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Some examples in the theory of Beurling's generalized prime numbers

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1. Introduction. In this article we shall construct various examples of generalized number systems in order to compare three major conditions for the validity of the prime number theorem (PNT) in the setting of Beurling's theory of generalized primes.

Beurling's abstract formulation of the PNT is as follows [1, 2]. A set of *generalized primes* is simply a sequence $P = \{p_k\}_{k=1}^{\infty}$ of real numbers tending to infinity with the only requirement that $1 < p_1 \le p_2 \le \cdots$. Its associated set of *generalized integers* is the non-decreasing sequence $1 = n_0 < n_1 \le n_2 \le \cdots$ arising as all possible finite products of the generalized primes (occurring in $\{n_k\}_{k=1}^{\infty}$ as many times as they can be represented by $p_{\nu_1}^{\alpha_1} \dots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$). Consider the counting functions of the generalized integers and primes,

$$(1.1)\qquad N(x) = N_P(x) = \sum_{n_k \le x} 1 \quad \text{and} \quad \pi(x) = \pi_P(x) = \sum_{p_k \le x} 1,$$

where one takes multiplicities into account. Beurling's problem is then to determine asymptotic requirements on N, as minimal as possible, which ensure the PNT in the form

$$(1.2)\qquad \pi(x) \sim \frac{x}{\log x}, \quad x \to \infty.$$

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Three chief conditions on N are the following. The first was found by Beurling in his seminal work [2]. He showed that

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right), \quad x \to \infty, \tag{1.3}$$

where $a > 0$ and $\gamma > 3/2$, suffices for the PNT (1.2) to hold. A significant extension to this result was achieved by Kahane [11]. He proved, giving a positive answer to a long-standing conjecture by Bateman and Diamond [1], that the L^2-hypothesis

$$\int_1^{\infty} \left| \frac{(N(t) - at)\log t}{t} \right|^2 \frac{dt}{t} < \infty, \tag{1.4}$$

for some $a > 0$, implies the PNT. We refer to the recent article [19] by Zhang for a detailed account of Kahane's proof of the Bateman–Diamond conjecture (see also the expository article [6]). Another condition yet for the PNT has recently been provided by Schlage-Puchta and Vindas [14], who have shown that

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \ \text{(C)}, \quad x \to \infty, \tag{1.5}$$

with $a > 0$ and $\gamma > 3/2$ is also sufficient to ensure the PNT. The symbol (C) stands for the *Cesàro sense* [7] and explicitly means that there is some (possibly large) $m \in \mathbb{N}$ such that the following average estimate holds:

$$\int_1^{x} \frac{N(t) - at}{t}\left(1 - \frac{t}{x}\right)^m dt = O\left(\frac{x}{\log^{\gamma} x}\right), \quad x \to \infty. \tag{1.6}$$

It is obvious that Beurling's condition (1.3) is a particular instance of both (1.4) and (1.5). Furthermore, Kahane's PNT also covers an earlier extension of Beurling's PNT by Diamond [3]. However, as pointed out in [14, 19], the relation between (1.4) and (1.5) is less clear.

Our main goal in this paper is to compare (1.4) and (1.5). We shall construct a family of sets of generalized primes fulfilling the conditions stated in the following theorem:

Theorem 1.1. *Let $1 < \alpha < 3/2$. There exists a generalized prime number system P_α whose generalized integer counting function N_{P_α} satisfies* (*for some $a_\alpha > 0$*)

$$N_{P_\alpha}(x) = a_\alpha x + O\left(\frac{x}{\log^{n} x}\right) \ \text{(C)}, \quad \textit{for } n = 1, 2, \dots, \tag{1.7}$$

but violates (1.4), *namely,*

$$\int_1^{\infty} \left| \frac{(N_{P_\alpha}(t) - a_\alpha t)\log t}{t} \right|^2 \frac{dt}{t} = \infty. \tag{1.8}$$

Moreover, these generalized primes satisfy the PNT with remainder

$$\pi_{P_\alpha}(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^\alpha x}\right). \tag{1.9}$$

Our method for establishing Theorem 1.1 is first to construct examples of *continuous* generalized number systems with the desired properties. For this, we shall translate in Section 2 the conditions (1.7) and (1.8) into analytic properties of zeta functions. Our continuous examples are actually inspired by the one Beurling gave in [2] to show that his theorem is sharp, that is, an example that satisfies (1.3) for $\gamma = 3/2$ but for which the PNT (1.2) fails. Concretely, in Section 3 we study the zeta functions $\zeta_{C,\alpha}$ associated to the family of absolutely continuous Riemann prime counting functions

$$\Pi_{C,\alpha}(x) = \int_1^x \frac{1 - \cos(\log^\alpha u)}{\log u}\, du, \qquad x \geq 1. \tag{1.10}$$

If $\alpha = 1$ in (1.10), this reduces to the example of Beurling, whose associated zeta function is $\zeta_{C,1}(s) = (1 + 1/(s-1)^2)^{1/2}$. In case $\alpha > 1$, explicit formulas for the zeta function of (1.10) are no longer available, which makes its analysis considerably more involved than that of Beurling's example. In the absence of an explicit formula, our method rather relies on studying qualitative properties of the zeta function, which will be obtained in Theorem 3.1 via the Fourier analysis of certain related singular oscillatory integrals. As we show, the condition $1 < \alpha < 3/2$ from Theorem 1.1 is connected with the asymptotic behavior of the derivative of $\zeta_{C,\alpha}(s)$ on $\Re e\, s = 1$.

The next step in our construction for the proof of Theorem 1.1 is to select a discrete set P_α of generalized primes whose prime counting function π_{P_α} is sufficiently close to (1.10). We follow here a discretization idea of Diamond, which he applied in [5] to produce a discrete example showing the sharpness of Beurling's theorem. We prove in Section 4 that the set of generalized primes

$$P_\alpha = \{p_k\}_{k=1}^\infty, \qquad p_k = \Pi_{C,\alpha}^{-1}(k), \tag{1.11}$$

satisfies all the requirements from Theorem 1.1.

Note that Diamond's example from [5] is precisely the case $\alpha = 1$ of (1.11). However, it should also be noticed that the analysis of our example (1.11) that we carry out in Section 4 is completely different from that in [5]. Our arguments rely on suitable bounds for the associated zeta functions and their derivatives. Moreover, our ideas lead to more accurate asymptotic information on the generalized integer counting function of Diamond's example. We give a proof of the following theorem in Section 5.

THEOREM 1.2. *Let P_1 be the set* (1.11) *of generalized primes corresponding to $\alpha = 1$. There are constants c, $\{d_j\}_{j=0}^\infty$, and $\{\theta_j\}_{j=0}^\infty$ such that N_{P_1}*

has asymptotic expansion

$$N_{P_1}(x) \sim cx + \frac{x}{\log^{3/2} x} \sum_{j=0}^{\infty} d_j \frac{\cos(\log x + \theta_j)}{\log^j x} = cx + d_0 \frac{x \cos(\log x + \theta_0)}{\log^{3/2} x} + O\left(\frac{x}{\log^{5/2} x}\right), \qquad x \to \infty, \tag{1.12}$$

with $c > 0$ and $d_0 \neq 0$, while the PNT does not hold for P_1.

We mention that Theorem 1.2 not only shows the sharpness of $\gamma > 3/2$ in Beurling's condition (1.3) for the PNT, but also that of $\gamma > 3/2$ in (1.5). In addition, (1.12) implies that all the Riesz means of the relative error $(N_{P_1}(x) - cx)/x$ satisfy

$$\int_1^x \frac{N_{P_1}(t) - ct}{t} \left(1 - \frac{t}{x}\right)^m dt = \Omega_{\pm}\left(\frac{x}{\log^{3/2} x}\right), \qquad x \to \infty,\ m = 0, 1, 2, \ldots.$$

Observe also that Theorem 1.1 in particular shows that the PNT by Schlage-Puchta and Vindas is a proper generalization of Beurling's result. They gave an example in [14, Sect. 6] to support this result, but their proof contains a few mistakes (there are gaps in the proof of [14, Lemma 6], and the proof of [14, Eq. (6.4)] turns out to be incorrect). The last section of this article will be devoted to correcting these mistakes; we prove there:

THEOREM 1.3. *There exists a set P^* of generalized primes such that $N_{P^*}(x) = x + \Omega(x/\log^{4/3} x)$, but $N_{P^*}(x) = x + O(x/\log^{5/3-\varepsilon} x)$ in Cesàro sense for arbitrary $\varepsilon > 0$. Furthermore, for this number system we have $\pi_{P^*}(x) = x/\log x + O(x/\log^{4/3-\varepsilon} x)$.*

1.1. Notation. We will often make use of standard Schwartz distribution calculus. See the textbooks [9, 17] for the theory of distributions and [7, 13] for asymptotic analysis of generalized functions. The standard test function spaces are denoted as usual by $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, while $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ stand for their topological duals, the spaces of distributions and tempered distributions. We fix the constants in the Fourier transform as $\hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx}\phi(x)\,dx$. Naturally, the Fourier transform is well defined on $\mathcal{S}'(\mathbb{R})$ via duality. If $f \in \mathcal{S}'(\mathbb{R})$ has support in $[0, \infty)$, its Laplace transform is $\mathcal{L}\{f; s\} = \langle f(u), e^{-su}\rangle$, $\Re e\, s > 0$, and its Fourier transform $\hat{f}$ is the distributional boundary value of $\mathcal{L}\{f; s\}$ on $\Re e\, s = 0$. We use the notation H for the Heaviside function, which is the characteristic function of $(0, \infty)$.

2. Auxiliary lemmas. We begin by defining some other helpful number-theoretic functions. As usual, the zeta function is indispensable for studying the prime number theorem in this context,

$$\zeta(s) = \int_{1^-}^{\infty} x^{-s}\, dN(x). \tag{2.1}$$

Besides the usual prime counting function π, we will also work with the Riemann prime counting function,

$$\Pi(x) = \sum_{n=1}^{\infty} \frac{\pi(x^{1/n})}{n}, \tag{2.2}$$

and we have the following link between Π and ζ:

$$\zeta(s) = \exp\Big(\int_{1^-}^{\infty} x^{-s}\, d\Pi(x)\Big). \tag{2.3}$$

We will consider an even broader definition of generalized primes [2], which also takes into account 'continuous' number systems. So, in this sense a generalized prime number system is merely a non-decreasing function Π that vanishes for $x \le 1$, where we assume that the integral involved in (2.3) is absolutely convergent in the half-plane $\Re e\, s > 1$. We normalize Π in such a way that it is right continuous. Clearly, the zeta function ζ from (2.3) can always be represented as (2.1) with a unique non-decreasing function N if we impose that N is right continuous; in fact, N is determined by $dN = \exp^*(d\Pi)$, where the exponential is taken with respect to the multiplicative convolution of measures [4]. The function π need not make sense in this framework. Note also that if N satisfies any of the three conditions (1.3)–(1.5), then $N(x) \sim ax$; consequently, if two of those conditions are simultaneously satisfied, the constant a should be the same. We remark as well that all the three PNT discussed in the introduction are valid in this more general setting.

In the rest of this section we connect (1.4) and (1.5) with the boundary behavior of $\zeta(s)$ on the line $\Re e\, s = 1$.

2.1. A sufficient condition for the Cesàro behavior. The following Tauberian lemma gives sufficient conditions on the zeta function for N to have the Cesàro behavior (1.5) with $\gamma = n \in \mathbb{N}$. The proof of this result makes use of the notion of the quasiasymptotic behavior of Schwartz distributions; for it, we use the notation of [13, Sect. 2.12, p. 160] (see also [14, p. 304]).

Lemma 2.1. *Let $n \in \mathbb{N}$. Suppose that the function $F(s) = \zeta(s) - a/(s-1)$ can be extended to the closed half-plane $\Re e\, s \ge 1$ as an n times continuously differentiable function. If for every $0 \le j \le n$ the functions $F^{(j)}(1+it)$ have at most polynomial growth with respect to the variable t, then N satisfies the*

Cesàro estimate

$$N(x) = ax + O\left(\frac{x}{\log^n x}\right) \quad (\mathrm{C}), \qquad x \to \infty. \tag{2.4}$$

Proof. We define a function R with support in $[0, \infty)$ in such a way that $N(x) = axH(x-1) + xR(\log x)$ (H is the Heaviside function). By the Wiener–Ikehara theorem (cf. [12, 18]), the assumptions imply $N(x) \sim ax$. This ensures that $R \in \mathcal{S}'(\mathbb{R})$. A quick computation shows that $F(s) = a + s\mathcal{L}\{R; s-1\}$ for $\Re e\, s > 1$. Let $\phi \in \mathcal{S}(\mathbb{R})$. We obtain

$$\begin{aligned}\langle R(u+h), \phi(u)\rangle &= \frac{1}{2\pi}\langle \hat{R}(t), \hat{\phi}(-t)e^{iht}\rangle \\ &= \frac{1}{2\pi}\lim_{\sigma\to 1^+}\int_{-\infty}^{\infty}\frac{F(\sigma+it)-a}{\sigma+it}\hat{\phi}(-t)e^{iht}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(1+it)-a}{1+it}\hat{\phi}(-t)e^{iht}\,dt.\end{aligned}$$

By using integration by parts n times, we can bound this last term as

$$\frac{(-1)^n}{2\pi}\int_{-\infty}^{\infty}\left(\frac{F(1+it)-a}{1+it}\hat{\phi}(-t)\right)^{(n)}\frac{e^{iht}}{i^n h^n}\,dt = O(h^{-n}), \qquad h \to \infty.$$

The last step is justified because all the derivatives of $F(1+it)$ have at most polynomial growth and any test function in $\mathcal{S}(\mathbb{R})$ decreases faster than any inverse power of $|t|$. We thus find that $\int_{-\infty}^{\infty} R(u+h)\phi(u)\,du = O(h^{-n})$. Assuming that $\phi \in \mathcal{D}(\mathbb{R})$ and writing $h = \log\lambda$ and $\varphi(x) = e^x\phi(e^x)$, we obtain the quasiasymptotic behavior

$$R(\log(\lambda x)) = O\left(\frac{1}{\log^n \lambda}\right), \qquad \lambda \to \infty, \text{ in } \mathcal{D}'(0,\infty), \tag{2.5}$$

which explicitly means that

$$\int_1^{\infty} R(\log(\lambda x))\varphi(x)\,dx = O\left(\frac{1}{\log^n\lambda}\right), \qquad \lambda \to \infty,$$

for every $\varphi \in \mathcal{D}(0,\infty)$. Using [15, Thm. 4.1], we find that the quasiasymptotic behavior (2.5) in $\mathcal{D}'(0,\infty)$ is equivalent to the same quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R})$, and, from the structural theorem for quasiasymptotic boundedness [13, Thm. 2.42, p. 163] (see also [15, 16]), we obtain the Cesàro behavior (2.4). ∎

2.2. Kahane's condition in terms of ζ. Note first that Kahane's condition (1.4) can be written as

$$N(x) = ax + \frac{x}{\log x}E(\log x), \qquad x \geq 1,$$

where $E \in L^2(\mathbb{R})$. We set $E(u) = 0$ for $u < 0$. Notice that $E(u)/u$ is right continuous at every point, as follows directly from its definition, and in particular it is integrable near $u = 0$.

In the rest of this subsection we consider a generalized number system which satisfies Kahane's condition (1.4) with $a > 0$. Since $N(x) \sim ax$, the abscissa of convergence of ζ is 1. Furthermore, $\zeta(1+it)$ always makes sense as a tempered distribution (the Fourier transform of the tempered measure $e^{-u}\,dN(e^u)$). With these ingredients we can compute the zeta function. We obtain

$$\zeta(s) = \frac{a}{s-1} + a + sG(s), \qquad \Re e\, s > 1, \tag{2.6}$$

with

$$G(s) = \int_0^\infty e^{-(s-1)u} \frac{E(u)}{u}\, du.$$

The function G admits a continuous and bounded extension to $\Re e\, s = 1$:

$$G(1+it) = \int_0^\infty e^{-itu} \frac{E(u)}{u}\, du.$$

Indeed, since $E(u)u^{-1} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, its Fourier transform $G(1+it)$ is in $C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Furthermore,

$$G'(1+it) = -\hat{E}(t) \in L^2(\mathbb{R}).$$

These observations lead to the following lemma. Recall that H is the Heaviside function, so that $H(|t|-1)$ below is the characteristic function of $(-\infty,-1) \cup (1,\infty)$.

Lemma 2.2. *Kahane's condition* (1.4) *holds if and only if the boundary value distribution of* $(\zeta(s) - a/(s-1))'$ *on* $\Re e\, s = 1$ *satisfies*

$$\frac{d}{ds}\left(\zeta(s) - \frac{a}{s-1}\right)\bigg|_{s=1+it} \in L^2_{\mathrm{loc}}(\mathbb{R}) \tag{2.7}$$

and

$$\left(\frac{\zeta(1+it)}{t}\right)' H(|t|-1) \in L^2(\mathbb{R}). \tag{2.8}$$

Naturally, the derivative in (2.8) is taken in the distributional sense with respect to the variable t.

Proof. We have already seen that Kahane's condition holds if and only if $G'(1+it) \in L^2(\mathbb{R})$, and that (2.7) and (2.8) are necessary for it. Assume these two conditions hold. Note that (2.7) is sufficient for $G(1+it) \in C(\mathbb{R})$, while (2.8) and (2.6) imply

$$G(1+it) = O(\sqrt{|t|}) \quad \text{for } |t| > 1,$$

because

$$|\zeta(1+it)| \ll |t| \int_{1\le|u|\le|t|} \left|\left(\frac{\zeta(1+iu)}{u}\right)'\right| du \ll |t|^{3/2},$$

by Hölder's inequality. So, we may take the continuity of $G(1+it)$ and the bound $G(1+it) = O(\sqrt{|t|})$ for granted in the rest of the proof. In view of (2.6), the function involved in (2.7) is precisely $G(1+it)+(1+it)G'(1+it)$; therefore, (2.7) yields $G'(1+it) \in L^2_{\mathrm{loc}}(\mathbb{R})$. It remains to show that $G'(1+it)$ is square integrable on $\mathbb{R}\setminus[-1,1]$. For $|t|>1$, appealing again to the defining equation (2.6), we have

$$\begin{aligned}\frac{i(1+it)}{t}G'(1+it) &= \left(\frac{\zeta(1+it)}{t}\right)' + \frac{2a}{it^3} + \frac{a+G(1+it)}{t^2}\\ &= \left(\frac{\zeta(1+it)}{t}\right)' + O\left(\frac{1}{|t|^{3/2}}\right) \in L^2(\mathbb{R}\setminus[-1,1]),\end{aligned}$$

which now gives $G'(1+it) \in L^2(\mathbb{R})$. ■

Our strategy in the next two sections to show Theorem 1.1 is to exhibit examples of generalized number systems which fail to meet the conditions of Lemma 2.2 but satisfy those of Lemma 2.1.

3. Continuous examples. We shall now study the family of absolutely continuous Riemann prime counting functions (1.10). For ease of writing, we drop α from the notation and we simply write

$$\Pi_C(x) = \Pi_{C,\alpha}(x) = \int_1^x \frac{1-\cos(\log^\alpha u)}{\log u}\, du, \qquad x \ge 1. \tag{3.1}$$

The number-theoretic functions associated with this example will also have the subscript C, that is, we denote them as N_C and ζ_C. As pointed out in the Introduction, when $\alpha = 1$ we recover the example of Beurling. For this reason, it is clear that $\alpha = 1$ will not yield an example for Theorem 1.1, as the prime number theorem is not even fulfilled and hence neither holds the Cesàro behavior (1.7) for N_C with $n > 3/2$. We therefore assume in this section that $\alpha > 1$. Now we calculate the function ζ_C of our continuous number system via formula (2.3):

$$\begin{aligned}\log\zeta_C(s) &= \int_1^\infty \frac{d\Pi_C(x)}{x^s} = \int_1^\infty \frac{1-\cos(\log^\alpha x)}{x^s \log x}\, dx\\ &= \int_0^\infty \frac{1-\cos u^\alpha}{u} e^{-(s-1)u}\, du\end{aligned}$$

$$= \text{F.p.}\int_0^\infty \frac{e^{-(s-1)u}}{u}\,du - \text{F.p.}\int_0^\infty \frac{\cos u^\alpha}{u} e^{-(s-1)u}\,du$$
$$= -\log(s-1) - \gamma - K(s), \qquad \Re e\, s > 1,$$

where $\gamma = 0.57721\ldots$ is (*from now on* in this article) the Euler–Mascheroni constant,

$$K(s) := \text{F.p.}\int_0^\infty \frac{\cos u^\alpha}{u} e^{-(s-1)u}\,du, \qquad \Re\text{e}\ s > 1, \tag{3.2}$$

and F.p. stands for the Hadamard finite part of a divergent integral [7, Sect. 2.4]. Summarizing, we have found that

$$\zeta_C(s) = \frac{e^{-\gamma} e^{-K(s)}}{s-1}, \qquad \Re e\, s > 1. \tag{3.3}$$

It is clear that we must investigate the properties of the function K in order to make further progress in understanding the zeta function ζ_C of (3.1). The next theorem is of independent interest; it reveals a number of useful analytic properties of the singular integral (3.2).

THEOREM 3.1. *Let $\alpha > 1$. The function K defined by* (3.2) *for $\Re e\, s > 1$ has the ensuing properties:*

(a) *K can be extended to the whole complex plane as an entire function.*
(b) *$K(1) = -\gamma/\alpha$.*
(c) *On the line $\Re e\, s = 1$ the function K and its derivatives have asymptotic behavior*

$$K(1+it) = -\log|t| - \gamma - \frac{\pi i}{2}\operatorname{sgn}(t) + O\Big(\frac{1}{|t|^\alpha}\Big) + O\Big(\frac{1}{|t|^{\frac{\alpha}{2(\alpha-1)}}}\Big), \tag{3.4}$$

$$K'(1+it) = A_{\alpha,1}|t|^{\frac{1-\alpha/2}{\alpha-1}} \exp\Big(-i\operatorname{sgn}(t)\Big(B_\alpha|t|^{\alpha/(\alpha-1)} - \frac{\pi}{4}\Big)\Big) + O\Big(\frac{1}{|t|}\Big), \tag{3.5}$$

and, for $m = 2, 3, \ldots,$

$$K^{(m)}(1+it) = A_{\alpha,m}|t|^{\frac{m-\alpha/2}{\alpha-1}} \exp\Big(-i\operatorname{sgn}(t)\Big(B_\alpha|t|^{\alpha/(\alpha-1)} - \frac{\pi}{4}\Big)\Big) + O(|t|^{\frac{m-3\alpha/2}{\alpha-1}}), \tag{3.6}$$

as $|t| \to \infty$, where

$$B_\alpha = (\alpha-1)\alpha^{-\alpha/(\alpha-1)} \quad \textit{and} \quad A_{\alpha,m} = (-1)^m \alpha^{\frac{1/2-m}{\alpha-1}}\sqrt{\frac{\pi}{2(\alpha-1)}}, \tag{3.7}$$

for $m = 1, 2, 3, \ldots.$

Proof. We shall obtain all the claimed properties of K from those of the analytic function

$$F(z) := \mathrm{F.\,p.} \int_0^\infty \frac{e^{-izu}}{u} e^{iu^\alpha}\, du, \qquad \Im m\, z < 0.$$

The two functions are obviously linked via the relation

$$K(1+iz) = \frac{F(z) + \overline{F(-\bar{z})}}{2}. \tag{3.8}$$

We need a (continuous) Littlewood–Paley partition of unity [10, Sect. 8.5]. So, pick an even function $\varphi \in \mathcal{D}(\mathbb{R})$ with the following properties: $\operatorname{supp} \varphi \subset (-1,1)$ and $\varphi(x) = 1$ for $x \in [-1/2, 1/2]$. Set $\psi(x) = -x\varphi'(x)$, an even test function with support in $(-1, 1/2] \cup [1/2, 1)$, so that we have the decomposition

$$1 = \varphi(x) + \int_0^1 \psi(yx)\, \frac{dy}{y}, \qquad x \in \mathbb{R}.$$

This leads to the continuous Littlewood–Paley decomposition

$$F(z) = \theta(z) + v(z), \qquad \Im m\, z < 0, \tag{3.9}$$

where

$$\theta(z) = \mathrm{F.p.} \int_0^\infty \frac{\varphi(u) e^{i(u^\alpha - zu)}}{u}\, du, \qquad v(z) = \int_0^1 \Phi(y,z) \frac{dy}{y},$$

and

$$\Phi(y,z) = \int_0^\infty \frac{\psi(yu) e^{i(u^\alpha - zu)}}{u}\, du, \qquad \Im m\, z < 0.$$

The formula for v still makes sense for $z = t \in \mathbb{R}$ if it is interpreted in the sense of tempered distributions, the integral with respect to y then being understood as a weak integral in $\mathcal{S}'(\mathbb{R})$. Observe that $\theta(z)$ and $\Phi(y,z)$ are entire functions of z, as follows at once from the well-known Paley–Wiener–Schwartz theorem [17]. The asymptotic behavior of θ and its derivatives on the real axis can be computed directly from the Estrada–Kanwal generalization of Erdélyi's asymptotic formula [7, p. 148]; indeed, employing only one term from that asymptotic formula, we obtain

$$\begin{aligned} \theta(t) &= -\log|t| - \gamma - \frac{\pi i}{2}\operatorname{sgn}(t) + O\left(\frac{1}{|t|^\alpha}\right), \\ \theta^{(m)}(t) &= \frac{(-1)^m (m-1)!}{t^m} + O\left(\frac{1}{|t|^{\alpha+m}}\right), \end{aligned} \tag{3.10}$$

for $m = 1, 2, \dots,$ as $|t| \to \infty$.

We now study the integral $\int_0^1 \Phi(y,z)y^{-1}\,dy$. If we consider $z=t+i\sigma$,we can write ($t\neq 0$)

$$\partial_z^m \Phi(y,z)$$
$$= y^{1-m}(-i)^m|t|^{1/(\alpha-1)}\int_0^\infty \rho_m(|t|^{1/(\alpha-1)}yx)e^{i|t|^{\alpha/(\alpha-1)}(x^\alpha-\operatorname{sgn}(t)x)+\sigma|t|^{1/(\alpha-1)}x}\,dx,$$

where $\rho_m(x)=x^{m-1}\psi(x)$ for $m\in\mathbb{N}$. We need to establish some asymptotic estimates for the above integrals

$$J_m(y,t;\sigma)=\int_0^\infty \rho_m(|t|^{1/(\alpha-1)}yx)e^{i|t|^{\alpha/(\alpha-1)}(x^\alpha-\operatorname{sgn}(t)x)+\sigma|t|^{1/(\alpha-1)}x}\,dx. \tag{3.11}$$

We shall show that for each $n\in\mathbb{N}$,

$$J_m(y,t;\sigma)=\begin{cases} O(y^n t^{-n}) & \text{if } t>0 \text{ and } t^{1/(\alpha-1)}y>2\alpha^{1/(\alpha-1)},\\ O(y^{n\alpha-1}t^{-1/(\alpha-1)}) & \text{if } t>0 \text{ and } t^{1/(\alpha-1)}y<1/2,\end{cases} \tag{3.12}$$

and

$$J_m(y,t;\sigma)=\begin{cases} O(y^n|t|^{-n}) & \text{if } t<0 \text{ and } |t|^{1/(\alpha-1)}y\geq 1,\\ O(y^{n\alpha-1}|t|^{-1/(\alpha-1)}) & \text{if } t<0 \text{ and } |t|^{1/(\alpha-1)}y<1,\end{cases} \tag{3.13}$$

where all big O-constants only depend on α, n, and the L^∞-norms of the derivatives of ρ_m. Notice that the estimates (3.12) and (3.13) yield, uniformly for z in compacts of $\mathbb{C}$,

$$|\partial_z^m\Phi(y,z)|=O_n(y^{n\alpha-m}) \quad \text{if } y|t|^{1/(\alpha-1)}<1/2,$$

for any n, which proves that the integrals $\int_0^1 \partial_z^m\Phi(y,z)y^{-1}\,dy$ are absolutely convergent in the space of entire functions, and thus $v(z)$ is entire. In particular, $F(z)$ is an entire function, hence so is $K(s)$ because of (3.8). Furthermore, using (3.13), one obtains at once

$$v^{(m)}(t)=O(|t|^{-n}) \quad \text{as } t\to-\infty,\ \forall n\in\mathbb{N}. \tag{3.14}$$

In order to prove (3.12) in the range $t^{1/(\alpha-1)}y>2\alpha^{1/(\alpha-1)}$, we rewrite (3.11) as

$$J_m(y,t;\sigma)=\int_0^\infty \frac{\rho_m(t^{1/(\alpha-1)}yx)}{g'(x)}g'(x)e^{it^{\alpha/(\alpha-1)}g(x)}\,dx,$$

where $g(x)=x^\alpha-x-i\sigma x/t$. The estimate (3.12) for $t^{1/(\alpha-1)}y>2\alpha^{1/(\alpha-1)}$ follows by integrating by parts n times and noticing that $|g'(x)|>1-2^{1-\alpha}>0$ for $x\in(0,\alpha^{-1/(\alpha-1)}/2)$. In fact, integrating by parts once gives

$$J_m(y,t;\sigma) \le \frac{\|g\|_{L^\infty} + \|g'\|_{L^\infty}}{(1-2^{1-\alpha})^2} t^{-\alpha/(\alpha-1)} \times \int_0^{(2\alpha^{1/(\alpha-1)})^{-1}} \big(yt^{1/(\alpha-1)}|\rho_m'(yt^{1/(\alpha-1)}x)| + |\rho_m(yt^{1/(\alpha-1)}x)|\big)\,dx \ll yt^{-1},$$

because $\rho(yt^{1/(\alpha-1)}x)$ vanishes for $x \ge (2\alpha^{1/(\alpha-1)})^{-1}$ and we have $t^{-\alpha/(\alpha-1)} \le (2\alpha^{1/(\alpha-1)})^{-1} yt^{-1}$. In the general case, we iterate this procedure n times to obtain $J_m(y,t;\sigma) = O(y^n t^{-n})$ where the O-constant only depends on α and $\|\rho_m\|_{L^\infty(\mathbb{R})}$, $\|\rho_m'\|_{L^\infty(\mathbb{R})}, \ldots, \|\rho_m^{(n)}\|_{L^\infty(\mathbb{R})}$. On the other hand, if $t^{1/(\alpha-1)}y < 1/2$, we integrate by parts n times in the integral

$$J_m(y,t;\sigma) = \frac{1}{t^{1/(\alpha-1)}y} \int_{1/2}^{1} \frac{\rho_m(x)}{f_y'(x)} f_y'(x) e^{iy^{-\alpha} f_y(x)}\,dx,$$

where $f_y(x) = x^\alpha - y^{\alpha-1}tx - i\sigma y^{\alpha-1}x$. The second part of (3.12) holds because $|f_y'(x)| \ge \Re e\, f_y'(x) > (\alpha-1)2^{1-\alpha}$ and the derivatives of f_y of order ≥ 2 are bounded on $(1/2, 1)$; once again, the O-constant only depends on α and $\|\rho_m\|_{L^\infty(\mathbb{R})}$, $\|\rho_m'\|_{L^\infty(\mathbb{R})}, \ldots, \|\rho_m^{(n)}\|_{L^\infty(\mathbb{R})}$. The estimate (3.13) is proved in a similar fashion.

We now obtain the asymptotic behavior of $v(t)$ and its derivatives as $t \to \infty$. Employing (3.12) we have, for each $n \in \mathbb{N}$,

$$\begin{aligned} v^{(m)}(t) &= (-i)^m t^{1/(\alpha-1)} \int_{t^{-1/(\alpha-1)}/2}^{2(\alpha/t)^{1/(\alpha-1)}} y^{-m} J_m(y,t;0)\,dy + O(t^{-n}) \\ &= (-i)^m t^{m/(\alpha-1)} \int_{1/2}^{2\alpha^{1/(\alpha-1)}} y^{-m} \int_0^\infty \rho_m(yx) e^{it^{\alpha/(\alpha-1)}(x^\alpha - x)}\,dx\,dy + O(t^{-n}) \end{aligned}$$

as $t \to \infty$. The asymptotic expansion of $\int_0^\infty \rho_m(yx) e^{it^{\alpha/(\alpha-1)}(x^\alpha-x)}\,dx$ can be derived directly from the stationary phase principle (cf. [9, Thm. 7.7.5]). The only critical point of $x^\alpha - x$ lies at $x = \alpha^{-1/(\alpha-1)}$, so the stationary phase principle leads, after a routine computation, to

$$\int_0^\infty \rho_m(yx) e^{it^{\alpha/(\alpha-1)}(x^\alpha-x)}\,dx = A_\alpha t^{-\frac{\alpha}{2(\alpha-1)}} e^{-i(\alpha-1)(t/\alpha)^{\alpha/(\alpha-1)}} \rho_m(\alpha^{1/(1-\alpha)}y) + O(t^{-\frac{3\alpha}{2(\alpha-1)}})$$

as $t \to \infty$, uniformly for $y \in (1/2, 2\alpha^{1/(\alpha-1)})$, where

$$A_\alpha = \sqrt{\frac{2\pi i}{\alpha^{1/(\alpha-1)}(\alpha-1)}}$$

and the big O-constant depends only on α, m, and the derivatives of order ≤ 2 of ψ. Observe also that

$$\int_{1/2}^{2\alpha^{1/(\alpha-1)}} y^{-m}\rho_m(\alpha^{1/(1-\alpha)}y)\,dy = \alpha^{\frac{1-m}{\alpha-1}}\int_{1/2}^{1}\frac{\psi(y)}{y}\,dy = \alpha^{\frac{1-m}{\alpha-1}}.$$

Hence,

(3.15)
$$v^{(m)}(t) = (-i)^m\alpha^{\frac{1/2-m}{\alpha-1}}t^{\frac{m-\alpha/2}{\alpha-1}}e^{-i(\alpha-1)(t/\alpha)^{\alpha/(\alpha-1)}}\sqrt{\frac{2\pi i}{\alpha-1}} + O(t^{\frac{m-3\alpha/2}{\alpha-1}})$$

as $t \to \infty$. The asymptotic estimates (3.4)–(3.6) with constants (3.7) follow by combining (3.8), (3.10), (3.14), and (3.15). Thus, the proofs of (a) and (c) are complete.

It remains to establish (b). Notice that $K(1) = \Re e\, F(0)$ because of (3.8). On the other hand, applying the Cauchy theorem to

$$\oint_{\mathsf{C}} \frac{e^{i\xi^\alpha}}{\xi}\,d\xi$$

for the contours $\mathsf{C} = [\varepsilon, r] \cup \{\xi = re^{i\vartheta} : \vartheta \in [0, \pi/(2\alpha)]\} \cup \{\xi = xe^{i\pi/(2\alpha)} : x \in [\varepsilon, r]\} \cup \{\xi = \varepsilon e^{i\vartheta} : \vartheta \in [0, \pi/(2\alpha)]\}$, one deduces that

$$\begin{aligned} F(0) &= \text{F.p.}\int_0^\infty \frac{e^{iu^\alpha}}{u}\,du \\ &= \text{F.p.}\int_0^\infty \frac{e^{-x^\alpha}}{x}\,dx + \lim_{\varepsilon\to 0^+} i\int_0^{\pi/(2\alpha)} e^{i\varepsilon^\alpha e^{i\alpha\vartheta}}\,d\vartheta \\ &= \frac{1}{\alpha}\,\text{F.p.}\int_0^\infty \frac{e^{-x}}{x}\,dx + \frac{i\pi}{2\alpha} = -\frac{\gamma}{\alpha} + \frac{i\pi}{2\alpha}. \ \blacksquare \end{aligned}$$

The previous theorem and (3.3) imply that ζ_C is analytic in $\mathbb{C} \setminus \{1\}$ and actually has a simple pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1}\zeta_C(s) = e^{-(1-1/\alpha)\gamma}.$$

Thus, in view of Theorem 3.1(c), the function N_C fulfills the hypotheses of Lemma 2.1 with $a = \exp(-\gamma(1 - 1/\alpha))$ for every n. Furthermore, (2.7) is also satisfied, as $\zeta_C(s) - a/(s - 1)$ is entire. Since we are interested in violating Kahane's condition, we must investigate (2.8). The Leibniz rule for differentiation gives

$$\left(\frac{\zeta_C(1+it)}{t}\right)' H(|t|-1) = \left(-\frac{e^{-K(1+it)-\gamma}K'(1+it)}{t^2} + \frac{2e^{-K(1+it)-\gamma}}{t^3}\right) iH(|t|-1).$$

Using (3.4) we see that the absolute value of the second term is asymptotic to $(2/t^2)H(|t|-1) \in L^2(\mathbb{R})$. Employing Lemma 2.2 and (3.4) once again, we find that Kahane's condition for N_C becomes equivalent to

$$\frac{K'(1+it)}{t} H(|t|-1) \in L^2(\mathbb{R}).$$

The asymptotic behavior of $t^{-1}K'(1+it)$ is given by (3.5):

$$\frac{K'(1+it)}{t} = A_{\alpha,1}|t|^{\frac{2-3\alpha/2}{\alpha-1}} \exp\left(-i\operatorname{sgn}(t)\left(B_\alpha |t|^{\alpha/(\alpha-1)} - \frac{\pi}{4}\right)\right) + O\left(\frac{1}{|t|^2}\right)$$

as $|t| \to \infty$. The second term above is L^2 for $|t| \geq 1$, while the first term is L^2 only for $\alpha > 3/2$. We summarize our results in the following proposition, which shows that our continuous number system satisfies the properties stated in Theorem 1.1. As usual, we set

$$\operatorname{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

PROPOSITION 3.2. *Let $\alpha > 1$. The functions N_C and Π_C satisfy*

$$N_C(x) = xe^{-\gamma(1-1/\alpha)} + O\left(\frac{x}{\log^n x}\right) \quad \text{(C)} \quad \textit{for } n = 1, 2, \ldots,$$

and

$$\Pi_C(x) = \operatorname{Li}(x) + O\left(\frac{x}{\log^\alpha x}\right). \tag{3.16}$$

Moreover,

$$\int_1^\infty \left|\frac{(N_C(x) - xe^{-\gamma(1-1/\alpha)})\log x}{x}\right|^2 \frac{dx}{x} = \infty$$

if and only if $1 < \alpha \leq 3/2$.

Proof. We only need to prove (3.16). This follows from a calculation:

$$\begin{aligned}\Pi_C(x) - \operatorname{Li}(x) &= -\int_2^x \frac{\cos(\log^\alpha u)}{\log u}\, du + O(1) \\ &= -\frac{1}{\alpha}\int_2^x \frac{u}{\log^\alpha u}\, d(\sin(\log^\alpha u)) + O(1)\end{aligned}$$

$$= \frac{1}{\alpha}\int_2^x \frac{\sin(\log^\alpha u)}{\log^\alpha u}\,du - \int_2^x \frac{\sin(\log^\alpha u)}{\log^{\alpha+1} u}\,du + O\biggl(\frac{x}{\log^\alpha x}\biggr)$$
$$= O\biggl(\frac{x}{\log^\alpha x}\biggr),$$

because
$$\int_2^x \frac{\sin(\log^\alpha u)}{\log^\alpha u}\,du \ll \int_{\sqrt{x}}^x \frac{du}{\log^\alpha u} + O(\sqrt{x}) \ll \frac{x}{\log^\alpha x},$$
and similarly the second integral has growth order $\ll x/\log^{\alpha+1} x$. ∎

4. Discrete examples: Proof of Theorem 1.1. We now discretize the family of continuous examples from the previous section. Let $\alpha > 1$. We recall that the functions of the continuous example were

$$\Pi_C(x) = \int_1^x \frac{1-\cos(\log^\alpha u)}{\log u}\,du \quad \text{and} \quad \zeta_C(s) = \frac{e^{-\gamma}e^{-K(s)}}{s-1},$$

where K is the entire function studied in Theorem 3.1. Our set P_α of generalized primes is defined as in the introduction, namely, its rth prime p_r is $\Pi_C^{-1}(r)$.

We shall now establish Theorem 1.1 for P_α. Throughout this section, π, ζ, N, and Π (cf. (2.2)) stand for the number-theoretic functions associated to P_α. We omit the subscript P_α not to overload the notation. As an easy consequence of the definition we obtain $0 \le \Pi_C(x) - \pi(x) \le 1$. By combining this observation with (3.16), we see at once that π satisfies the PNT

$$\pi(x) = \mathrm{Li}(x) + O\biggl(\frac{x}{\log^\alpha x}\biggr), \tag{4.1}$$

where the only requirement is $\alpha > 1$.

This shows that the asymptotic formula (1.9) from Theorem 1.1 holds for $1 < \alpha \le 2$. Naturally, (4.1) implies that our set P_α of generalized primes satisfies a version of Mertens' second theorem, which we state in the next lemma because we shall need it below. The proof is a simple application of integration by parts, the relation $\pi(x) = \Pi_C(x) + O(1)$, and the explicit formula for Π_C; we therefore omit it. Notice that the asymptotic estimate is even valid for $0 < \alpha \le 1$, with the obvious extension of the definition of P_α for these parameters.

LEMMA 4.1. *Let $\alpha > 0$. The generalized prime number system P_α satisfies the Mertens type asymptotic estimate*

$$\sum_{p_r \le x} \frac{1}{p_r} = \log\log x + M + O\biggl(\frac{1}{\log^\alpha x}\biggr)$$

for some constant $M = M_\alpha$.

We now concentrate on showing (1.7) and (1.8). We will prove that they hold with the constant

$$a_\alpha = \exp\Big(-\gamma\Big(1-\frac{1}{\alpha}\Big) + \int_1^\infty x^{-1}\, d(\Pi - \Pi_C)(x)\Big). \tag{4.2}$$

We express the zeta function of this prime number system in terms of ζ_C. We find

$$\zeta(s) = \zeta_C(s)\exp\Big(\int_1^\infty x^{-s}\, d(\Pi - \Pi_C)(x)\Big). \tag{4.3}$$

Note that $\int_1^\infty x^{-s}\, d(\Pi - \Pi_C)(x)$ is analytic on the half-plane $\Re e\, s > 1/2$ because $\Pi(x) - \Pi_C(x) = \Pi(x) - \pi(x) + \pi(x) - \Pi_C(x) = O(x^{1/2}) + O(1)$. Employing Theorem 3.1, we see that, when $\alpha > 1$, ζ is also analytic in $\Re e\, s > 1/2$ except at $s = 1$, and that

$$\mathrm{Res}_{s=1}\zeta(s) = a_\alpha,$$

where a_α is given by (4.2). Hence, the hypothesis (2.7) from Lemma 2.2 is satisfied with a_α for all $\alpha > 1$. We also mention that P_α satisfies the Riemann hypothesis in the form $\zeta(s) \neq 0$ for $\Re e\, s > 1/2$, $s \neq 1$. (This follows from the factorizations (4.3), (3.3), and Theorem 3.1(a).)

As we are interested in the growth behavior of ζ on the line $\Re e\, s = 1$, we will try to control the term $\int_1^\infty x^{-1-it}\, d(\Pi - \Pi_C)(x)$. The following lemma gives a useful bound for it, and this section will mostly be dedicated to its proof.

LEMMA 4.2. *Let $\alpha \geq 1$. The discrete prime number system P_α satisfies*

$$\Big|\Re e \int_1^\infty x^{-1-it}\, d(\Pi - \Pi_C)(x)\Big| = \Big|\int_1^\infty \frac{\cos(t\log x)}{x}\, d(\Pi - \Pi_C)(x)\Big| = O(\log\log|t|).$$

The same bound holds for the imaginary part, and the proof is exactly the same. We first give a Hoheisel–Ingham type estimate for the gaps between consecutive primes from P_α.

LEMMA 4.3. *Let $\alpha \geq 1$. Then $p_{r+1} - p_r < p_r^{2/3}\log p_r$ for sufficiently large r.*

Proof. Set $d = p_r^{2/3}\log p_r$. It suffices to show that for p_r sufficiently large we have

$$\int_{p_r}^{p_r+d} \frac{1-\cos(\log^\alpha u)}{\log u}\, du > 1,$$

which is certainly implied by $\int_{p_r}^{p_r+d}(1-\cos(\log^\alpha u))\, du > 2\log p_r$. If $p_r < u < p_r + d$, then

$$\log^\alpha\Big(u + \frac{d}{4}\Big) - \log^\alpha u \geq \frac{\alpha d\log^{\alpha-1} u}{4(u+d/4)} \geq \frac{d}{5p_r}.$$

Since $\cos t \le 1 - t^2/3$ for $|t| < \pi/4$, this implies that among the four intervals $[p_r, p_r + d/4]$, ..., $[p_r + 3d/4, p_r + d]$ there is one, which we call I, such that

$$\cos(\log^\alpha u) \le 1 - \frac{d^2}{75p_r^2}$$

for all $u \in I$. The integrand in question is non-negative for all u, so we may restrict the range of integration to I and obtain

$$\int_I (1 - \cos(\log^\alpha u))\, du > \frac{d}{4} \cdot \frac{d^2}{75p_r^2} = \frac{\log^3 p_r}{300} > 2\log p_r.$$

Hence our claim follows. ■

Proof of Lemma 4.2. First we are going to change the integration measure:

$$\left|\int_1^\infty \frac{\cos(t\log x)}{x}\, d(\Pi - \Pi_C)(x)\right| \le \left|\int_1^\infty \frac{\cos(t\log x)}{x}\, d(\Pi - \pi)(x)\right| + \left|\int_1^\infty \frac{\cos(t\log x)}{x}\, d(\pi - \Pi_C)(x)\right|.$$

We can estimate the first integral as follows:

$$\left|\int_1^\infty \frac{\cos(t\log x)}{x}\, d(\Pi - \pi)(x)\right| \le \int_1^\infty \frac{1}{x}\, d(\Pi - \pi)(x) < \infty,$$

where we have used the fact that $d(\Pi - \pi)$ is a positive measure and $\Pi(x) - \pi(x) = O(x^{1/2})$. To estimate the second integral, we split it into integrals over $[p_r, p_{r+1})$. Such an interval contributes

$$\left|\int_{[p_r,p_{r+1})} \frac{\cos(t\log x)}{x}\, d(\pi - \Pi_C)(x)\right| = \left|\int_{p_r}^{p_{r+1}} \left(\frac{\cos(t\log p_r)}{p_r} - \frac{\cos(t\log x)}{x}\right) d\Pi_C(x)\right|,$$

since $\int_{p_r}^{p_{r+1}} d\Pi_C(x) = 1$. The above can be further estimated by

$$\begin{aligned}\left|\int_{p_r}^{p_{r+1}} \left(\frac{\cos(t\log p_r)}{p_r} - \frac{\cos(t\log x)}{x}\right) d\Pi_C(x)\right| &\le \int_{p_r}^{p_{r+1}} \left|\frac{\cos(t\log p_r)}{p_r} - \frac{\cos(t\log x)}{p_r}\right| d\Pi_C(x) \\ &\quad + \int_{p_r}^{p_{r+1}} \left|\frac{\cos(t\log x)}{p_r} - \frac{\cos(t\log x)}{x}\right| d\Pi_C(x).\end{aligned}$$

The second of these integrals is bounded by

$$\int_{p_r}^{p_{r+1}} \left(\frac{1}{p_r} - \frac{1}{p_{r+1}}\right) d\Pi_C(x) = \frac{p_{r+1} - p_r}{p_r p_{r+1}} \le \frac{p_r^{2/3+\varepsilon}}{p_r^2},$$

by Lemma 4.3, and after summation on r this gives a contribution which is finite and does not depend on t. We now bound the other integral. By the mean value theorem, we have

$$\int_{p_r}^{p_{r+1}} \left|\frac{\cos(t\log p_r)}{p_r} - \frac{\cos(t\log x)}{p_r}\right| d\Pi_C(x)$$
$$\le \frac{|t\log p_{r+1} - t\log p_r|}{p_r} \le \frac{|t|}{p_r}\log\left(1 + \frac{p_r^{2/3+\varepsilon}}{p_r}\right) \le \frac{|t|}{p_r^{4/3-\varepsilon}} \le \frac{1}{p_r^{5/4}}$$

for $p_r \ge |t|^{13}$ and p_r sufficiently large. As the sum over finitely many small p_r is $O(1)$, the latter condition is inessential. After summation on r we see that these integrals deliver a finite contribution which does not depend on t. Finally, it remains to bound the integrals for $p_r \le |t|^{13}$. Here we apply Corollary 4.1:

$$\sum_{p_r \le |t|^{13}} \int_{p_r}^{p_{r+1}} \left|\frac{\cos(t\log p_r)}{p_r} - \frac{\cos(t\log x)}{p_r}\right| d\Pi_C(x) \le \sum_{p_r \le |t|^{13}} \frac{2}{p_r}$$
$$= O(\log\log|t|). \blacksquare$$

With the same techniques the following bounds can also be established:

$$\int_1^\infty x^{-1-it}\log^n x\, d(\Pi - \Pi_C)(x) = O(\log^n|t|), \qquad n = 1, 2, \ldots. \tag{4.4}$$

We have set the ground for the remaining part of the proof of Theorem 1.1. With the above bounds it is clear that $\zeta(1+it), \zeta'(1+it), \zeta''(1+it), \ldots$ have at most polynomial growth. By Lemma 2.1 the counting function N of this discrete prime number system has Cesàro behavior (1.7) with the constant (4.2) whenever $\alpha > 1$. For Kahane's condition we calculate $(\zeta(1+it)t^{-1})'$ by the Leibniz rule. All the terms involved are L^2 except possibly for

$$\frac{e^{-K(1+it)}K'(1+it)\exp(\int_1^\infty x^{-1-it}\, d(\Pi - \Pi_C)(x))}{t^2}. \tag{4.5}$$

Using the fact that there exists an $m \in \mathbb{N}$ such that ([1])

$$\left|\exp\Big(\int_1^\infty x^{-1-it}\, d(\Pi - \Pi_C)(x)\Big)\right| \gg \frac{1}{\log^m|t|} \qquad \text{for } |t| \gg 1,$$

([1]) The proof of Lemma 4.2 shows that $m = 2$ suffices.

and applying Theorem 3.1, exactly as in the discussion of Section 3, we find that (4.5) is not L^2 when $1 < \alpha < 3/2$. Lemma 2.2 yields (1.8) for $1 < \alpha < 3/2$, and so Theorem 1.1 has been established for P_α.

REMARK 4.4. If $\alpha > 3/2$ then P_α does satisfy Kahane's condition, as also follows from the above argument. In contrast to Proposition 3.2, whether Kahane's condition holds true or false for $P_{3/2}$ is an open question.

5. On the examples of Diamond and Beurling. Proof of Theorem 1.2. In the previous section we extracted a discrete example from a continuous one by applying Diamond's discretization procedure used in [5] to show the sharpness of Beurling's PNT. However, our technique used to prove that our family of discrete examples have the desired properties from Theorem 1.1 was quite different (Diamond's technique is based on operational calculus for the multiplicative convolution of measures).

In this section we show how our method can also be applied to provide an alternative analysis of the Diamond–Beurling examples for the sharpness of the condition $\gamma = 3/2$ in Beurling's theorem. In fact, our technique below leads to a more precise asymptotic formula for the generalized integer counting function of Diamond's example. So, the goal of this section is to prove Theorem 1.2.

We recall that Beurling's example provided in [2] is the Riemann prime counting function

$$\Pi_{C,1}(x) = \int_1^x \frac{1-\cos(\log u)}{\log u}\, du,$$

corresponding to the case $\alpha = 1$ in (1.10). Its associated zeta function is

$$\zeta_{C,1}(s) := \left(1 + \frac{1}{(s-1)^2}\right)^{1/2} = \exp\Big(\int_1^\infty x^{-s}\, d\Pi_{C,1}(x)\Big).$$

Diamond's example P_1 is then the case $\alpha = 1$ of (1.11). We immediately get

$$\Pi_{C,1}(x) = \frac{x}{\log x}\left(1 - \frac{\sqrt{2}}{2}\cos\left(\log x - \frac{\pi}{4}\right)\right) + O\left(\frac{x}{\log^2 x}\right),$$

and since $\pi_{P_1}(x) = \Pi_{C,1}(x) + O(1)$,

$$\pi_{P_1}(x) = \frac{x}{\log x}\left(1 - \frac{\sqrt{2}}{2}\cos\left(\log x - \frac{\pi}{4}\right)\right) + O\left(\frac{x}{\log^2 x}\right), \tag{5.1}$$

whence neither $\Pi_{C,1}$ nor π_{P_1} satisfy the PNT.

To study $N_{C,1}$ and N_{P_1}, we need a number of properties of their zeta functions on $\Re e\, s = 1$. We control $\zeta_{C,1}$ completely. On this line $\zeta_{C,1}$ is analytic except for a simple pole at $s = 1$ with residue 1, and two branch singularities at $s = 1 + i$ and $s = 1 - i$, where $\zeta_{C,1}$ is still continuous. Writing $\zeta_{C,1}(s) = (s-1-i)^{1/2}(s-1+i)^{1/2}(s-1)^{-1}$, we have around $1 \pm i$ the expansions

$$\zeta_{C,1}(s) = (1-i)(s-1-i)^{1/2} + \sum_{k=1}^{\infty} a_k (s-1-i)^{k+1/2}, \qquad |s-1-i| < 1, \tag{5.2}$$

$$\zeta_{C,1}(s) = (1+i)(s-1+i)^{1/2} + \sum_{k=1}^{\infty} \overline{a}_k (s-1+i)^{k+1/2}, \qquad |s-1+i| < 1, \tag{5.3}$$

where explicitly $a_k = (1-i)i^k \sum_{j=0}^{k} \binom{1/2}{j}(-1/2)^j$. On the other hand, the function $\int_1^\infty x^{-s}\, d(\Pi_{P_1} - \Pi_{C,1})(x)$ is analytic on the half-plane $\Re e\, s > 1/2$, where Π_{P_1} is the Riemann generalized prime counting function associated to P_1. So,

$$\zeta_{P_1}(s) = \left(1 + \frac{1}{(s-1)^2}\right)^{1/2} \exp\Big(\int_1^\infty x^{-s}\, d(\Pi_{P_1} - \Pi_{C,1})(x)\Big), \tag{5.4}$$

and we find that ζ_{P_1} shares some analytic properties with $\zeta_{C,1}$, namely, it has a simple pole at $s = 1$ with residue

$$c := \mathrm{Res}_{s=1}\zeta_{P_1}(s) = \exp\Big(\int_1^\infty x^{-1}\, d(\Pi_{P_1} - \Pi_{C,1})(x)\Big) > 0, \tag{5.5}$$

and two branch singularities at $s = 1 \pm i$. We also have the expansions at $s = 1 \pm i$,

$$\zeta_{P_1}(s) = b_0(s-1-i)^{1/2} + \sum_{k=1}^{\infty} b_k (s-1-i)^{k+1/2}, \qquad |s-1-i| < 1/2, \tag{5.6}$$

$$\zeta_{P_1}(s) = \overline{b}_0(s-1+i)^{1/2} + \sum_{k=1}^{\infty} \overline{b}_k (s-1+i)^{k+1/2}, \qquad |s-1+i| < 1/2, \tag{5.7}$$

where $b_0 = (1-i)\exp(\int_1^\infty x^{-1-i}\, d(\Pi_{P_1} - \Pi_{C,1})(x)) \neq 0$ and the rest of the constants b_j come from (5.2) and the Taylor expansion of the function $\exp(\int_0^\infty x^{-s}\, d(\Pi_{P_1} - \Pi_{C,1})(x))$ at $s = 1+i$.

We shall deduce a full asymptotic series for $N_{P_1}(x)$ and $N_{C,1}(x)$ simultaneously from the ensuing general result.

THEOREM 5.1. *Let N be non-decreasing and vanishing for $x \le 1$ with zeta function $\zeta(s) = \int_{1^-}^\infty x^{-s}\, dN(x)$ convergent for $\Re e\, s > 1$. Suppose there are constants $a, r_1, \ldots, r_n \in [0,\infty)$ and $\theta_1, \ldots, \theta_n \in [0, 2\pi)$ such that*

$$G(s) = \zeta(s) - \frac{a}{s-1} - s\sum_{j=1}^{n}\big(r_j e^{\theta_j i}(s-1-i)^{j-1/2} + r_j e^{-\theta_j i}(s-1+i)^{j-1/2}\big) \tag{5.8}$$

admits a C^n-extension to the line $\Re e\, s = 1$ and

$$|G^{(j)}(1+it)| = O(|t|^{\beta+n-j}), \qquad |t| \to \infty,\ j = 0, 1, \ldots, n,$$

for $\beta \ge 0$. Then

$$N(x) = ax + \frac{2x}{\log^{1/2} x} \sum_{j=1}^{n} \frac{r_j \cos(\log x + \theta_j)}{\Gamma(-j+1/2)\log^j x} + O\biggl(\frac{x}{\log^{n/(1+\beta)} x}\biggr), \qquad x \to \infty.$$

Proof. Set

$$T(x) := ae^x + 2e^x \sum_{j=1}^{n} \bigl(r_j \cos(\theta_j)\cos(x) - r_j \sin(\theta_j)\sin(x)\bigr) \frac{x_+^{-j-1/2}}{\Gamma(-j+1/2)}$$

and define $R(x) := e^{-x}(N(e^x) - T(x))$. The tempered distributions $x_+^{-j-1/2}$ are those defined in [7, Sect. 2.4], i.e., the extension to $[0,\infty)$ of the singular functions $x^{-j-1/2}H(x)$ at $x = 0$ via Hadamard finite part regularization. By the classical Wiener–Ikehara theorem we know that $N(x) \sim ax$, and this implies $R(x) = o(1)$. We have to show that $R(x) = O(x^{-n/(1+\beta)})$ as $x \to \infty$. Since $\mathcal{L}\{\cos(x)x_+^{-j-1/2}; s\} = (\Gamma(-j+1/2)/2)[(s-i)^{j-1/2} + (s+i)^{j-1/2}]$ and $\mathcal{L}\{\sin(x)x_+^{-j-1/2}; s\} = (\Gamma(-j+1/2)/(2i))[(s-i)^{j-1/2} - (s+i)^{j-1/2}]$, we have $s\mathcal{L}\{R; s-1\} = G(s) - a$. Letting $\Re e\, s \to 1^+$, we deduce that $\hat{R}(t) = (1+it)^{-1}(G(1+it) - a)$ in $\mathcal{S}'(\mathbb{R})$.

We now derive a useful relation for R. Notice that there exists a B such that $|T'(x)| \le Be^x$ for $x \ge 1$. Applying the mean value theorem to T and using the fact that N is non-decreasing, we obtain

$$R(y) \ge \frac{N(e^x) - T(x)}{e^x} \frac{e^x}{e^y} - B(y-x) \ge \frac{R(x)}{4}$$

if $x \le y \le x + \min\{R(x)/(2B), \log(4/3)\}$ and $R(x) > 0$. Similarly,

$$-R(y) \ge -\frac{R(x)}{2} \qquad \text{if } R(x) < 0 \text{ and } x + \frac{R(x)}{2B} \le y \le x.$$

We now estimate R if $R(x) > 0$. The case $R(x) < 0$ can be treated similarly. We choose an $\varepsilon \le \min\{R(x)/(2B), \log(4/3)\}$ and a test function $\phi \in \mathcal{D}(0,1)$ such that $\phi \ge 0$ and $\int_{-\infty}^{\infty} \phi(y)\, dy = 1$. Using the inequality derived for R and the estimates on the derivatives of G, we obtain

$$\begin{aligned} R(x) &\le \frac{4}{\varepsilon} \int_0^{\varepsilon} R(y+x)\phi\Bigl(\frac{y}{\varepsilon}\Bigr)\, dy = \frac{2}{\pi} \int_{-\infty}^{\infty} \hat{R}(t) e^{ixt} \hat{\phi}(-\varepsilon t)\, dt \\ &= \frac{2}{(ix)^n \pi} \int_{-\infty}^{\infty} e^{ixt} \bigl(\hat{R}(t)\hat{\phi}(-\varepsilon t)\bigr)^{(n)}\, dt \\ &= O(1) x^{-n} \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{\infty} (1+|t|)^{\beta-1+n-j} \varepsilon^{n-j} |\hat{\phi}^{(n-j)}(-\varepsilon t)|\, dt = O(1) x^{-n} \varepsilon^{-\beta}, \end{aligned}$$

where we have used Parseval's relation in the distributional sense. If we choose [2] $\varepsilon = R(x)/(2B)$, we get $R(x) = O(x^{-n/(1+\beta)})$. A similar reasoning gives the result for $R(x) < 0$. This concludes the proof of Theorem 5.1. ∎

We can apply this theorem directly to N_C. Indeed, employing (5.2) and (5.3), one concludes that

$$(5.9)\qquad N_C(x) \sim x - \frac{x\sin(\log x)}{\sqrt{\pi}\log^{3/2} x} + \frac{x}{\log^{5/2} x}\sum_{j=0}^{\infty} c_j \frac{\cos(\log x + \vartheta_j)}{\log^j x}$$
$$= x - \frac{x\sin(\log x)}{\sqrt{\pi}\log^{3/2} x} + O\left(\frac{x}{\log^{5/2} x}\right), \quad x \to \infty,$$

for some constants c_j and ϑ_j.

To show that N_{P_1} has a similar asymptotic series, we need to look at the growth of ζ_{P_1} on $\Re e\, s = 1$. This can be achieved with the aid of Lemma 4.2 and the bounds (4.4). In fact, if we combine those estimates with (5.4), we see at once that $\zeta_{P_1}^{(n)}(1+it) = O(\log^{n+2}|t|)$ for $|t| > 2$. This and the expansions (5.6) and (5.7) allow us to apply Theorem 5.1 to conclude that $N_{P_1}(x)$ has an asymptotic series (1.12) as $x \to \infty$, where the constant c is given by (5.5) and

$$d_0 = \frac{1}{\sqrt{\pi}}\exp\left(\int_1^{\infty} \frac{\cos(\log x)}{x}\, d(\Pi_{P_1} - \Pi_{C,1})(x)\right) > 0,$$

$$\theta_0 = \frac{\pi}{2} - \int_1^{\infty} \frac{\sin(\log x)}{x}\, d(\Pi_{P_1} - \Pi_{C,1})(x).$$

The proof of Theorem 1.2 is now complete.

Remark 5.2. The asymptotic formula $N_{C,1}(x) = x + O(x/\log^{3/2} x)$ was first obtained by Beurling [2] via the Perron inversion formula and contour integration. The asymptotic expansion (5.9) already appears in Diamond's paper [5]. He refined Beurling's computation and also deduced from (5.9) the first order approximation $N_{P_1}(x) = cx + O(x/\log^{3/2} x)$ via convolution techniques. On the other hand, the asymptotic formula (1.12) is new, and our proof, in contrast to those of Diamond and Beurling, avoids any use of information about the zeta functions on $\Re e\, s < 1$.

6. Proof of Theorem 1.3. In this section we amend the arguments from [14] and show that the number system constructed in [14, Sect. 6] does satisfy the requirements from Theorem 1.3. This generalized prime number

[2] Since $R(x) = o(1)$, we may assume that $R(x)/(2B) \le \log(4/3)$ for x large enough.

system is denoted here by P^* and is constructed by removing and doubling suitable blocks of ordinary rational primes. Throughout this section we write $\pi = \pi_{P^*}$ and $N = N_{P^*}$, once again not to overload the notation. For completeness, some parts of this section overlap with [14]. What differs here from [14, Sect. 6] is the crucial [14, Lemma 6.3] and the proof of [14, Prop. 6.2], which require substantially new technical work.

For the construction of our set of generalized primes, we begin by selecting a sequence of integers x_i, where x_1 is chosen so large that for all $x > x_1$ the interval $[x, x + x/\log^{1/3} x]$ contains more than $x/(2\log^{4/3} x)$ ordinary rational prime numbers and $x_{i+1} = \lfloor 2^{\sqrt[4]{x_i}} \rfloor$. One has $i = O(\log\log x_i)$, and thus we may assume that $i \le \log^{1/6} x_i$.

We associate to each x_i four disjoint intervals $I_{i,1}, \dots, I_{i,4}$. We start with $I_{i,2} = [x_i, x_i + x_i/\log^{1/3} x_i]$ and define $I_{i,3}$ as the contiguous interval starting at $x_i + x_i/\log^{1/3} x_i$ which contains as many (ordinary rational) prime numbers as $I_{i,2}$. It is important to notice that $I_{i,2}$ and $I_{i,3}$ each have at least $x_i/(2\log^{4/3} x_i)$ ordinary rational prime numbers. Therefore, the length of $I_{i,3}$ is also at most $O(x_i/\log^{1/3} x_i)$, in view of the classical PNT. We now choose $I_{i,1}$ and $I_{i,4}$ so that they have the properties of the following lemma, whose proof was given in [14].

LEMMA 6.1. *There are intervals $I_{i,1}$ and $I_{i,4}$ such that $I_{i,1}$ has upper bound x_i, $I_{i,4}$ has lower bound equal to the upper bound of $I_{i,3}$, and $I_{i,1}$ and $I_{i,4}$ contain the same number of* (*ordinary rational*) *primes, and*

$$\prod_{\nu=1}^{i} \prod_{p\in I_{\nu,1}\cup I_{\nu,3}} \left(1 - \frac{1}{p}\right)^{(-1)^{\nu+1}} \prod_{p\in I_{\nu,2}\cup I_{\nu,4}} \left(1 - \frac{1}{p}\right)^{(-1)^{\nu}} = 1 + O\left(\frac{1}{x_i}\right).$$

In addition, the lengths of $I_{i,1}$ and $I_{i,4}$ are $O(ix_i/\log^{1/3} x_i)$ and each of them contains $O(ix_i/\log^{4/3} x_i)$ (*ordinary rational*) *primes.*

We define x_k^- to be the least integer in $I_{k,1}$, and x_k^+ the largest integer in $I_{k,4}$. It follows that $x_k/\log^{1/3} x_k \le x_k^+ - x_k^- = O(kx_k/\log^{1/3} x_k)$. Since $k < \log^{1/6} x_k$, we have $x_k^+ < 2x_k$ and $x_k^- > 2^{-1}x_k$, for sufficiently large k. We may thus assume that these properties hold for all k.

The sequence $P^* = \{p_\nu\}_{\nu=1}^{\infty}$ of generalized primes is then constructed as follows. We use the term 'prime number' for the ordinary rational primes and 'prime element' for the elements of P^*. Take one prime element p for each prime number p which is not in any of the intervals $I_{i,j}$. If i is even, take no prime elements in $I_{i,2} \cup I_{i,4}$ and two prime elements p for all prime numbers p which are in one of the intervals $I_{i,1}, I_{i,3}$. If i is odd, take no prime elements in $I_{i,1} \cup I_{i,3}$ and two prime elements for all prime numbers p which belong to one of the intervals $I_{i,2}, I_{i,4}$. As previously mentioned, we simplify the notation and write $\pi(x) = \pi_{P^*}(x)$ and $N(x) = N_{P^*}(x)$ for the counting functions of P^* and its associated generalized integer counting function. The rest of the section is dedicated to proving that N and π have the properties stated in Proposition 1.3. We actually show something stronger:

PROPOSITION 6.2. *We have $N(x) = x + \Omega(x/\log^{4/3} x)$; however, for an arbitrary $\varepsilon > 0$,*

$$N(x) = x + O\left(\frac{x}{\log^{5/3-\varepsilon} x}\right) \quad (\mathrm{C}, 1),$$

i.e., its first order Cesàro mean $\bar{N}$ has asymptotics

$$\bar{N}(x) := \int_1^x \frac{N(t)}{t}\,dt = x + O\left(\frac{x}{\log^{5/3-\varepsilon} x}\right). \tag{6.1}$$

For this system,

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x \log\log x}{\log^{4/3} x}\right).$$

The asymptotic bound for the prime counting function π of our generalized prime set P^* follows immediately from the definition of P^* and the classical prime number theorem. The non-trivial part in the proof of Proposition 6.2 is to establish the asymptotic formulas for N and $\bar{N}$.

To progress further, we introduce a family of generalized prime number systems approximating P^*. We define the generalized prime set P_k^* by applying the same construction used for P^*, but only taking the intervals $I_{i,j}$ with $i \le k$ into account; furthermore, we write $N_k(x) = N_{P_k^*}(x)$.

We first try to control the growth $N_k(x)$ on suitable large intervals. For this we will use the theory of integers without large prime factors [8]. This theory studies the function

$$\Psi(x, y) = \#\{1 \le n \le x : P(n) \le y\},$$

where $P(n)$ denotes the largest prime factor of n, with the convention $P(1) = 1$. This function is well studied [8] and we will only use the simple estimate [8, (1.4)]

$$\Psi(x, y) \ll x e^{-\log x/(2\log y)} \log y. \tag{6.2}$$

A weaker version of the following lemma was stated in [14], but the proof given there contains a mistake. Furthermore, the range of validity for the estimates in [14, Lemma 6.3] appears to be too weak to lead to a proof of the Cesàro estimate (6.1). We correct the error in the proof and show the assertions in a broader range.

LEMMA 6.3. *Let $\eta > 1$. If* $\exp(\log^\eta x_k) \le x < \exp(x_k^{3/5})$, *then*

$$N_k(x) = x + O\left(\frac{x}{\log^{5/3} x}\right) \tag{6.3}$$

and

$$\bar{N}_k(x) := \int_1^x \frac{N_k(t)}{t}\, dt = x + O\left(\frac{x}{\log^{5/3} x}\right), \tag{6.4}$$

for all sufficiently large k.

Proof. Let $f(n)$ be the number of representations of n as a finite product of elements of P_k^*. Note that $N_k(x) = \sum_{n\le x} f(n)$. On setting $f(1) = 1$, the function $f(n)$ becomes multiplicative and we have

$$f(p^\alpha) = \begin{cases} \alpha+1 & \text{if } \exists 2i \le k : p \in I_{2i,1} \cup I_{2i,3}, \\ 0 & \text{if } \exists 2i \le k : p \in I_{2i,2} \cup I_{2i,4}, \\ 0 & \text{if } \exists 2i+1 \le k : p \in I_{2i+1,1} \cup I_{2i+1,3}, \\ \alpha+1 & \text{if } \exists 2i+1 \le k : p \in I_{2i+1,2} \cup I_{2i+1,4}, \\ 1 & \text{otherwise.} \end{cases}$$

We also introduce the multiplicative function $g(n) = \sum_{d\mid n} \mu(n/d) f(d)$. The values of g at powers of prime numbers are easily seen to be

$$g(p^\alpha) = \begin{cases} 1 & \text{if } f(p) = 2, \\ -1 & \text{if } f(p) = 0 \text{ and } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{H}_k$ the set of all integers all of whose prime divisors are in $\bigcup_{i\le k} I_{i,j}$, and for each integer n, let $n_{\mathcal{H}_k}$ be the largest divisor of n belonging to $\mathcal{H}_k$. We have

$$\begin{aligned} N_k(x) &= \sum_{m\in\mathcal{H}_k} \sum_{\substack{n\le x \\ n_{\mathcal{H}_k}=m}} f(m) = \sum_{m\in\mathcal{H}_k} \sum_{\substack{n\le x \\ m\mid n}} g(m) = \sum_{m\in\mathcal{H}_k} g(m) \left[\frac{x}{m}\right] \\ &= x \sum_{m\in\mathcal{H}_k} \frac{g(m)}{m} - x \sum_{\substack{m\in\mathcal{H}_k \\ d>x}} \frac{g(m)}{m} + O(|\mathcal{H}_k \cap [1,x]|), \end{aligned}$$

and since

$$\sum_{m\in\mathcal{H}_k} \frac{g(m)}{m} = \prod_{i=1}^{k} \prod_{p\in I_{i,1}\cup I_{i,3}} \left(1-\frac{1}{p}\right)^{(-1)^{i+1}} \prod_{p\in I_{i,2}\cup I_{i,4}} \left(1-\frac{1}{p}\right)^{(-1)^{i}},$$

we obtain

$$N_k(x) = x + O\left(\frac{x}{x_k}\right) + O(|\mathcal{H}_k \cap [1,x]|) - x \sum_{\substack{m\in\mathcal{H}_k \\ d>x}} \frac{g(m)}{m}.$$

The first error term is negligible because $x < \exp(x_k^{3/5})$. To estimate the remaining two terms we use the function Ψ. Any element of $\mathcal{H}_k$ has only prime divisors below $2x_k$. Using this observation and employing the estimate (6.2), we find that

$$\begin{aligned}|\mathcal{H}_k \cap [1,x]| &\ll x^{1-\frac{1}{2\log(2x_k)}} \log x_k \\ &\ll \frac{x}{\log^{5/3} x}\biggl(\frac{(x_k \log x_k)^{2\log(2x_k)}}{x}\biggr)^{\frac{1}{2\log(2x_k)}} \\ &\ll \frac{x}{\log^{5/3} x} \quad \text{for } \exp(8\log^2 x_k) \le x < \exp(x_k^{3/5}).\end{aligned}$$

Similarly, we can extend the bound to a broader region:

$$\begin{aligned}|\mathcal{H}_k \cap [1,x]| &\ll x^{1-\frac{1}{2\log(2x_k)}} \log x_k \\ &\ll \frac{x}{\log^{5/3} x}\biggl(\frac{(\log^{13/3} x_k)^{2\log(2x_k)}}{x}\biggr)^{\frac{1}{2\log(2x_k)}} \\ &\ll \frac{x}{\log^{5/3} x} \quad \text{for } \exp(\log^{\eta} x_k) \le x < \exp(8\log^2 x_k),\end{aligned}$$

which is valid for all sufficiently large k. For the other term,

$$\begin{aligned}\biggl|\sum_{\substack{d\in\mathcal{H}_k\\ d>x}} \frac{g(d)}{d}\biggr| &\le \sum_{\substack{P(d)\le 2x_k\\ d>x}} \frac{1}{d} = \int_{x^-}^{\infty} \frac{1}{t}\, d\Psi(t,2x_k) \\ &= \lim_{t\to\infty} \frac{\Psi(t,2x_k)}{t} - \frac{\Psi(x,2x_k)}{x} + \int_x^{\infty} \frac{\Psi(t,2x_k)}{t^2}\, dt.\end{aligned}$$

The limit term equals 0 because it is $O(t^{-1/2\log(2x_k)} \log 2x_k)$ by (6.2). The second term is negligible because it is a negative term in a positive result. It remains to bound the integral:

$$\begin{aligned}\int_x^{\infty} \frac{\Psi(t,2x_k)}{t^2}\, dt &\ll \int_x^{\infty} t^{-1-\frac{1}{2\log(2x_k)}} \log 2x_k \, dt \ll x^{-\frac{1}{2\log(2x_k)}} \log^2 x_k \\ &\ll \frac{1}{\log^{5/3} x} \quad \text{because } \exp(\log^{\eta} x_k) \le x \le \exp(x_k^{3/5}),\end{aligned}$$

where the last inequality is deduced in the same way as above. This concludes the proof of (6.3).

We now address the Cesàro estimate. Using the estimates already found for $N_k(x)$, we find

$$\overline{N}_k(x) - x = O\biggl(\frac{x}{\log^{5/3} x}\biggr) + O\biggl(\int_1^x \frac{|\mathcal{H}_k \cap [1,t]|}{t}\, dt\biggr) + O\biggl(\int_1^x\int_t^{\infty} \frac{\Psi(s,2x_k)}{s^2}\, ds\, dt\biggr).$$

We bound the double integral in the given range; the other term can be treated similarly. We obtain

$$\int_1^x \int_t^\infty \frac{\Psi(s, 2x_k)}{s^2}\, ds\, dt \ll \int_1^x \int_t^\infty s^{-1-\frac{1}{2\log(2x_k)}} \log(2x_k)\, ds\, dt$$
$$= \int_1^x t^{-\frac{1}{2\log(2x_k)}} 2\log^2(2x_k)\, dt$$
$$\ll x^{1-\frac{1}{2\log(2x_k)}} \log^3(2x_k) = O\Big(\frac{x}{\log^{5/3} x}\Big),$$

where again the last step is shown by considering the regions $\exp(8\log^2 x_k) \le x < \exp(x_k^{3/5})$ and $\exp(\log^\eta x_k) \le x < \exp(8\log^2 x_k)$ separately. ∎

Proof of Proposition 6.2. We choose η smaller than $\frac{5/3}{5/3-\varepsilon}$ in Lemma 6.3. The Ω-estimate for $N(x)$ follows almost immediately from (6.3). For $x < x_{k+1}^+$, we have $N(x) = N_k(x)$ with the exception of the missing and doubled primes from $[x_{k+1}^-, x_{k+1}^+]$. Observe that, because $x_{k+1} = \lfloor \exp(x_k^{1/4} \log 2)\rfloor$,

$$[x_{k+1}^-, x_{k+1}^+] \subset [\exp(\log^\eta x_k), \exp(x_k^{3/5})).$$

Since we changed more than $x/(4\log^{4/3} x)$ primes when x is the upper bound of either the interval $I_{k+1,1}$ or $I_{k+1,2}$, we deduce from Lemma 6.3 that $|N(x) - x|$ becomes as large as $x/(8\log^{4/3} x)$ infinitely often as $x \to \infty$.

It remains to show (6.1). We bound the Cesàro means of N in the range $x_k^- \le x < x_{k+1}^-$. We start by observing that $N(x) = N_k(x)$ within this range, so (6.4) gives (6.1) for $\exp(\log^\eta x_k) \le x < x_{k+1}^-$. Assume now that

$$x_k^- \le x < \exp(\log^\eta x_k).$$

Lemma 6.3 implies that

$$\bar{N}_{k-1}(x) = \int_1^x \frac{N_{k-1}(t)}{t}\, dt = x + O\Big(\frac{x}{\log^{5/3} x}\Big),$$

because, by construction of the sequence, the interval $[x_k^-, \exp(\log^\eta x_k)]$ is contained in $[\exp(\log^\eta x_{k-1}), \exp(x_{k-1}^{3/5})]$. Therefore, it suffices to prove that

$$\bar{N}_k(x) - \bar{N}_{k-1}(x) = \int_{x_k^-}^x \frac{N_k(t) - N_{k-1}(t)}{t}\, dt \tag{6.5}$$

has growth order $O(x/\log^{5/3-\varepsilon} x)$ in the interval $[x_k^-, \exp(\log^\eta x_k)]$. Note that only the intervals $\nu \cdot (I_{k,1} \cup \cdots \cup I_{k,4})$ contribute to the integral (6.5) with ν a generalized integer from the number system generated by P_k^*. Only

$\nu \leq x/x_k^-$ deliver a contribution. There are at most $O(x/x_k^-) = O(x/x_k)$ such integers. The contribution of one such generalized integer is

$$O\left(\frac{kx_k}{\log^{4/3} x_k}\right)\int_{\nu x_k^-}^{\nu x_k^+} \frac{dt}{t} = O\left(\frac{k^2 x_k}{\log^{5/3} x_k}\right),$$

where we have used the fact that the length of $I_{k,i}$ is $O(kx_k \log^{-1/3} x_k)$, as derived in Lemma 6.1. In total, the integral is bounded by

$$O\left(\frac{x}{x_k}\right)O\left(\frac{k^2 x_k}{\log^{5/3} x_k}\right) = O\left(\frac{k^2 x}{\log^{5/(3\eta)} x}\right) = O\left(\frac{x}{\log^{5/3-\varepsilon} x}\right). \blacksquare$$

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