b)+I_v^-(m;b)$ with $I_v^+(m;b)$ and $I_v^-(m;b)$ being the integrals of $\widehat{\Phi_{vm}}\big(\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\big)\eta_v(t)$ with respect to the measure $d^\times t$ over $|t|_v\le 1$ and over $|t|_v>1$, respectively. From [21, Lemma 11.4],

$$\text{(10.4)}\qquad \delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\in H_v\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]\mathbf{K}_v\quad\text{with}\quad |x|_v=\begin{cases}|t|_v^{-1}|b|_v & (|t|_v\le 1),\\ |t|_v|b+1|_v & (|t|_v>1).\end{cases}$$

Hence, by Lemma 10.1, $I_v^+(m;b)$ becomes the sum of

$$\text{(10.5)}\qquad \int_{\substack{|t|_v\le 1\\ \max(1,|t|_v^{-1}|b|_v)=q_v^m}}(-q_v^{-m/2})\eta_v(t)\,d^\times t$$

and

(10.6)
$$\sum_{l=0}^{m-1} \int_{\substack{|t|_v\le 1\\ \max(1,|t|_v^{-1}|b|_v)=q_v^l}} \{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\eta_v(t)\,d^\times t.$$

The condition $|t|_v \le 1$, $\max(1,|t|_v^{-1}|b|_v) = q_v^l$ is equivalent to $|b|_v \le q_v^l$, $|t|_v = q_v^{-l}|b|_v$ if $l>0$, and to $|b|\le|t|\le 1$ if $l=0$. Hence, (10.5) is equal to $-q_v^{-m/2}\operatorname{vol}(\mathfrak{o}_v^\times)\delta_m^{\eta_v}(b)$, and (10.6) equals

$$\operatorname{vol}(\mathfrak{o}_v^\times)\sum_{l=\max(0,-\operatorname{ord}_v(b))}^{m-1}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\delta_l^{\eta_v}(b).$$

This completes the evaluation of the integral $I_v^+(m;b)$. In the same way, the integral $I_v^-(m;b)$ is calculated in a similar form; from the resulting expression, one can see that $I_v^-(m;b)=\eta_v(\varpi_v)I_v^+(m;\varpi_v^{-1}(b+1))$. This settles the case of $m>0$. The case $m=0$ is similar. ∎

From Lemma 10.2, we have a useful estimate for $J_v(b,\alpha_v)$.

Lemma 10.3. *Let* $\alpha_v^{(m)}(s_v) = q_v^{ms_v/2}+q_v^{-ms_v/2}$ *with* $m\in\mathbb{N}_0$. *If* $m>0$, *then*

$$|J_v^{\eta_v}(b,\alpha_v^{(m)})|$$
$$\ll (m+1)^2\{\delta(|b|_v\le q_v^{m-1})q_v^{1-m/2}+\delta(|b|_v=q_v^m)q_v^{-m/2}\},\qquad b\in F_v^\times-\{-1\},$$

with the implied constant independent of v *and* m. *If* $m=0$, *then*

$$J_v^{\eta_v}(b,\alpha_v^{(0)}) = -2\operatorname{vol}(\mathfrak{o}_v^\times)\Lambda_v^{\eta_v}(b),$$

where $\Lambda_v^{\eta_v}$ *is the function on* $F_v^\times-\{-1\}$ *defined by*

$$\Lambda_v^{\eta_v}(b) = \delta(b\in\mathfrak{o}_v)\delta_0^{\eta_v}(b(b+1)). \tag{10.7}$$

Proof. To infer the estimate from Lemma 10.2 in the case when $m>0$, it suffices to note that $I_m^+(\varpi_v^{-1}(b+1))=0$ if $|b|_v\ge q_v^m$, or equivalently if $|\varpi_v^{-1}(b+1)|_v\ge q_v^{m+1}$. The formula for $J_v^{\eta_v}(b,\alpha_v^{(0)})$ is obtained by noting the relation $\Lambda_v^{\eta_v}(b)=\delta_0^{\eta_v}(b)+\eta_v(\varpi_v)\delta_0^{\eta_v}(\varpi_v^{-1}(b+1))$. ∎

Lemma 10.4. *Let* $v\in\Sigma_{\mathrm{fin}}-S\cup S(\mathfrak{nf})$. *Then*

$$J_v^{\eta_v}(b) = \operatorname{vol}(\mathfrak{o}_v^\times)\Lambda_v^{\eta_v}(b)$$

with $\Lambda_v^{\eta_v}(b)$ *being defined by* (10.7).

Proof. This is proved in the same way as the case $m=0$ in Lemma 10.3. ∎

LEMMA 10.5. *Let* $v \in S(\mathfrak{n}) - S(\mathfrak{f})$. *If* $\eta_v(\varpi_v) = 1$, *then*

$$J_v^{\eta_v}(b) = \mathrm{vol}(\mathfrak{o}_v^\times)\delta(b \in \mathfrak{n}\mathfrak{o}_v)\{\mathrm{ord}_v(b) - \mathrm{ord}_v(\mathfrak{n}) + 1\}.$$

If $\eta_v(\varpi_v) = -1$, *then*

$$J_v^{\eta_v}(b) = \mathrm{vol}(\mathfrak{o}_v^\times)\delta(b \in \mathfrak{n}\mathfrak{o}_v)2^{-1}(\eta_v(b) + (-1)^{\mathrm{ord}_v(\mathfrak{n})}).$$

Proof. This is proved in the same way as Lemma 10.4. We only have to remark that the assertion in the last sentence of [21, Lemma 11.4] is relevant here. ∎

10.2. An evaluation of non-archimedean integrals (for ramified η_v). We shall calculate the integral $J_v^{\eta_v}(b)$ at finitely many places $v \in S(\mathfrak{f})$. In this subsection, we fix $v \in S(\mathfrak{f})$ and set $f = f(\eta_v)$; thus f is a positive integer. For $l \in \mathbb{Z}$, consider the following subsets of $F_v^\times$ depending on $b \in F_v^\times - \{-1\}$:

$$D_l(b) = \{t \in F_v^\times \mid |t|_v = q_v^{-l},\ |1 + t\varpi_v^{-f}|_v\,|b + t\varpi_v^{-f}(b+1)|_v \le q_v^{-l}\} \qquad (l \in \mathbb{Z} - \{f\}),$$

$$D_f(b) = \big\{t \in F_v^\times \;\big|\; -t \in \varpi_v^f(\mathfrak{o}_v^\times - U_v(f)),\ |1 + t\varpi_v^{-f}|_v\,|b + t\varpi_v^{-f}(b+1)|_v \le q_v^{-f}\big\},$$

where $U_v(m) = 1 + \mathfrak{p}_v^m$ for any positive integer m.

LEMMA 10.6. *Let* $l > f$. *Then* $D_l(b) = \emptyset$ *unless* $l = f - \mathrm{ord}_v(b+1) + \mathrm{ord}_v(b)$, *in which case* $\mathrm{ord}_v(b) > 0$, $\mathrm{ord}_v(b+1) = 0$ *and*

$$\int_{t \in D_l(b)} \eta_v(t)\, d^\times t = \eta_v\big(-\varpi_v^f b(b+1)^{-1}\big)(1 - q_v^{-1})^{-1} q_v^{-f-d_v/2}.$$

Proof. By the change of variable $t = \varpi_v^l t'$, we have

$$\int_{t \in D_l(b)} \eta_v(t)\, d^\times t = \eta_v(\varpi_v^l) \int_{t' \in D'} \eta_v(t')\, d^\times t'$$

with $D' = \{t' \in \mathfrak{o}_v^\times \mid |1 + t'\varpi_v^{l-f}|_v\,|b + t'\varpi_v^{l-f}(b+1)|_v \le q_v^{-l}\}$. Let $t' \in \mathfrak{o}_v^\times$. Then the condition

$$|1 + t'\varpi_v^{l-f}|_v\,|b + t'\varpi_v^{l-f}(b+1)|_v \le q_v^{-l}$$

is equivalent to

$$t' \in \varpi_v^{f-l}\frac{-b}{b+1}(1 + \varpi_v^l b^{-1}\mathfrak{o}_v). \tag{10.8}$$

If $|\varpi_v^l b^{-1}|_v > 1$, then $1 + \varpi_v^l b^{-1}\mathfrak{o}_v = \varpi_v^l b^{-1}\mathfrak{o}_v$. Hence, from (10.8),

$$1 = |t'|_v \le \left|\varpi_v^{f-l}\frac{-b}{b+1} \cdot \varpi_v^l b^{-1}\right|_v = \left|\frac{\varpi_v^f}{b+1}\right|_v,$$

and so $b+1 \in \mathfrak{p}_v^f$. Since $f > 0$, we obtain $|b|_v = 1$, which, combined with $|\varpi_v^l b^{-1}|_v > 1$, implies $|\varpi_v^l|_v > 1$, contradicting $l > f > 0$.

If $|\varpi_v^l b^{-1}|_v = 1$, then $b \in \varpi_v^l \mathfrak{o}_v^\times$; thus, $|b+1|_v = 1$ because $l > f > 0$. Hence, from (10.8),

$$1 = |t'|_v \le \left|\varpi_v^{f-l} \frac{-b}{b+1}\right|_v = |\varpi_v^f|_v = q_v^{-f},$$

which is impossible due to $f > 0$. From the considerations so far, we have the inequality $|\varpi_v^l b^{-1}|_v < 1$, which yields $1 + \varpi_v^l b^{-1} \mathfrak{o}_v \subset \mathfrak{o}_v^\times$. Hence, from (10.8), we have the second equality of

$$1 = |t'|_v = \left|\varpi_v^{f-l} \frac{-b}{b+1}\right|_v,$$

which implies $l = f - \operatorname{ord}_v(b+1) + \operatorname{ord}_v(b)$. From this and $l > f$, we get $\operatorname{ord}_v(b+1) < \operatorname{ord}_v(b)$, which holds if and only if $\operatorname{ord}_v(b) > 0$ and $\operatorname{ord}_v(b+1) = 0$.

If we set $t' = \varpi_v^{f-l} \frac{-b}{b+1} r$, then (10.8) becomes $r \in 1 + \varpi_v^l b^{-1} \mathfrak{o}_v = 1 + \varpi_v^f \mathfrak{o}_v$; thus

$$\int_{t \in D_l(b)} \eta_v(t)\, d^\times t = \eta_v(\varpi_v^l) \eta_v\left(\varpi_v^{f-l} \frac{-b}{b+1}\right) \int_{r \in 1+\varpi_v^f \mathfrak{o}_v} \eta_v(r)\, d^\times r$$
$$= \eta_v\left(\varpi_v^f \frac{-b}{b+1}\right) q_v^{-f-d_v/2} (1 - q_v^{-1})^{-1}. \ \blacksquare$$

LEMMA 10.7. *Let $l < f$. Then $D_l(b) = \emptyset$ unless $l = f - \operatorname{ord}_v(b+1) + \operatorname{ord}_v(b)$, in which case $\operatorname{ord}_v(b+1) > 0$, $\operatorname{ord}_v(b) = 0$ and*

$$\int_{t \in D_l(b)} \eta_v(t)\, d^\times t = \eta_v\big(-\varpi_v^f b (b+1)^{-1}\big)(1 - q_v^{-1})^{-1} q_v^{-f-d_v/2}.$$

Proof. This is proved in the same way as the previous lemma. ∎

LEMMA 10.8. *The set $D_f(b)$ is empty unless $\operatorname{ord}_v(b) = \operatorname{ord}_v(b+1) \le 0$, in which case*

$$\int_{t \in D_f(b)} \eta_v(t)\, d^\times t = \delta\big(b(b+1)^{-1} \in \mathfrak{o}_v^\times\big) \eta_v\big(-\varpi_v^f b (b+1)^{-1}\big)$$
$$\times (1 - q_v^{-1})^{-1} q_v^{-f + \operatorname{ord}_v(b) - d_v/2}.$$

Proof. As $t = -\varpi_v^f t'$, the set $D_f(b)$ is mapped bijectively onto the set of t' such that

$$t' \in \mathfrak{o}_v^\times - U_v(f), \tag{10.9}$$

$$|1 - t'|_v \, |b - t'(b+1)|_v \le q_v^{-f}. \tag{10.10}$$

We shall show that (10.9) and (10.10) are equivalent to the following conditions:

$$t' \in \frac{b}{b+1}(1+\varpi_v^f b^{-1}\mathfrak{o}_v), \tag{10.11}$$

$$\frac{b}{b+1} \in \mathfrak{o}_v^\times, \quad b \notin \mathfrak{p}_v. \tag{10.12}$$

Noting that, under the condition (10.12), the sets $U_v(1)$ and $\frac{b}{b+1}(1+\varpi_v^f b^{-1}\mathfrak{o}_v)$ are disjoint, we easily see that (10.11) and (10.12) imply (10.9) and (10.10). For the converse, we first observe that (10.9) is equivalent to $t' \in \mathfrak{o}_v^\times$ and $|t'-1|_v > q_v^{-f}$. Hence by (10.10),

$$|b - t'(b+1)|_v \le q_v^{-f}|t'-1|_v^{-1} < 1,$$

or equivalently

$$b - t'(b+1) \in \mathfrak{p}_v. \tag{10.13}$$

If $b \in \mathfrak{p}_v$, then $b+1 \in \mathfrak{o}_v^\times$. From these and (10.13), $t' \in \frac{b}{b+1} + \mathfrak{p}_v = \mathfrak{p}_v$; this contradicts $t' \in \mathfrak{o}_v^\times$. Thus $b \notin \mathfrak{p}_v$. From (10.13), we have $t'\frac{b+1}{b} \in 1+\mathfrak{p}_v \subset \mathfrak{o}_v^\times$. Since $t' \in \mathfrak{o}_v^\times$ by (10.9), we get $\frac{b}{b+1} \in \mathfrak{o}_v^\times$. From (10.13),

$$t' \in \frac{b}{b+1} + \frac{1}{b+1}\mathfrak{p}_v = \frac{b}{b+1}(1+b^{-1}\mathfrak{p}_v).$$

Since $b^{-1} \in \mathfrak{o}_v$, we have $t' \in \frac{b}{b+1}U_v(1)$, which yields $t' \in \mathfrak{o}_v^\times - U_v(1)$ because $\frac{b}{b+1}U_v(1) \cap U_v(1) = \emptyset$. Thus $|t'-1|_v = 1$. Combining this with (10.10), we obtain (10.11). This settles the desired converse implication. Consequently,

$$\begin{aligned}
&\int_{t\in D_f(b)} \eta_v(t)\,d^\times t = \eta_v(-\varpi_v^f)\int \eta_v(t')\,d^\times t' \\
&\quad = \delta\left(\frac{b}{b+1} \in \mathfrak{o}_v^\times, b \notin \mathfrak{p}_v\right)\eta_v(-\varpi_v^f)\eta_v\left(\frac{b}{b+1}\right)\int_{r\in 1+\varpi_v^f b^{-1}\mathfrak{o}_v} \eta_v(r)\,d^\times r \\
&\quad = \delta\left(\frac{b}{b+1} \in \mathfrak{o}_v^\times\right)\eta_v\left(\varpi_v^f\frac{-b}{b+1}\right)\delta(b \in \mathfrak{o}_v^\times)q_v^{-f+\mathrm{ord}_v(b)-d_v/2}(1-q_v^{-1})^{-1}. \ \blacksquare
\end{aligned}$$

LEMMA 10.9. *Let η_v be a character of $F_v^\times$ of order 2 and of conductor $f > 0$. Then, for $b \in F_v^\times - \{-1\}$,*

$$\begin{aligned}
J_v^{\eta_v}(b) = \delta(b \in \mathfrak{p}_v^{-f})\big\{&\eta_v(-1) + \big(\delta(b \in \mathfrak{o}_v) + \delta(b \notin \mathfrak{o}_v)q_v^{\mathrm{ord}_v(b)}\big)\eta_v(-b(b+1))\big\} \\
&\times q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}.
\end{aligned}$$

Proof. From [21, Lemmas 11.4 and 11.5],

$$J_v^{\eta_v}(b) = \delta(b \in \mathfrak{p}_v^{-f})(J_{v,1}^{\eta_v}(b) + J_{v,2}^{\eta_v}(b))$$

with

$$J_{v,1}^{\eta_v}(b) = \int\limits_{\substack{-t\in\varpi_v^f U_v(f)\\ |t|_v|b+1|_v\le 1}} \eta_v(-1)\, d^\times t,$$

$$J_{v,2}^{\eta_v}(b) = \int\limits_{\substack{-t\in F_v^\times-\varpi_v^f U_v(f)\\ |1+t\varpi_v^{-f}|_v\,|b+t\varpi_v^{-f}(b+1)|_v\le|t|_v}} \eta_v(t\varpi_v^{-f})\, d^\times t.$$

If $b\in\mathfrak{p}_v^{-f}$, then $t\in-\varpi_v^f U_v(f)$ implies $|b+1|_v\le q_v^f=|t|_v^{-1}$; thus,

$$\begin{aligned} J_{v,1}^{\eta_v}(b) &= \eta_v(-1)\,\mathrm{vol}(-\varpi_v^f U_v(f); d^\times t) = \eta_v(-1)\,\mathrm{vol}(U_v(f); d^\times t)\\ &= \eta_v(-1)q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}. \end{aligned}$$

The integral domain of $J_{v,2}^{\eta_v}(b)$ is a disjoint union of the sets $D_l(b)(l\in\mathbb{Z})$. From Lemmas 10.6–10.8, we have

$$J_{v,2}^{\eta_v}(b) = \big(\delta(b\in\mathfrak{o}_v)+\delta(b\notin\mathfrak{o}_v)q_v^{\mathrm{ord}_v(b)}\big)\eta_v\left(\frac{-b}{b+1}\right)q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}. \ \blacksquare$$

Lemma 10.10. *Let η be an idele class character of $F^\times$ with conductor $\mathfrak{f}$ such that $\eta^2=\mathbf{1}$. There exists a constant $C>1$ independent of η such that*

$$|J_v^{\eta_v}(b)|\le C\delta(|b|_v\le q_v^{f(\eta_v)})q_v^{-f(\eta_v)}$$

for any $v\in S(\mathfrak{f})$ and for any $b\in F_v^\times-\{-1\}$.

Proof. This is obvious from the previous lemma. Indeed, $C=4$ is sufficient. ∎

Corollary 10.11. *For any $\epsilon>0$,*

$$\Big|\prod_{v\in S(\mathfrak{f})} J_v^{\eta_v}(b)\Big| \ll_\epsilon \Big\{\prod_{v\in S(\mathfrak{f})}\delta(b\in\mathfrak{p}_v^{-f(\eta_v)})\Big\}\mathrm{N}(\mathfrak{f})^{-1+\epsilon},\qquad b\in F^\times-\{-1\},$$

with the implied constant independent of η and $b\in F^\times-\{-1\}$.

Proof. Given $\epsilon>0$, let $P(\epsilon)$ be the set of $v\in\Sigma_{\mathrm{fin}}$ such that $q_v\le C^{1/\epsilon}$, where $C>1$ is the constant in the previous lemma. Then, from the lemma,

$$\begin{aligned} |J_v^{\eta_v}(b)| &\le C\delta(|b|_v\le q_v^{f(\eta_v)})q_v^{-f(\eta_v)+\epsilon} && \text{if } v\in S(\mathfrak{f})\cap P(\epsilon),\\ |J_v^{\eta_v}(b)| &\le \delta(|b|_v\le q_v^{f(\eta_v)})q_v^{-f(\eta_v)+\epsilon} && \text{if } v\in S(\mathfrak{f})-P(\epsilon). \end{aligned}$$

Taking the product of these inequalities, we obtain

$$\begin{aligned} \Big|\prod_{v\in S(\mathfrak{f})} J_v^{\eta_v}(b)\Big| &= \Big\{\prod_{v\in S(\mathfrak{f})\cap P(\epsilon)}|J_v^{\eta_v}(b)|\Big\}\Big\{\prod_{v\in S(\mathfrak{f})-P(\epsilon)}|J_v^{\eta_v}(b)|\Big\}\\ &\le C^{\#S(\mathfrak{f})}\Big\{\prod_{v\in S(\mathfrak{f})}\delta(b\in\mathfrak{p}_v^{-f(\eta_v)})\Big\}\mathrm{N}(\mathfrak{f})^{-1+\epsilon}. \ \blacksquare \end{aligned}$$

10.3. An evaluation of archimedean integrals. In this subsection, we explicitly evaluate the integral

$$J^\eta(l;b) = \int_{\mathbb{R}^\times} \Psi^{(0)}\big(l;\delta_b \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\big)\eta(t)\, d^\times t, \qquad b \in \mathbb{R}^\times - \{-1\}, \tag{10.14}$$

where $\eta : \mathbb{R}^\times \to \{\pm 1\}$ is a character and $\Psi^{(0)}(l;-)$ is the holomorphic Shintani function of weight l (≥ 4).

Lemma 10.12. *We have*

$$J^\eta(l;b) = \int_{\mathbb{R}^\times} (1-it)^{-l/2}(1+b+t^{-1}bi)^{-l/2}\eta(t)\, d^\times t, \qquad b \in \mathbb{R}^\times - \{-1\}.$$

Proof. From Lemma 3.2,

$$\Psi^{(0)}\big(l;\delta_b\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\big) = e^{il\theta}(1+ix)^{-l/2} \quad \text{if } \delta_b\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \in T\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] k_\theta.$$

From the proof of Lemma 7.3, we have $e^{i\theta} = \frac{1+it}{\sqrt{t^2+1}}$ and $x = bt^{-1} + t(b+1)$. Thus,

$$\begin{aligned} J^\eta(l;b) &= \int_{\mathbb{R}^\times} \left(\frac{1+it}{\sqrt{t^2+1}}\right)^l \{1+i(bt^{-1}+t(b+1))\}^{-l/2}\eta(t)\, d^\times t \\ &= \int_{\mathbb{R}^\times} (1-it)^{-l/2}(1+b+t^{-1}bi)^{-l/2}\eta(t)\, d^\times t. \ \blacksquare \end{aligned}$$

Lemma 10.13. *Define*

$$J_+(l;b) = i^{l/2}(1+b)^{-l/2}\int_0^\infty (t+i)^{-l/2}\left(t+\frac{bi}{b+1}\right)^{-l/2} t^{l/2-1}\, dt.$$

Then

$$J^{\mathbf{1}}(l;b) = J_+(l;b) + \overline{J_+(l;b)}, \qquad J^{\mathrm{sgn}}(l;b) = J_+(l;b) - \overline{J_+(l;b)}.$$

Proof. By dividing the integral $J^\eta(l;b)$ into two parts according to $t>0$ and $t<0$, we obtain the assertions immediately. ∎

Lemma 10.14. *Suppose $b(b+1) > 0$. Then*

$$\begin{aligned} J_+(l;b) &= (1+b)^{-l/2}\int_0^1 u^{l/2-1}(1-u)^{l/2-1}\left(\frac{-1}{b+1}u+1\right)^{-l/2} du \\ &= (1+b)^{-l/2}\Gamma(l/2)^2\Gamma(l)^{-1}\,{}_2F_1\big(l/2, l/2; l; (b+1)^{-1}\big) \\ &= 2Q_{l/2-1}(2b+1), \end{aligned}$$

where $Q_n(x)$ is the Legendre function of the second kind.

Proof. If we set

$$f(z) = i^{l/2}(1+b)^{-l/2}(z+i)^{-l/2}\{z+bi/(b+1)\}^{-l/2}z^{l/2-1},$$

then $f(z)$ is a meromorphic function on $\mathbb{C}$ with poles only at $z = -i$ and $-bi/(1+b)$, both in the lower half-plane $\mathrm{Im}(z) < 0$. For $R > 0$, let Q_R denote the rectangle $0 \le \mathrm{Im}(z) \le R$, $0 \le \mathrm{Re}(z) \le R$. Regarding ∂Q_R as a contour with counterclockwise orientation, by Cauchy's theorem we have $\int_{\partial Q_R} f(z)\,dz = 0$. From this,

$$J_+(l;b) = \int_{0i}^{i\infty} f(z)\,dz - \lim_{R\to\infty} \int_{\partial Q_R - [0,R]\cup i[0,R]} f(z)\,dz = \int_{0i}^{i\infty} f(z)\,dz$$
$$= (1+b)^{-l/2} \int_0^\infty (t+1)^{-l/2} \left(t + \frac{b}{b+1}\right)^{-l/2} t^{l/2-1}\,dt.$$

By the change of variable $t + 1 = u^{-1}$, this becomes

$$(1+b)^{-l/2} \int_0^1 u^{l/2-1}(1-u)^{l/2-1} \left(\frac{-1}{b+1}u + 1\right)^{-l/2} du.$$

By using the integral representation of ${}_2F_1(a,b;c,z)$ in [10, p. 54], we obtain

$$J_+(l;b) = (1+b)^{-l/2}\Gamma(l/2)^2\Gamma(l)^{-1}\,{}_2F_1\big(l/2, l/2; l; (b+1)^{-1}\big).$$

If we further apply the formula

$$2^{-n}(2n+1)!(n!)^{-2}Q_n(x) = (1+x)^{-(n+1)}\,{}_2F_1\left(n+1, n+1; 2n+2; \frac{2}{x+1}\right)$$

([10, p. 233]) with $n = l/2 - 1$ and $x = 2b + 1$, we obtain $J_+(l;b) = 2Q_{l/2-1}(2b+1)$ as desired. ∎

Lemma 10.15.

(1) *If* $b(b+1) > 0$, *then*

$$J^{\mathbf{1}}(l;b) = (1+b)^{-l/2} 2\Gamma(l/2)^2\Gamma(l)^{-1}\,{}_2F_1\big(l/2, l/2; l; (b+1)^{-1}\big),$$
$$J^{\mathrm{sgn}}(l;b) = 0.$$

(2) *If* $b(b+1) < 0$, *then*

$$J^{\mathbf{1}}(l;b) = 2\log\left|\frac{b+1}{b}\right| P_{l/2-1}(2b+1) - \sum_{m=1}^{[l/4]} \frac{8(l-4m+1)}{(2m-1)(l-2m)} P_{l/2-2m}(2b+1), \tag{10.15}$$

$$J^{\mathrm{sgn}}(l;b) = 2\pi i P_{l/2-1}(2b+1), \tag{10.16}$$

where $P_n(z)$ *denotes the Legendre polynomial of degree* n.

Proof. First suppose $b \in \mathbb{R}$ and $b(b+1) > 0$. Then from the previous lemma, $J_+(l;b)$ is a real number. Thus, $J^{\mathbf{1}}(l;b) = 2J^+(l;b)$ and $J^{\mathrm{sgn}}(l;b) = 0$ by Lemma 10.13.

From $J_+(l;b) = 2Q_{l/2-1}(2b+1)$, applying the formula in [10, p. 234], we obtain

$$J_+(l;b) = \log\left(\frac{b+1}{b}\right)P_{l/2-1}(2b+1) - \sum_{m=1}^{[l/4]} \frac{4(l-4m+1)}{(2m-1)(l-2m)} P_{l/2-2m}(2b+1) \tag{10.17}$$

for $b \in \mathbb{R}$ such that $b(b+1) > 0$. From the defining formula for $J_+(l;b)$, the function $b \mapsto J_+(l;b)$ on $\mathbb{R}^\times - \{-1\}$ has a holomorphic continuation to the whole complex b-plane away from the set $S = \{b \in \mathbb{C} \mid bi/(b+1) \in (-\infty, 0)\} \cup \{0, -1\}$, which is the upper half of the circle centered at $-1/2$ of radius $1/2$ with the edge points included. Thus, if we choose the branch of $\log((1+b)/b)$ on the domain $\mathbb{C} - S$ so that it is real for $b > 0$, then the formula (10.17) remains valid on $\mathbb{C} - S$ by analytic continuation. Let $b \in \mathbb{R}$ such that $b(b+1) < 0$. Then $b \in \mathbb{C} - S$. Hence, by taking the sum of (10.17) and its complex conjugate, we obtain the formula for $J^{\mathbf{1}}(l;b)$. As for $J^{\mathrm{sgn}}(l;b)$, we have

$$\begin{aligned} J^{\mathrm{sgn}}(l;b) &= J_+(l;b) - \overline{J_+(l;b)} = \left\{\log\left(\frac{b+1}{b}\right) - \overline{\log\left(\frac{b+1}{b}\right)}\right\} P_{l/2-1}(2b+1) \\ &= 2\pi i P_{l/2-1}(2b+1). \ \blacksquare \end{aligned}$$

11. Explicit formula for the unipotent term. Let $v \in S$. The aim of this section is to evaluate the integrals

$$U_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i} \int\limits_{L_v(c)} (1 - \eta_v(\varpi_v) q_v^{-(s+1)/2})^{-1} (1 - q_v^{(s+1)/2})^{-1} \alpha_v(s)\, d\mu_v(s), \tag{11.1}$$

$$U_v'(\alpha_v) = \frac{\log q_v}{2\pi i} \int\limits_{L_v(c)} (1 - q_v^{(s+1)/2})^{-2} (1 - q_v^{-(s+1)/2})^{-1} \alpha_v(s)\, d\mu_v(s), \tag{11.2}$$

for the test functions α given by (9.2), where $L_v(c) = c + i\left[\frac{-2\pi}{\log q_v}, \frac{2\pi}{\log q_v}\right]$.

PROPOSITION 11.1. *Let* $\alpha_v(s) = q_v^{ms/2} + q_v^{-ms/2}$ *with* $m \in \mathbb{N}_0$. *Then*

$$U_v^{\eta_v}(\alpha_v) = \begin{cases} \delta(m>0) q_v^{1-m/2}\{(m-1) - (m+1)q_v^{-1}\} - 2\delta(m=0) & \\ & (\eta_v(\varpi_v) = 1), \\ \delta(m \in 2\mathbb{N}) q_v^{1-m/2}(1 - q_v^{-1}) - 2\delta(m=0) & (\eta_v(\varpi_v) = -1), \end{cases}$$

$$U_v'(\alpha_v) = -2^{-1}(\log q_v) q_v^{1-m/2} \delta(m>0)\{(m-1)(m-2) - m(m+1)q_v^{-1}\}.$$

Proof. We give an indication of the proof for (11.1) when $\eta_v(\varpi_v) = -1$; the remaining cases are similar. By a change of variable,

$$\begin{aligned} &U_v^{\eta_v}(\alpha_v) \\ &\quad = \frac{1}{2\pi i} \oint_{|z|=q_v^{c/2}} (1+q_v^{-1/2}z^{-1})^{-1}(1-q_v^{1/2}z)^{-1}(z^m+z^{-m})q_v^{1/2}(z-z^{-1})\,\frac{dz}{z} \\ &\quad = \{\mathrm{Res}_{z=q_v^{-1/2}} + \mathrm{Res}_{z=-q_v^{-1/2}} + \mathrm{Res}_{z=0}\}\phi(z), \end{aligned}$$

where $\phi(z) = \frac{(z^2-1)(z^m+z^{-m})}{1-q_v z^2}\,\frac{q_v}{z}$. By evaluating the residues, we are done. ∎

12. Subconvexity estimates in the weight aspect. In this section we prove Theorem 1.3 by using the relative trace formula (Theorem 9.1); we take a particular test function $\alpha_S^\pi \in \mathcal{A}_S$ depending on a fixed cuspidal representation π with varying S. To have a good control of the term $\mathbb{J}^\eta_{\mathrm{hyp}}(\mathfrak{n}|\alpha_S^\pi)$ revealing the dependence on S, our formula for local orbital integrals (Lemma 10.3) is indispensable. In this section, $\theta \in [0,1]$ is such that the spectral radius of the Satake parameter $A_v(\pi)$ of $\pi \in \mathit{\Pi}_{\mathrm{cus}}(l,\mathfrak{n})$ at $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi)$ is no greater than $q_v^{\theta/2}$ for all $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi)$. Since the Ramanujan conjecture for the holomorphic Hilbert cusp forms is known to hold [1], we can actually take $\theta = 0$; however, we let θ be unspecified until the very end to be able to keep track of the dependence on the Ramanujan exponent θ in various estimations.

In this section, we abuse the symbol $\mathfrak{p}_v$ to designate the global ideal $\mathfrak{p}_v \cap \mathfrak{o}$.

12.1. An auxiliary estimate of semilocal terms. Let S be a finite set of finite places v such that $\eta_v(\varpi_v) = -1$. For a decomposable function $\alpha_S(\mathbf{s}) = \prod_{v\in S}\alpha_v(s_v)$ in $\mathcal{A}_S$, we set

$$J_S(b;\alpha_S) = \prod_{v\in S} J_v(b;\alpha_v), \qquad b \in F^\times - \{-1\},$$

where we simply write $J_v(b;\alpha_v)$ in place of $J_v^{\eta_v}(b;\alpha_v)$. Extending this linearly, we have a linear functional $\alpha_S \mapsto J_S(b;\alpha_S)$ on the space $\bigotimes_{v\in S}\mathcal{A}_v$. Given $\pi \in \mathit{\Pi}_{\mathrm{cus}}(l,\mathfrak{n})$, set

$$\lambda_v(\pi) = \mathrm{tr}\, A_v(\pi), \qquad v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi),$$

with $A_v(\pi) \in \mathrm{GL}(2,\mathbb{C})$ the Satake parameter of π_v. Then we define an element of $\mathcal{A}_S$ depending on the automorphic representation π as follows:

$$\alpha_S^\pi(\mathbf{s}) = \Big(\sum_{v\in S}\{\lambda_v(\pi)(z_v+z_v^{-1}) - (z_v^2+z_v^{-2}+1)\}\Big)^2,$$

where $z_v = q_v^{s_v/2}$ for each $v \in S$. We need an estimate of $J_S(b;\alpha_S^\pi)$ with varying b. For an $\mathfrak{o}$-ideal $\mathfrak{a}$ such that $S(\mathfrak{a}) \subset S$, let us define a function $D_S(\mathfrak{a};-)$ on $F^\times - \{-1\}$ by

$$D_S(\mathfrak{a};b) = \Big\{ \prod_{w\in S-S(\mathfrak{a})} \Lambda_w(b) \Big\} \Big\{ \prod_{v\in S(\mathfrak{a})} \delta(|b|_v \le q_v^{\mathrm{ord}_v(\mathfrak{a})}) \Big\},$$

where $\Lambda_w(b) = \delta(b \in \mathfrak{o}_w)(\mathrm{ord}_w(b(b+1))+1)$.

PROPOSITION 12.1. *Set* $P = \{(v_1,v_2) \in S^2 \mid v_1 \ne v_2\}$. *Then*

$$\begin{aligned}|J_S(b;\alpha_S^\pi)| \ll \sum_{v\in S} \big\{ & D_S(\mathfrak{o};b) q_v^{(\theta+1)/2} + D_S(\mathfrak{p}_v;b) q_v^{\theta} \\ & + D_S(\mathfrak{p}_v^2;b) q_v^{\theta-1} + D_S(\mathfrak{p}_v^3;b) q_v^{-1} + D_S(\mathfrak{p}_v^4;b) q_v^{-2} \big\} \\ + \sum_{(v_1,v_2)\in P} \big\{ & D_S(\mathfrak{o};b) q_{v_1}^{(\theta+1)/2} q_{v_2}^{(\theta+1)/2} + D_S(\mathfrak{p}_{v_1};b) q_{v_2}^{(\theta+1)/2} + D_S(\mathfrak{p}_{v_1}\mathfrak{p}_{v_2};b) \\ & + D_S(\mathfrak{p}_{v_1}^2\mathfrak{p}_{v_2};b) q_{v_1}^{-1} + D_S(\mathfrak{p}_{v_1}^2;b) q_{v_1}^{-1} q_{v_2}^{(\theta+1)/2} + D_S(\mathfrak{p}_{v_1}^2\mathfrak{p}_{v_2}^2;b) q_{v_1}^{-1} q_{v_2}^{-1} \big\}\end{aligned}$$

for $b \in F^\times - \{-1\}$, *where the implied constant is absolute.*

Proof. Set $Z_v = \lambda_v(\pi)(z_v + z_v^{-1}) - (z_v^2 + z_v^{-2} + 1)$ for any $v \in S$. By expanding the square, we have $\alpha_S^\pi(\mathbf{s}) = \sum_{v\in S} Z_v^2 + \sum_{(v_1,v_2)\in P} Z_{v_1} Z_{v_2}$, which, together with Lemma 10.3, gives

$$\begin{aligned}(12.1)\qquad & J_S(b;\alpha_S^\pi) \\ & = \sum_{v\in S} \Big\{ \prod_{w\in S-\{v\}} J_w(b;1) \Big\} J_v(b;Z_v^2) \\ & \quad + \sum_{(v_1,v_2)\in P} \Big\{ \prod_{w\in S-\{v_1,v_2\}} J_w(b;1) \Big\} J_{v_1}(b;Z_{v_1}) J_{v_2}(b;Z_{v_2}) \\ & = \sum_{v\in S} \Big\{ \prod_{w\in S-\{v\}} -\mathrm{vol}(\mathfrak{o}_w^\times) \Lambda_w^{\eta_w}(b) \Big\} J_v(b;Z_v^2) \\ & \quad + \sum_{(v_1,v_2)\in P} \Big\{ \prod_{w\in S-\{v_1,v_2\}} -\mathrm{vol}(\mathfrak{o}_w^\times) \Lambda_w^{\eta_w}(b) \Big\} J_{v_1}(b;Z_{v_1}) J_{v_2}(b;Z_{v_2}).\end{aligned}$$

Let us estimate the integral $J_v(b;Z_v^2)$. By expanding the square,

$$\begin{aligned}Z_v^2 &= \lambda_v(\pi)^2(z_v^2 + z_v^{-2} + 2) + (z_v^4 + z_v^{-4} + 2) + 2(z_v^2 + z_v^{-2}) + 1 \\ &\quad - 2\lambda_v(\pi)(z_v^3 + z_v^{-3}) - 4\lambda_v(\pi)(z_v + z_v^{-1}) \\ &= \lambda_v(\pi)^2(\alpha_v^{(2)} + \alpha_v^{(0)}) + \alpha_v^{(4)} + 2\alpha_v^{(2)} + \tfrac{3}{2}\alpha_v^{(0)} - 2\lambda_v(\pi)\alpha_v^{(3)} - 4\lambda_v(\pi)\alpha_v^{(1)}.\end{aligned}$$

Noting that $\Lambda_v^{\eta_v}(b) \le \Lambda_v(b)$, by this expression and by the estimates in Lemma 10.3, we obtain

$$
\begin{aligned}
(12.2)\qquad & |J_v(b;Z_v^2)|\,\mathrm{vol}(\mathfrak{o}_v^\times)^{-1} \\
&\ll \delta(|b|_v \le 1)\{|\lambda_v(\pi)|q_v^{1/2} + \Lambda_v(b)(1+|\lambda_v(\pi)|^2)\} \\
&\quad + \delta(|b|_v \le q_v)\{|\lambda_v(\pi)|q_v^{-1/2} + 1 + |\lambda_v(\pi)|^2\} \\
&\quad + \delta(|b|_v \le q_v^2)\{|\lambda_v(\pi)|q_v^{-1/2} + q_v^{-1} + |\lambda_v(\pi)|^2 q_v^{-1}\} \\
&\quad + \delta(|b|_v \le q_v^3)\{|\lambda_v(\pi)|q_v^{-3/2} + q_v^{-1}\} + \delta(|b|_v \le q_v^4)q_v^{-2} \\
&\ll \delta(|b|_v \le 1)\{q_v^{(\theta+1)/2} + \Lambda_v(b)q_v^\theta\} + \delta(|b|_v \le q_v)q_v^\theta \\
&\quad + \delta(|b|_v \le q_v^2)q_v^{\theta-1} + \delta(|b|_v \le q_v^3)q_v^{-1} + \delta(|b|_v \le q_v^4)q_v^{-2},
\end{aligned}
$$

where to show the second inequality we use the estimate $|\lambda_v(\pi)| \le 2q_v^{\theta/2}$ as well as the inequalities $-1 \le (\theta-1)/2 \le \theta \le (\theta+1)/2$, $(\theta-3)/2 \le -1$. For $J_v(b;Z_v)$, directly from Lemma 10.3, we have

$$
\begin{aligned}
(12.3)\qquad |J_v(b;Z_v)|\,\mathrm{vol}(\mathfrak{o}_v^\times)^{-1} &\ll \delta(|b|_v \le 1)\{q_v^{(\theta+1)/2} + \Lambda_v(b)\} \\
&\quad + \delta(|b|_v \le q_v) + \delta(|b|_v \le q_v^2)q_v^{-1}.
\end{aligned}
$$

From (12.1)–(12.3), we immediately obtain the desired estimate. ∎

12.2. A basic majorant for the hyperbolic term (odd case). For $b \in F^\times - \{-1\}$, viewing b as a real number, say b_v, by the mapping $F \hookrightarrow F_v \cong \mathbb{R}$ for each $v \in \Sigma_\infty$, we define

$$
\mathfrak{m}_\infty(l;b) = \prod_{v\in\Sigma_\infty} |J^{\mathrm{sgn}}(l_v;b_v)|,
$$

where $J^{\mathrm{sgn}}(l_v;b_v)$ is the integral (10.14). For relatively prime integral ideals $\mathfrak{n}$ and $\mathfrak{a}$ and for $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{Z}_{>2})^{\Sigma_\infty}$, we set

$$
\mathfrak{I}(l,\mathfrak{n},\mathfrak{a}) := \sum_{b\in\mathfrak{n}\mathfrak{o}(S)-\{0,-1\}} \Big\{ \prod_{v\in\Sigma_{\mathrm{fin}}-S} \Lambda_v(b) \Big\} D_S(\mathfrak{a};b)\mathfrak{m}_\infty(l;b),
$$

where S is a finite set of places such that $S(\mathfrak{a}) \subset S \subset \Sigma_{\mathrm{fin}} - S(\mathfrak{n})$, and $\mathfrak{o}(S)$ is the S-integer ring. We note that the right-hand side is independent of S. Indeed, if we set

$$
\tau^{S(\mathfrak{a})}(b) = \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a})} \Lambda_v(b) \prod_{v\in S(\mathfrak{a})} \delta(|b|_v \le q_v^{\mathrm{ord}_v(\mathfrak{a})}),
$$

then

$$
(12.4)\qquad \mathfrak{I}(l,\mathfrak{n},\mathfrak{a}) = \sum_{b\in\mathfrak{n}\mathfrak{a}^{-1}-\{0,-1\}} \tau^{S(\mathfrak{a})}(b)\mathfrak{m}_\infty(l;b).
$$

LEMMA 12.2. *Let $\mathfrak{a}$ and $\mathfrak{n}$ be relatively prime ideals. Then, for any $\epsilon > 0$,*

$$\mathfrak{I}(l, \mathfrak{n}, \mathfrak{a}) \ll_\epsilon \Big\{ \prod_{v \in \Sigma_\infty} l_v \Big\}^{-1/2} \mathrm{N}(\mathfrak{a})^{5/4+\epsilon}$$

with the implied constant depending on ϵ while independent of $(l, \mathfrak{n}, \mathfrak{a})$. If $\mathfrak{a}$ is trivial, then $\mathfrak{I}(l, \mathfrak{n}, \mathfrak{o}) = 0$ for any $\mathfrak{n}$ and l.

12.3. The proof of Lemma 12.2

LEMMA 12.3. *For any $\epsilon > 0$,*

$$\tau^{S(\mathfrak{a})}(b) \ll_\epsilon \big(\mathrm{N}(\mathfrak{a})^2 |\mathrm{N}(b(b+1))|\big)^\epsilon, \qquad b \in \mathfrak{a}^{-1} - \{0, -1\},$$

with the implied constant independent of b.

Proof. Let $b \in \mathfrak{a}^{-1} - \{0\}$; then $(b(b+1))\mathfrak{a}^2 = \mathfrak{b} \prod_{j=1}^r \mathfrak{p}_j^{e_j}$ for some $e_j \in \mathbb{N}$, where $\mathfrak{p}_j$ are prime ideals of $\mathfrak{o}$ relatively prime to $\mathfrak{a}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{o}$ dividing $\mathfrak{a}$. For each j, there exist a prime number p_j and $d_j \in \mathbb{N}$ such that $\mathrm{N}(\mathfrak{p}_j) = p_j^{d_j}$. By taking norms, we see that $\mathrm{N}(\mathfrak{a})^2 |\mathrm{N}(b(b+1))|$ equals $\mathrm{N}(\mathfrak{b}) \prod_{j=1}^r \mathrm{N}(\mathfrak{p}_j)^{e_j} = \mathrm{N}(\mathfrak{b}) \prod_{j=1}^r p_j^{d_j e_j}$, and hence

$$d\big(\mathrm{N}(\mathfrak{a})^2 |\mathrm{N}(b(b+1))|\big) = d(\mathrm{N}(\mathfrak{b})) \prod_{j=1}^r (e_j d_j + 1) \geq \prod_{j=1}^r (e_j + 1) = \tau^{S(\mathfrak{a})}(b),$$

where, for a natural number m, $d(m)$ denotes the number of positive divisors of m. Invoking the well-known bound $d(m) \ll_\epsilon m^\epsilon$, we obtain the desired estimate. ∎

From Lemmas 12.3 and 10.15,

$$\mathfrak{I}(l, \mathfrak{n}, \mathfrak{a}) \ll_\epsilon \mathrm{N}(\mathfrak{a})^{2\epsilon} \sum_{b \in \mathfrak{n}\mathfrak{a}^{-1} \cap \mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^\epsilon \prod_{v \in \Sigma_\infty} |2\pi P_{l_v/2-1}(2b_v + 1)|,$$

where $\mathcal{Q}_\infty$ denotes the cube $(-1, 0)^{\Sigma_\infty}$ in $\prod_{v \in \Sigma_\infty} \mathbb{R}$. Invoking the inequality $|P_n(x)| \leq (1 - x^2)^{-1/2} n^{-1/2}$ for $|x| < 1$, $n \in \mathbb{N}$ ([10, p. 237]), we have

(12.5)

$$\mathfrak{I}(l, \mathfrak{n}, \mathfrak{a}) \ll_\epsilon \pi^{d_F} \mathrm{N}(\mathfrak{a})^{2\epsilon} \sum_{b \in \mathfrak{n}\mathfrak{a}^{-1} \cap \mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^{-1/4+\epsilon} \prod_{v \in \Sigma_\infty} (l_v/2 - 1)^{-1/2}.$$

To estimate $\sum_{b \in \mathfrak{n}\mathfrak{a}^{-1} \cap \mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^{-1/4+\epsilon}$, we need several lemmas.

LEMMA 12.4. *For a positive integer c, let $\nu(c)$ be the number of $\mathfrak{o}$-ideals $\mathfrak{c}$ such that $\mathrm{N}(\mathfrak{c}) = c$. Then, for any $\epsilon > 0$, $\nu(c) \ll_\epsilon c^\epsilon$ with the implied constant independent of c.*

Proof. Suppose c is a prime power p^t. Then an ideal $\mathfrak{c}$ such that $\mathrm{N}(\mathfrak{c}) = p^t$ must be a power of a prime ideal $\mathfrak{p}$ lying above p. The number of choices for $\mathfrak{p}$ is at most $d_F = [F : \mathbb{Q}]$. If $\mathfrak{c} = \mathfrak{p}^e$, then $\mathrm{N}(\mathfrak{c}) = p^t$ is equivalent to

$p^{me} = p^t$, where $\mathrm{N}(\mathfrak{p}) = p^m$. Hence $e = t/m \le t \le t\log_2 p \le \log_2 p^t$. From this, we have $\nu(p^t) \le d_F \log_2 p^t$.

Given $\epsilon > 0$, let $x(\epsilon) > 1$ be such that $d_F \log_2 x \le x^\epsilon$ for any $x \ge x(\epsilon)$. Let $Q(\epsilon)$ be the set of prime powers p^t such that $p^t \le x(\epsilon)$. Noting that $Q(\epsilon)$ is a finite set, we set $C(\epsilon) = \prod_{q\in Q(\epsilon)} \nu(q)$, which is a constant depending only on ϵ. Let c' (resp. c'') be the product of the prime powers $p_i^{t_i}$ such that $p_i^{t_i} \in Q(\epsilon)$ (resp. $p_i^{t_i} \notin Q(\epsilon)$) in the prime factorization $c = \prod_i p_i^{t_i}$ of c. Since ν is multiplicative, we have

$$\nu(c) = \nu(c')\nu(c'') \le \prod_{q\in Q(\epsilon)} \nu(q) \prod_{i:\, p_i | c''} d_F \log_2 p_i^{t_i} \le C(\epsilon) \prod_{i:\, p_i|c''} p_i^{t_i\epsilon}$$
$$\le C(\epsilon)\Big(\prod_i p_i^{t_i}\Big)^\epsilon \le C(\epsilon) c^\epsilon. \ \blacksquare$$

LEMMA 12.5. *Let $C = \{C_v\}_{v\in\Sigma_\infty}$ be a family of positive real numbers. For any $\epsilon > 0$, we have*

$$\#\{u \in \mathfrak{o}^\times \mid |u_v| < C_v \ (\forall v \in \Sigma_\infty)\} \ll_\epsilon \Big(\prod_{v\in\Sigma_\infty} C_v\Big)^\epsilon$$

with the implied constant independent of C.

Proof. For simplicity, we set $d = d_F$. By the Dirichlet unit theorem, there exist fundamental units ε_j $(1 \le j \le d-1)$ such that any $\gamma \in \mathfrak{o}^\times$ is written uniquely in the form $\gamma = \pm\varepsilon_1^{n_1} \cdots \varepsilon_{d-1}^{n_{d-1}}$ with integers $n_j \in \mathbb{Z}$. From this, the inequality $|\gamma_v| < C_v$ can be written as

$$\sum_{j=1}^{d-1} n_j \log|(\varepsilon_j)_v| < \log C_v \quad (v \in \Sigma_\infty). \tag{12.6}$$

Let $\mathfrak{U}(C)$ be the set of $u \in \mathfrak{o}^\times$ such that $|u_v| < C_v$ for all $v \in \Sigma_\infty$. Thus, $\#\mathfrak{U}(C)$ is bounded from above by the number of points $(n_j)_{1\le j\le d-1} \in \mathbb{Z}^{d-1}$ lying on the Euclidean domain $D(C)$ in $\mathbb{R}^{d-1}$ defined by the system of linear inequalities (12.6). Fix an enumeration $\Sigma_\infty = \{v_1, \dots, v_d\}$ and let $E_i = (\log|(\varepsilon_j)_{v_i}|)_{1\le j\le d-1} \in \mathbb{R}^{d-1}$ for $1 \le i \le d$. The first $d-1$ vectors E_i $(1 \le i \le d-1)$ form a basis of $\mathbb{R}^{d-1}$; let E_j^* $(1 \le j \le d-1)$ be its dual basis. From the relation $|\mathrm{N}(\varepsilon_j)| = 1$, we have $\sum_{i=1}^d E_i = 0$. Hence, if we write a general point $y \in \mathbb{R}^{d-1}$ as $y = \sum_{i=1}^{d-1}(\log C_{v_i} - y_i)E_i^*$, then $y \in D(C)$ if and only if

$$y_i > 0 \quad (1 \le i \le d-1), \qquad \sum_{i=1}^{d-1} y_i < \sum_{j=1}^{d} \log C_{v_j}.$$

The volume of this region in the y-space with respect to the Euclidean measure is $\frac{1}{r_F(d-1)!}(\sum_{j=1}^d \log C_{v_j})^d$, where r_F is the regulator of F. Thus $\mathrm{vol}(D(C)) \ll (\log\prod_v C_v)^d \ll_\epsilon (\prod_v C_v)^\epsilon$, and we are done. $\blacksquare$

LEMMA 12.6. *Let $\mathfrak{a}$ be an integral ideal and c a positive integer. For any $\epsilon, \epsilon' > 0$,*

$$\#\{b \in \mathfrak{a}^{-1} \cap \mathcal{Q}_\infty \mid \mathrm{N}((b)\mathfrak{a}) = c\} \ll_{\epsilon,\epsilon'} c^{\epsilon'-\epsilon}\,\mathrm{N}(\mathfrak{a})^{\epsilon}$$

with the implied constant independent of $\mathfrak{a}$ and c.

Proof. Let $\mathfrak{c}$ be an integral ideal such that $\mathrm{N}(\mathfrak{c}) = c$. From Lemma 12.4, the number of such $\mathfrak{c}$ is bounded by $c^{\epsilon'}$ for any $\epsilon' > 0$. If $\mathfrak{c}\mathfrak{a}^{-1}$ is a principal ideal, say (ξ), then using Lemma 12.5 we have

$$\begin{aligned}\#\{b \in \mathfrak{a}^{-1} \cap \mathcal{Q}_\infty \mid \mathfrak{c} = (b)\mathfrak{a}\} &= \#\{u \in \mathfrak{o}^\times \mid |u_v| < |\xi_v|^{-1}\ (\forall v \in \Sigma_\infty)\} \\ &\ll_\epsilon \Big(\prod_{v\in\Sigma_\infty} |\xi_v|^{-1}\Big)^{\epsilon} = (|\mathrm{N}(\xi)|^{-1})^{\epsilon} = (c^{-1}\mathrm{N}(\mathfrak{a}))^{\epsilon}. \ \blacksquare\end{aligned}$$

12.3.1. *The completion of the proof of Lemma 12.2.* From (12.5), we have

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a}) \ll_\epsilon \mathrm{N}(\mathfrak{a})^{2\epsilon}\Big\{\prod_{v\in\Sigma_\infty} l_v\Big\}^{-1/2} \sum_{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^{-1/4+\epsilon} \tag{12.7}$$

with the implied constant independent of $(l,\mathfrak{n},\mathfrak{a})$. Setting $\mathrm{N}((b)\mathfrak{a}) = c$, we rewrite the last sum as

$$\begin{aligned}&\sum_{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^{-1/4+\epsilon} \\ &\qquad = \mathrm{N}(\mathfrak{a})^{1/4-\epsilon}\sum_{c=1}^{\infty} c^{-1/4+\epsilon} \sum_{\substack{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty \\ |\mathrm{N}((b)\mathfrak{a})|=c}} |\mathrm{N}(b+1)|^{-1/4+\epsilon}.\end{aligned}$$

The range of c is reduced to $1 \le c \le \mathrm{N}(\mathfrak{a})$ by the condition $b \in \mathcal{Q}_\infty$. Since $(0) \ne (b+1)\mathfrak{a} \subset \mathfrak{o}$, we have $\mathrm{N}((b+1)\mathfrak{a}) \ge 1$, so the last sum over b is trivially bounded by $\mathrm{N}(\mathfrak{a})^{1/4-\epsilon}\#\{b \in \mathfrak{a}^{-1} \cap \mathcal{Q}_\infty \mid |\mathrm{N}((b)\mathfrak{a})| = c\}$ for any $\epsilon \in (0, 1/4)$. Combining these considerations with Lemma 12.6, we obtain

$$\begin{aligned}&\sum_{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^{-1/4+\epsilon} \\ &\qquad \ll_{\epsilon,\delta,\delta'} \mathrm{N}(\mathfrak{a})^{1/2-2\epsilon}\sum_{c=1}^{\mathrm{N}(\mathfrak{a})} c^{-1/4+\epsilon}c^{\delta'-\delta}\,\mathrm{N}(\mathfrak{a})^{\delta} \\ &\qquad \ll_{\epsilon,\delta,\delta'} \mathrm{N}(\mathfrak{a})^{1/2-2\epsilon}\,\mathrm{N}(\mathfrak{a})^{\delta} \times \mathrm{N}(\mathfrak{a})^{3/4+\epsilon+\delta'-\delta}\log\mathrm{N}(\mathfrak{a}) \\ &\qquad = \mathrm{N}(\mathfrak{a})^{5/4-\epsilon+\delta'}\log\mathrm{N}(\mathfrak{a})\end{aligned} \tag{12.8}$$

for any sufficiently small $\delta, \delta' > 0$. Consequently, we have the desired estimate from (12.7) and (12.8).

It remains to show the second assertion of Lemma 12.2. Suppose that $b \in \mathfrak{n} \cap \mathcal{Q}_\infty$. The integrality of b yields $\mathrm{N}(b) \in \mathbb{Z}$. Since $b \in \mathcal{Q}_\infty$, we have $0 < |b_v| < 1$ for all $v \in \Sigma_\infty$, so $0 < |\mathrm{N}(b)| < 1$. Thus, if $\mathfrak{a} = \mathfrak{o}$, then the sum on the right-hand side of (12.5) is empty. This completes the proof.

12.4. An estimate of the hyperbolic term. Let η be a quadratic idele class character of $F^\times$ with conductor $\mathfrak{f}$ such that $\eta_v(-1) = -1$ for all $v \in \Sigma_\infty$. Given an integral ideal $\mathfrak{n}$, for a large $K \geq 2$, let $S = S_K^{\mathfrak{n},\eta} = \{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}) \mid \eta_v(\varpi_v) = -1,\ K \leq q_v \leq 2K\}$, and consider the test function $\alpha_S^\pi(\mathbf{s})$ depending on $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$ (see §12.1).

LEMMA 12.7. *There exists a constant $C > 1$ independent of $\mathfrak{n}$ and η such that*

$$C^{-1}K(\log K)^{-1} < \sharp S < CK(\log K)^{-1} \quad \text{for all } K \geq 2.$$

Proof. This follows from an analogue of Dirichlet's theorem on arithmetic progressions for number fields. ∎

PROPOSITION 12.8. *Let $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$. For any $\epsilon > 0$,*

$$|\mathbb{J}_{\mathrm{hyp}}^\eta(l, \mathfrak{n}|\alpha_S^\pi)| \ll_\epsilon \Big\{\prod_{v\in\Sigma_\infty} l_v\Big\}^{-1/2} \mathrm{N}(\mathfrak{f})^{1/4+\epsilon} K^{5+\epsilon}$$

with the implied constant independent of l, $\mathfrak{n}$, π, η and K.

Proof. Set $P = \{(v_1, v_2) \in S^2 \mid v_1 \neq v_2\}$. From Lemmas 10.4 and 10.5,

$$|\mathbb{J}_{\mathrm{hyp}}^\eta(l, \mathfrak{n}|\alpha_S^\pi)| \ll \sum_{b\in F^\times - \{-1\}} |J_S(b; \alpha_S^\pi)| \Big\{\prod_{v\in\Sigma_{\mathrm{fin}}-S\cup S(\mathfrak{f})} \Lambda_v(b)\Big\}\Big\{\prod_{v\in S(\mathfrak{f})} |J_v^{\eta_v}(b)|\Big\}\mathfrak{m}_\infty(l; b).$$

Combining this with Corollary 10.11 and Proposition 12.1, we see that this is majorized by $\mathrm{N}(\mathfrak{f})^{-1+\epsilon}$ multiplied by

$$\begin{aligned}
&\Big\{\sum_{v\in S} q_v^{(\theta+1)/2}\Big\}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{f}) + \sum_{v\in S} q_v^\theta \mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_v\mathfrak{f}) + \sum_{v\in S} q_v^{\theta-1}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_v^2\mathfrak{f}) \\
&\quad + \sum_{v\in S} q_v^{-1}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_v^3\mathfrak{f}) + \sum_{v\in S} q_v^{-2}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_v^4\mathfrak{f}) \\
&\quad + \Big\{\sum_{(v_1,v_2)\in P} q_{v_1}^{(\theta+1)/2} q_{v_2}^{(\theta+1)/2}\Big\}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{f}) + \sum_{(v_1,v_2)\in P} q_{v_1}^{(\theta+1)/2}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_{v_2}\mathfrak{f}) \\
&\quad + \sum_{(v_1,v_2)\in P} \mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_{v_1}\mathfrak{p}_{v_2}\mathfrak{f}) + \sum_{(v_1,v_2)\in P} q_{v_1}^{-1}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_{v_1}^2\mathfrak{p}_{v_2}\mathfrak{f}) \\
&\quad + \sum_{(v_1,v_2)\in P} q_{v_1}^{-1} q_{v_2}^{(\theta+1)/2}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_{v_1}^2\mathfrak{f}) + \sum_{(v_1,v_2)\in P} q_{v_1}^{-1} q_{v_2}^{-1}\mathfrak{I}(l, \mathfrak{n}, \mathfrak{p}_{v_1}^2\mathfrak{p}_{v_2}^2\mathfrak{f}).
\end{aligned}$$

Invoking the bound $\sharp S \ll K$ obtained from Lemma 12.7 and applying Lemma 12.2, we estimate each term occurring above. Thus, after a power saving, we obtain $|\mathbb{J}_{\mathrm{hyp}}^\eta(l, \mathfrak{n}|\alpha_S^\pi)| \ll_\epsilon \mathrm{N}(\mathfrak{f})^{-1+\epsilon}\varphi(l, K)$, where $L = \prod_{v\in\Sigma_\infty} l_v$

and $\varphi(l,K)$ is

$$\mathrm{N}(\mathfrak{f})^{5/4+\epsilon}L^{-1/2}(K^{(\theta+3)/2}+K^{\theta+9/4+\epsilon}+K^{\theta+5/2+2\epsilon}+K^{15/4+3\epsilon}+K^{4+4\epsilon}$$
$$+K^{3+\theta}+K^{(2\theta+15)/4+\epsilon}+K^{9/2+2\epsilon}+K^{19/4+3\epsilon}+K^{(\theta+8)/2+2\epsilon}+K^{5+4\epsilon}).$$

Since $\theta\in[0,1]$, this is bounded by $\mathrm{N}(\mathfrak{f})^{1/4+2\epsilon}L^{-1/2}K^{5+4\epsilon}$. This completes the proof. ∎

12.5. An estimate of the unipotent term. Set $S=S_K^{\mathfrak{n},\eta}$ with $K\geq 2$.

PROPOSITION 12.9. *Let $\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})$. For any $\epsilon>0$,*

$$|\tilde{\mathbb{J}}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha_S^{\pi})|\ll_{\epsilon}|\mathcal{G}(\eta)|\mathrm{N}(\mathfrak{f})^{\epsilon}K^{1+\theta}$$

with the implied constant independent of l, $\mathfrak{n}$, π, η and K.

Proof. We use the same notation as in the proof of Proposition 12.1. By substituting the expression $\alpha_S^{\pi}(\mathbf{s})=\sum_{v\in S}Z_v^2+\sum_{(v_1,v_2)\in P}Z_{v_1}Z_{v_2}$, we obtain

$$\begin{aligned}
&|\tilde{\mathbb{J}}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha_S^{\pi})|\\
&\ll \mathbf{C}_F^{\eta}(l,\mathfrak{n})\Big(\sum_{v\in S}\Big\{\prod_{w\in S-\{v\}}|U_w^{\eta_w}(1)|\Big\}|U_v^{\eta_v}(Z_v^2)|\\
&\qquad+\sum_{(v_1,v_2)\in P}\Big\{\prod_{w\in S-\{v_1,v_2\}}|U_w^{\eta_w}(1)|\Big\}|U_{v_1}^{\eta_{v_1}}(Z_{v_1})|\,|U_{v_2}^{\eta_{v_2}}(Z_{v_2})|\Big)\\
&\ll L_{\mathrm{fin}}(1,\eta)\Big(\sum_{v\in S}|U_v^{\eta_v}(Z_v^2)|+\sum_{(v_1,v_2)\in P}|U_{v_1}^{\eta_{v_1}}(Z_{v_1})|\,|U_{v_2}^{\eta_{v_2}}(Z_{v_2})|\Big),
\end{aligned}$$

where to simplify, we use $U_w^{\eta_w}(1)=-1$ from Proposition 11.1. As in the proof of Proposition 12.1, using Proposition 11.1, we compute each term as follows:

$$\begin{aligned}
U_v^{\eta_v}(Z_v^2)&=\lambda_v(\pi)^2\{U_v^{\eta_v}(\alpha_v^{(2)})+U_v^{\eta_v}(\alpha_v^{(0)})\}\\
&\quad+U_v^{\eta_v}(\alpha_v^{(4)})+2U_v^{\eta_v}(\alpha_v^{(2)})+\tfrac{3}{2}U_v^{\eta_v}(\alpha_v^{(0)})\\
&=\lambda_v(\pi)^2\{(1-q_v^{-1})-2\}+q_v^{-1}(1-q_v^{-1})+2(1-q_v^{-1})-3.
\end{aligned}$$

Since $|\lambda_v(\pi)|\ll q_v^{\theta/2}$ with $\theta\in[0,1]$, the above yields $|U_v^{\eta_v}(Z_v^2)|\ll q_v^{\theta}(1+q_v^{-1})+q_v^{-2}+q_v^{-1}+1\ll q_v^{\theta}$. In a similar way, $U_v^{\eta_v}(Z_v)=q_v^{-1}$. Applying these, we continue the estimate of $\tilde{\mathbb{J}}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha_S^{\pi})$ as follows:

$$\begin{aligned}
L_{\mathrm{fin}}(1,\eta)\Big(\sum_{v\in S}q_v^{\theta}+\sum_{(v_1,v_2)\in P}q_v^{-1}q_v^{-1}\Big)&\ll_{\epsilon}\mathrm{N}(\mathfrak{f})^{\epsilon}\Big\{\frac{K}{\log K}K^{\theta}+\Big(\frac{K}{\log K}\Big)^2K^{-2}\Big\}\\
&\ll\mathrm{N}(\mathfrak{f})^{\epsilon}K^{\theta+1}.
\end{aligned}$$

We remark that $L_{\mathrm{fin}}(1,\eta)\ll_{\epsilon}\mathrm{N}(\mathfrak{f})^{\epsilon}$ [3, Theorem 2]. This completes the proof. ∎

12.6. A subconvexity bound (odd case). Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$. For a family $l = (l_v)_{v\in\Sigma_\infty}$ of positive even integers, let $\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ denote the set of all cuspidal automorphic representations $\pi \cong \bigotimes_{v\in\Sigma_F}\pi_v$ of $\mathrm{PGL}(2,\mathbb{A})$ such that $\mathfrak{f}_\pi = \mathfrak{n}$ and π_v is isomorphic to the discrete series representation D_{l_v} of minimal $\mathbf{K}^0_v$-type l_v for each $v\in\Sigma_\infty$.

THEOREM 12.10. *Let η be a quadratic idele class character of $F^\times$ with conductor $\mathfrak{f}$ such that $\eta_v(-1) = -1$ for all $v\in\Sigma_\infty$. Let $\mathfrak{n}$ be an integral ideal relatively prime to $\mathfrak{f}$. Assume that $l_v \geq 6$ for all $v\in\Sigma_\infty$. Then, for any $\epsilon > 0$,*

$$|L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta)| \ll_\epsilon (\mathrm{N}(\mathfrak{n}\mathfrak{f})KL)^\epsilon\,\mathrm{N}(\mathfrak{n})\big(LK^{\theta-1} + \mathrm{N}(\mathfrak{f})^{3/4}L^{1/2}K^3\big),$$

where $L = \prod_{v\in\Sigma_\infty} l_v$ and the implied constant is independent of l, $\mathfrak{n}$, η, $K\geq 2$ and $\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$.

Proof. Let $\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ and let $S = S_K^{\mathfrak{n},\eta}$. By applying Theorem 9.1 for the test function $\alpha_S^\pi(\mathbf{s})$, we have

$$|C(l,\mathfrak{n},S)|\Big|\sum_{\pi'\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\mathbb{I}^\eta_{\mathrm{cus}}(\pi';l,\mathfrak{n})\alpha_S^\pi(\nu_S(\pi'))\Big| \leq |\tilde{\mathbb{J}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha_S^\pi)| + |\mathbb{J}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha_S^\pi)|$$

with

$$C(l,\mathfrak{n},S) = (-1)^{\#S}2^{-1}D_F^{-1}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\prod_{v\in\Sigma_\infty}2\pi\Gamma(l_v-1)/\Gamma(l_v/2)^2.$$

From Proposition 6.6 and the non-negativity of $\mathbb{I}^\eta_{\mathrm{cus}}(\pi';l,\mathfrak{n})/(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)$ by Lemma 6.2, the left-hand side becomes

$$|C(l,\mathfrak{n},S)|\,|\mathcal{G}(\eta)| \times\sum_{\pi'\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\frac{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{f}_{\pi'})]}{\mathrm{N}(\mathfrak{f}_{\pi'})}w^\eta_{\mathfrak{n}}(\pi')\,\frac{L(1/2,\pi')L(1/2,\pi'\otimes\eta)}{L^{S_{\pi'}}(1,\pi';\mathrm{Ad})}\alpha_S^\pi(\nu_S(\pi')),$$

which is greater than the summand corresponding to π by non-negativity again.

Let us examine the π-term more closely. First, from the explicit formula, $w^\eta_{\mathfrak{n}}(\pi) = 1$ for $\mathfrak{f}_\pi = \mathfrak{n}$. Let $A_v(\pi) = \mathrm{diag}(z_v, z_v^{-1})$ be the Satake parameter of our π. Then, using Lemma 12.7, we obtain

$$\alpha_S^\pi(\nu_S(\pi)) = \Big(\sum_{v\in S}\{(z_v+z_v^{-1})^2 - (z_v^2+z_v^{-2}+1)\}\Big)^2 = (\#S)^2 \gg_\epsilon K^{2-\epsilon}.$$

Separating the gamma factors from the L-functions, we get

$$\begin{aligned}|C(l,\mathfrak{n},S)|\,&\frac{L_\infty(1/2,\pi)L_\infty(1/2,\pi\otimes\eta)}{L_\infty(1,\pi;\mathrm{Ad})}\\ &\asymp [\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\Big\{\prod_{v\in\Sigma_\infty}\frac{2\pi\Gamma(l_v-1)}{\Gamma(l_v/2)^2}\Big\}\Big\{\prod_{v\in\Sigma_\infty}\frac{\Gamma_{\mathbb{C}}(l_v/2)^2}{\Gamma_{\mathbb{C}}(l_v)}\Big\}\\ &\asymp [\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\prod_{v\in\Sigma_\infty}(l_v-1)^{-1},\end{aligned}$$

where all the implied constants only depend on F. The remaining factors in the π-term are easily seen to be bounded from below by a constant independent of $(l,\mathfrak{n},\pi,\eta)$. The considerations so far yield

$$\begin{aligned}(12.9)\qquad |\mathcal{G}(\eta)|K^{2-\epsilon}\,\mathrm{N}(\mathfrak{n})^{-1}L^{-1}&\frac{L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta)}{L_{\mathrm{fin}}^{S_\pi}(1,\pi;\mathrm{Ad})}\\ &\ll_\epsilon |\tilde{\mathbb{J}}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha_S^\pi)|+|\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha_S^\pi)|.\end{aligned}$$

From Propositions 12.8 and 12.9, the right-hand side is estimated by

$$\ll_\epsilon |\mathcal{G}(\eta)|\mathrm{N}(\mathfrak{f})^{\epsilon}K^{1+\theta}+\mathrm{N}(\mathfrak{f})^{1/4+\epsilon}L^{-1/2}K^{5+\epsilon}.$$

To complete the proof, we invoke the bound $L_{\mathrm{fin}}^{S_\pi}(1,\pi;\mathrm{Ad})\ll_\epsilon(\mathrm{N}(\mathfrak{n})L)^\epsilon$, which is known to hold for a general class of L-series [3, Theorem 2]. We remark that $|\mathcal{G}(\eta)|=D_F^{-1/2}\,\mathrm{N}(\mathfrak{f})^{-1/2}\prod_{v\in S(\mathfrak{f})}(1-q_v^{-1})^{-1}\geq D_F^{-1/2}\,\mathrm{N}(\mathfrak{f})^{-1/2}$. ∎

THEOREM 12.11. *Let η be a quadratic idele class character of $F^\times$ such that $\eta_v(-1)=-1$ for all $v\in\Sigma_\infty$. Let $\mathfrak{n}$ be an integral ideal relatively prime to $\mathfrak{f}$. Assume that $l_v\geq 6$ for all $v\in\Sigma_\infty$. Then, for any $\epsilon>0$,*

$$|L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta)|\ll_\epsilon \mathrm{N}(\mathfrak{f})^{3/4+\epsilon}\,\mathrm{N}(\mathfrak{n})^{1+\epsilon}\Big\{\prod_{v\in\Sigma_\infty}l_v\Big\}^{(7-\theta)/(8-2\theta)+\epsilon}$$

with the implied constant independent of l, $\mathfrak{n}$, η and $\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{n})$.

Proof. We apply the estimate in Theorem 12.10 with K such that $LK^{\theta-1}\asymp L^{1/2}K^3$, or equivalently $K\asymp L^{1/(8-2\theta)}$. Then we obtain the desired estimate. ∎

If $\theta\in[0,1)$, the estimate in Theorem 12.11 breaks the convex bound $L_{\mathrm{fin}}(1/2,\pi)L_{\mathrm{fin}}(1/2,\pi\otimes\eta)\ll_\epsilon\{C(\pi)C(\pi\otimes\eta)\}^{1/4+\epsilon}\ll(\prod_{v\in\Sigma_\infty}l_v)^{1+\epsilon}$ in the weight aspect with a fixed level $\mathfrak{n}$ and a fixed character η. To deduce Theorem 1.3, we only have to invoke the Ramanujan bound $\theta=0$ (see [1]) in Theorem 12.11.

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