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Can we define Taylor polynomials on algebraic curves?

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Abstract. We study the problem of finding the correct definition of a Taylor polynomial of degree d at a point $\mathbf{a}$ for a function defined on an irreducible algebraic curve V in $\mathbb{C}^2$. We show that a satisfactory definition can be given if and only if the point $\mathbf{a}$ is d-Taylorian, which holds for all but finitely many points of V. We provide an application to the study of the limit of certain Lagrangian interpolation operators when points coalesce.

1. Introduction. There are different useful ways of looking at a Taylor polynomial

$$\mathbf{T}_a^d(f)(x) = \sum_{j=0}^{d} \frac{f^{(j)}(a)}{j!}(x-a)^j \tag{1.1}$$

of a univariate function f. Most fundamental is the approximation point of view: $\mathbf{T}_a^d(f)(x)$ is the *local polynomial approximant* of f around a of degree d in the sense that, when well defined,

$$f(x) - \mathbf{T}_a^d(f)(x) = o(x-a)^d \quad (x \to a).$$

A second point of view, which we may call the *numerical analysis point of view*, is to consider $\mathbf{T}_a^d(f)(x)$, for a sufficiently smooth function f, as the limit of Lagrange interpolation polynomials of f when the interpolation nodes tend to a, that is,

$$\mathbf{T}_a^d(f)(x) = \lim_{A \to a} \mathbf{L}[A; f](x), \tag{1.2}$$

where $A = \{a_0, \dots, a_d\}$, $a_i \neq a_k$, $A \to a$ means that $\max_{i=0,\dots,d} |a_i - a| \to 0$ and $\mathbf{L}[A; f](x)$ is the Lagrange interpolation polynomial of f at points of A.

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This result, which is crucial in computational mathematics since it provides the simplest way of computing (approximate) Taylor polynomials, is a well known consequence of the Newton formula for Lagrange interpolation and is essentially equivalent to the fact that divided differences furnish suitable approximations of derivatives.

A third approach, which, at first, may look somewhat abstract, provides a natural link between the first two points of view and will prove useful what follows. We give an informal statement which will be clarified later: the map $f \mapsto \mathbf{T}^d_a(f)$ defined, say, on the space of holomorphic functions on a neighbourhood of a, is a *polynomial projector* (a linear map which coincides with the identity on its range) whose *interpolation conditions are supported on* $\{a\}$ and whose *kernel is an ideal.*

In the multivariate seting, *in general*, (1.2) no longer holds true,

$$\text{in } \mathbb{C}^n,\ n>1,\ \textit{in general}, \quad \mathbf{T}^d_{\mathbf{a}}(f)(x) \neq \lim_{A\to\mathbf{a}} \mathbf{L}[A;f](x), \tag{1.3}$$

which should be understood as follows. First, we can find sequences of unisolvent arrays (i.e. sets of points for which multivariate Lagrange interpolation is well defined) whose points tend to $\mathbf{a}$ and for which the limit on the right hand side of (1.3) does not exist. Moreover, for a given function f, the limit in (1.3) may exist and be different from the Taylor polynomial of f. Such examples can be found in [1]. Of course, there are many important cases, that is, particular classes of arrays A, for which equality does hold in (1.3). One can even derive necessary and sufficient conditions on A ensuring that $\mathbf{L}[A;f](x)$ approaches $\mathbf{T}^d_{\mathbf{a}}(f)(x)$ (for regular functions f). Such conditions can be found in [1] together with references to earlier results. We also refer to [6] for a study in the case where interpolation is done at natural lattices. However, and this is the important point for the present paper, in contrast with the univariate case, equality in (1.3) requires specific assumptions on the interpolation points A which are much stronger than mere convergence to a single limit point.

Here, we wish to analyze the numerical analysis interpretation of Taylor polynomials in the case of (irreducible) algebraic curves in $\mathbb{C}^2$ which is, in some sense, intermediary between the univariate and multivariate cases. On such a curve V, there is a natural space $\mathscr{P}_d(V)$ of polynomials of degree at most d (see (1.6)), and one may consider Lagrange interpolation, thus obtaining projectors, say $\mathbf{L}_V[A,\cdot](\mathbf{x})$ for suitably chosen sets of interpolation points A on V. A natural definition of a Taylor polynomial $\mathbf{T}^d_{V,\mathbf{a}}(f)(\mathbf{x})$ on the curve V would be

$$\mathbf{T}^d_{V,\mathbf{a}}(f)(\mathbf{x}) := \lim_{A\to\mathbf{a}} \mathbf{L}_V[A;f](\mathbf{x}), \tag{1.4}$$

provided, of course, that the limit exists. This is the theoretical motivation for our study of the limit of Lagrangian operators on curves. Difficulties however arise immediately. First, in general, not every set A containing the correct number of points, i.e. dim $\mathscr{P}_d(V)$ points, enables interpolation so that the condition $A \to \mathbf{a}$ should be clarified. Further, more seriously, it does not take much time to encounter an example for which the limit does not exist:

EXAMPLE 1.1. Let $V = \{y = x^3\} \subset \mathbb{C}^2$, $\mathbf{a} = (0,0)$ and $f(x,y) = x^2$. If $A = \{(x_0, x_0^3), (x_1, x_1^3), (x_2, x_2^3)\}$ with $x_0 = t$, $x_1 = 2t$, $x_2 = -3t + t^2$, $t \neq 0$, then the points of A tend to $(0,0) \in V$ as $t \to 0$ and $\mathbf{L}_V[A; \cdot]$ is well defined, but a calculation shows that

$$\mathbf{L}_V[A; f](x,y) = -2(-3t + t^2) + (-7 + 3t)x + y/t^2, \tag{1.5}$$

which has no limit as $t \to 0$. To check (1.5), we just need to observe that the right hand side is a polynomial of degree 1 on V which takes on the same value as f at each point of A.

At this point, one might conjecture that, just as in the multivariate case, the existence of the limit in (1.4) will require stronger assumptions on the way the points of A tend to $\mathbf{a}$. We will show that this is not true. Our main result says that the supplementary assumption is an assumption on the limit point $\mathbf{a} \in V$ rather than on the sets of interpolation points A. This is summarized as

$$\left.\begin{array}{c} A \to \mathbf{a} \\ \text{a reasonable assumption on } \mathbf{a} \end{array}\right\} \Rightarrow \lim_{A \to \mathbf{a}} \mathbf{L}_V[A; f](\mathbf{x}) \text{ exists.}$$

It turns out that the required property on $\mathbf{a}$ was introduced by Bos and Calvi [5] from a different perspective. The limit point $\mathbf{a}$ has to be d-Taylorian where d is the degree of interpolation. The definition of a d-Taylorian point is recalled below. Here, let us just point out that all but finitely many points on V are d-Taylorian. We will also show that the limit $\mathbf{T}^d_{V,\mathbf{a}}(f)$ is a natural Hermitian projector introduced in [4], thus showing that such a projector provides the correct definition of a Taylor polynomial on an algebraic curve. In the last section, we will apply our result to the study of the limit of certain (ordinary) bivariate Lagrange interpolation polynomials. We restrict ourselves to the study of the complex case in order to stick to the formalism introduced in [5]. However, with suitable translations, everything remains true in the real case.

Let us point out that the restriction to irreducible curves is natural. For, clearly, the local behaviour of a function f around a point $\mathbf{a}$ on a curve V is related to the connected component of V which contains $\mathbf{a}$, and is independent of the possible other components.

It is likely that some of the results in this paper can be extended to a more general setting, in particular to algebraic curves in $\mathbb{C}^N$, $N > 2$, but certainly with a less elementary treatment.

Notation. Bold letters are used to denote points in $\mathbb{C}^2$, thus, for instance, $\mathbf{x} = (x_1, x_2) \in \mathbb{C}^2$ where $x_i \in \mathbb{C}$.

We let $\mathscr{P} = \mathscr{P}(\mathbb{C}^2)$ denote the space of all polynomials on $\mathbb{C}^2$ and $\mathscr{P}_d = \mathscr{P}_d(\mathbb{C}^2)$ its subspace of polynomials of degree at most d, spanned by the $m_d = (d+2)(d+1)/2$ monomials $\mathbf{x}^\alpha$ with $|\alpha| \le d$ where $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$ for $\mathbf{x} = (x_1, x_2)$ and $\alpha = (\alpha_1, \alpha_2)$. We will also consider the spaces

$$\mathscr{P}_d(V) = \{p_{|V} : p \in \mathscr{P}_d\} \tag{1.6}$$

when V is an algebraic curve of the form

$$V = V(q) = \{\mathbf{x} \in \mathbb{C}^2 : q(\mathbf{x}) = 0\} \quad \text{or, briefly,} \quad V = \{q = 0\}, \tag{1.7}$$

where q is an irreducible polynomial of positive degree s. The dimension $m_d(V)$ of $\mathscr{P}_d(V)$ is $m_d - m_{d-s}$ where $m_{d-m} = 0$ for $d < m$. The spaces $\mathscr{P}_d$ and $\mathscr{P}_d(V)$ are endowed with their usual topology of finite-dimensional normed space. Unless otherwise specified, the degree d will be fixed throughout the paper.

If X is open (or compact) in $\mathbb{C}$ or in a complex manifold, then $\mathscr{A}(X)$ denotes the space of holomorphic functions on X (or on a neighbourhood of X). When $X = \{a\}$ we write $\mathscr{A}(a)$ instead of $\mathscr{A}(\{a\})$. These spaces are endowed with their usual topology. The space of continuous linear forms on $\mathscr{A}(a)$, often called analytic functionals, is denoted by $\mathscr{A}'(a)$.

The open disk of centre a and radius ρ is $D(a, \rho)$ while $B(\mathbf{a}, \rho)$ denotes the open euclidean ball of centre $\mathbf{a}$ and radius ρ in $\mathbb{C}^2$.

2. Lagrange interpolation and Vandermonde determinants. We recall basic facts of interpolation theory and prove some simple but crucial properties of Vandermonde determinants.

2.1. Lagrange interpolation. A subset $X = \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ of m distinct points in a given set χ is said to be *unisolvent* for a (vector) space F of functions defined on χ if, for every function f defined on X, there exists a unique $P \in \mathrm{F}$ such that $P(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in X$. This function is called the (generalized) *Lagrange interpolation function* of f at X and is denoted by $\mathbf{L}[X; f]$.

Generalized Vandermonde determinants are fundamental in interpolation theory. Given a basis $\mathscr{B} = \{p_1, \dots, p_m\}$ for F, the equations

$$\mathrm{V}(\mathscr{B}; X) = (p_i(\mathbf{x}_j))_{1 \le i,j \le m} \quad \text{and} \quad \mathrm{VDM}(\mathscr{B}; X) = \det \mathrm{V}(\mathscr{B}; X) \tag{2.1}$$

respectively define the *Vandermonde matrix* and the *Vandermonde determinant* corresponding to $\mathscr{B}$ and X. Note that here j is used for the row index of the matrix. The values of $\mathrm{VDM}(\mathscr{B}; X)$ for two different bases only differ by the determinant of the basis change matrix, i.e. $\mathrm{VDM}(\mathscr{B}_1; X) = \det(M)\mathrm{VDM}(\mathscr{B}_2; X)$ where M is the transition matrix giving B_2 in terms of B_1. We will always tacitly order the elements in a matrix according to their index. A set X is unisolvent if and only if $\mathrm{VDM}(\mathscr{B}; X) \neq 0$. We may use Vandermonde determinants to derive the following dual expressions for the Lagrange interpolation function:

$$\mathbf{L}[X; f](\mathbf{x}) = \sum_{i=1}^{m} f(\mathbf{x}_i) \frac{\mathrm{VDM}(\mathscr{B}; X[\mathbf{x}_i \leftarrow \mathbf{x}])}{\mathrm{VDM}(\mathscr{B}; X)} = \sum_{i=1}^{m} \frac{\mathrm{VDM}(\mathscr{B}[f_i \leftarrow f]; X)}{\mathrm{VDM}(\mathscr{B}; X)} f_i(\mathbf{x}) \tag{2.2}$$

where $X[\mathbf{x}_i \leftarrow \mathbf{x}]$ means that we substitute $\mathbf{x}$ for $\mathbf{x}_i$ in X, and likewise for $\mathscr{B}[f_i \leftarrow f]$. We will use interpolation when $\mathrm{F} = \mathscr{P}_d$ or $\mathrm{F} = \mathscr{P}_d(V)$. In the latter case, we will write $\mathbf{L}_V[X; f]$ rather than $\mathbf{L}[X; f]$.

2.2. Limits of Vandermonde determinants. We use classical facts on divided differences (see e.g [2] or [8, Chapter 6] for a classical treatment). Let us just recall that if $T = \{t_0, \ldots, t_d\}$ is a set of $d+1$ not necessarily distinct points in $\mathbb{C}$ and h is a function defined on T and holomorphic on a neighbourhood of any repeated point, then the divided difference $h[t_0, \ldots, t_d]$ of h at T is the leading coefficient of the Lagrange–Hermite interpolation polynomial of h at T. When the points are pairwise distinct, the Lagrange interpolation formula implies

$$h[t_0, \ldots, t_d] = \sum_{j=0}^{d} h(t_j) \prod_{\substack{i=0 \\ i \neq j}}^{d} \frac{1}{t_j - t_i}, \tag{2.3}$$

whereas when all points coincide, $h[t, \ldots, t] = f^{(d)}(t)/d!$, so that since divided differences of a holomorphic function are continuous (even holomorphic) functions of the nodes, we have

$$\lim_{t_0, \ldots, t_d \to t} h[t_0, \ldots, t_d] = \frac{h^{(d)}(t)}{d!}, \qquad f \in \mathscr{A}(t). \tag{2.4}$$

Lemma 2.1. *Let $\mathscr{F} = \{f_0, \ldots, f_d\}$ be a family of functions defined on $T = \{t_0, \ldots, t_d\} \subset \mathbb{C}$, where the t_j are pairwise distinct. Then*

$$\mathrm{VDM}(\mathscr{F}; T) = \begin{vmatrix} f_0[t_0] & f_1[t_0] & \cdots & f_d[t_0] \\ f_0[t_0, t_1] & f_1[t_0, t_1] & \cdots & f_d[t_0, t_1] \\ \vdots & \vdots & \ddots & \vdots \\ f_0[t_0, \ldots, t_d] & f_1[t_0, \ldots, t_d] & \cdots & f_d[t_0, \ldots, t_d] \end{vmatrix} \prod_{0 \le i < j \le d} (t_j - t_i). \tag{2.5}$$

Proof. Equation (2.3) can be viewed as a matrix product. Namely, if $M = (M_{ij})_{1\le i,j\le m}$ with $M_{ij} = f_j[t_0,\dots,t_i]$ and $V = \mathrm{V}(\mathscr{F};T)$ so that $V_{kj} = f_j(t_k)$, then

$$M_{ij} = \sum_{k=0}^{d} L_{ik}V_{kj} \quad \text{or} \quad M = L\cdot V, \tag{2.6}$$

where L is the lower triangular matrix defined by

$$L_{ij} = \begin{cases} \prod_{s=0,\, s\neq j}^{i} \frac{1}{t_j - t_s}, & i \le j, \\ 0, & i > j, \end{cases} \qquad 0 \le i,j \le d.$$

Here an empty product is taken to be 1, so that $L_{00} = 1$. We obtain the result by taking the determinant on both sides of (2.6). ∎

Now, we may write (2.5) as

$$\mathrm{VDM}(\mathscr{F};T) = \mathrm{VDM}(T)\ \cdot\ \Phi(\mathscr{F};t_0,\dots,t_d), \tag{2.7}$$

where $\mathrm{VDM}(T)$ is the classical Vandermonde determinant and Φ is the function defined by the determinant on the right hand side. Observe that if Ω is a simply connected domain containing the points t_j, then Φ is holomorphic on Ω^{d+1} when the functions f_i are holomorphic on Ω.

In view of (2.5), for functions f_i in $\mathscr{A}(t)$, Φ is holomorphic in a neighbourhood of $(t,\dots,t)$ and the value of Φ at $(t,\dots,t)$ is given by the determinant of the matrix of $f_i^{(j)}(t)/j!$, which is the Wronskian of the family $\mathscr{F}$ at the point t,

$$\Phi(\mathscr{F};t,\dots,t) = W(\mathscr{F},t). \tag{2.8}$$

Note that ordinary Wronskians are defined without the normalization by $j!$ for the jth derivatives, but this difference is irrelevant in this paper. Now, if f is another function of the same regularity, we denote by $\mathscr{F}[f_k \leftarrow f]$ the family $\mathscr{F}$ in which we substitute f for f_k. We have

$$\lim_{\substack{t_0,\dots,t_d\to t\\ t_i\neq t_j}} \frac{\mathrm{VDM}(\mathscr{F}[f_k\leftarrow f];T)}{\mathrm{VDM}(\mathscr{F};T)} = \frac{W(\mathscr{F}[f_k\leftarrow f],t)}{W(\mathscr{F},t)}, \qquad k = 0,\dots,d, \tag{2.9}$$

provided that the denominator does not vanish. Indeed, using (2.5) both for $\mathrm{VDM}(\mathscr{F};T)$ and $\mathrm{VDM}(\mathscr{F}[f_k\leftarrow f];T)$, the common factor $\mathrm{VDM}(T)$ vanishes, and (2.8) for $\mathscr{F}$ and $\mathscr{F}[f_k\leftarrow f]$ then yields (2.9).

In the application given in Section 6.3 below, we will need the fact that the convergence in (2.9) holds uniformly in f. This is easily seen as follows. In fact, if $t_j^n \to t$ as $n\to\infty$ for $j = 0,\dots,d$, then for n large enough all the t_j^n will be in the interior of a closed disk $\overline{D}(t,\rho)\subset\Omega$. Assume that g_n

converges to g in $\mathscr{A}(\Omega)$. We have

$$g_n[t_0^n, \dots, t_j^n] = \frac{1}{2i\pi} \int_\gamma \frac{g_n(z)}{(z - t_0^n) \cdots (z - t_j^n)}\, dz,$$

where γ is the circle of centre t and radius ρ with positive orientation. The uniform convergence of g_n to g on the circle implies

$$\lim_{n\to\infty} g_n[t_0^n, \dots, t_j^n] = \frac{1}{2i\pi} \int_\gamma \frac{g(z)}{(z-t)^{j+1}} dz = \frac{g^{(j)}(t)}{j!}, \qquad j = 0, \dots, d.$$

This readily yields the following lemma.

Lemma 2.2. *Let $g_n \to g$ in $\mathscr{A}(\Omega)$. If, for each n, $T_n = \{t_0^n, \dots, t_d^n\}$ is a set of $d+1$ pairwise distinct elements in Ω with $t_i^n \to t \in \Omega$ for $i = 0, \dots, d$ and $W(\mathscr{F}, t) \neq 0$, then*

$$\lim_{n\to\infty} \frac{\mathrm{VDM}(\mathscr{F}[f_k \leftarrow g_n]; T_n)}{\mathrm{VDM}(\mathscr{F}; T_n)} = \frac{W(\mathscr{F}[f_k \leftarrow g], t)}{W(\mathscr{F}, t)}, \quad k = 0, \dots, d. \tag{2.10}$$

3. Algebraic curves around Taylorian points

3.1. d-Taylorian points. Let q be an irreducible polynomial in $\mathbb{C}^2$. We denote by $V = V(q)$ the algebraic curve generated by q (see (1.7)). A *regular* (or *smooth*) point is a point on V for which the gradient of q does not vanish. The set of regular points, denoted by V^0, forms a complex manifold of dimension 1 [11, Th. 2 and Th. I, §24]. The implicit function theorem gives a local parametrization $z_2 = \phi(z_1)$ when $\partial_2 q(\mathbf{a}) \neq 0$ (or $z_1 = \phi(z_2)$ when $\partial_1 q(\mathbf{a}) \neq 0$) where ϕ is holomorphic in a neighbourhood of a_1 (or of a_2) with $\mathbf{a} = (a_1, a_2)$. More generally, a (local) parametrization of V at $\mathbf{a} \in V^0$ is any local chart $\mathscr{L} = (a, U, R)$ at $\mathbf{a}$ of V^0, where $a \in \mathbb{C}$, U is an open neighbourhood of a and $R = (R_1, R_2) : U \to \mathbb{C}^2$ is such that $R(a) = \mathbf{a}$ and $R(U) \subset V^0$. Every such R coincides with ϕ_1 (or ϕ_2) composed with a holomorphic diffeomorphism from a neighbourhood of a onto a neighbourhood of a_1 (or of a_2). In particular, the map R defines a homeomorphism from U onto $R(U) \subset V$.

Given such a local parametrization, we consider the space

$$\mathscr{P}_d^{\mathscr{L}} := \mathscr{P}_d \circ R = \mathscr{P}_d(V) \circ R.$$

This is a space of holomorphic functions defined on U. The least part $f_\downarrow$ of an element $f \in \mathscr{P}_d^{\mathscr{L}}$ at a is the first nonzero element in the series expansion of f at a, i.e., if $f(t) = c_k(t-a)^k + \sum_{j>k} c_j(t-a)^j$ with $c_k \neq 0$ then $f_\downarrow(t) = c_k(t-a)^k$. We set

$$\mathscr{P}_{d\downarrow}^{\mathscr{L}} = \mathrm{span}\{f_\downarrow : f \in \mathscr{P}_d^{\mathscr{L}}\}.$$

Any element $(t-a)^k$ in $\mathscr{P}^{\mathscr{L}}_{d\downarrow}$ is called a *least a-monomial* (for $\mathscr{L}$). A fundamental object is

$$\mathbf{pow}(\mathbf{a}, d) = \{k \in \mathbb{N} : (t-a)^k \text{ is a least } a\text{-monomial}\}.$$

It is known [5, p. 548] that

$$\sharp\mathbf{pow}(\mathbf{a}, d) = \dim \mathscr{P}^{\mathscr{L}}_d = m_d(V).$$

It is proved in [5, Lemma 3.1] that, in conformity with the notation, $\mathbf{pow}(\mathbf{a}, d)$ does not depend on $\mathscr{L}$ but only on $\mathbf{a}$. From now on, we will assume—without loss of generality—that $a = 0$.

The interesting case occurs when $\mathbf{pow}(\mathbf{a}, d)$ is gap-free, that is,

$$\mathbf{pow}(\mathbf{a}, d) = \{0, 1, \dots, m_d(V) - 1\}.$$

In that case, the point $\mathbf{a}$ is said to be *d-Taylorian* (for V).

Many examples can be found in [5]. A simple explicit example is treated in Example 3.3 below.

We now recall the most important property that will be used in this paper. It provides the key connection between Vandermonde determinants (and Lagrange interpolation) as studied in Section 2 and the property of being d-Taylorian.

THEOREM 3.1 (Bos–Calvi [5, Theorem 4.5]). *Let $V = V(q)$ be an irreducible algebraic curve and $\mathbf{a}$ a regular point on V. Then $\mathbf{a}$ is d-Taylorian if and only if for one (and hence every) parametrization $\mathscr{L} = (a, U, R)$ at $\mathbf{a} \in V^0$, the Wronskian $W(\mathscr{F}, a)$ does not vanish, where $\mathscr{F} = \{p_i \circ R : i = 0, \dots, m_d(V) - 1\}$ and the p_i form a basis of $\mathscr{P}_d(V)$.*

Further properties of Taylorian points will be used (and recalled) below. Here, let us just point out that, as shown in [5], every point on a line or an irreducible quadric in $\mathbb{C}^2$ is d-Taylorian for every $d \geq 1$; and, more generally, for each $d \geq 1$, all but finitely many points on an irreducible algebraic curve in $\mathbb{C}^2$ are d-Taylorian.

3.2. Unisolvent sets of points on a curve. The following theorem gives an important property of Taylorian points.

THEOREM 3.2. *Let q be an irreducible polynomial in $\mathbb{C}^2$ and $V = V(q)$. If $\mathbf{a}$ is a d-Taylorian point then there exists a neighbourhood Ω of $\mathbf{a}$ in V^0 such that any set of $m_d(V)$ distinct points in Ω is unisolvent for $\mathscr{P}_d(V)$.*

Proof. Choose a parametrization $\mathscr{L} = (0, U, R)$ of V at $\mathbf{a}$ and apply Lemma 2.1 in the form (2.7) with $\mathscr{F} = \{p_i \circ R : i = 0, \dots, m-1\}$ where $m = m_d(V)$ and the m polynomials p_i form a basis of $\mathscr{P}_d(V)$. For a set $X = \{R(t_i) : i = 0, \dots, m-1\}$ of m distinct points in $R(U)$, we have

$$\mathrm{VDM}(\mathscr{F}; X) = \mathrm{VDM}(T) \cdot \Phi(\mathscr{F}; t_0, \dots, t_{m-1}). \tag{3.1}$$

Since the functions in $\mathscr{F}$ are holomorphic on a neighbourhood of 0, according to (2.8) the value at 0 of $\Phi(\mathscr{F}; t_0, \ldots, t_{m-1})$ is the Wronskian $W(\mathscr{F}, 0)$. Now, in view of Theorem 3.1, this Wronskian does not vanish when $\mathbf{a}$ is d-Taylorian. By continuity, $\Phi(\mathscr{F}; t_0, \ldots, t_{m-1})$ will not vanish for t_i close enough to zero, say $t_i \in D(0, \delta) \subset U$ for $i = 0, \ldots, m-1$. So it suffices to take $\Omega = R(D(0, \delta))$. ∎

The above result says that the space of polynomials $\mathscr{P}_d(V)$ restricted to Ω is a Haar space and we may reword the above result by saying that $\mathscr{P}_d(V)$ is locally Haar around d-Taylorian points.

As shown by the following example, the assumption that $\mathbf{a}$ is d-Taylorian cannot be removed.

EXAMPLE 3.3. Let $V = \{y = x^3\}$ and $\mathscr{L}$ be the trivial parametrization of V at $0 = (0, 0)$ (i.e., $R(x) = (x, x^3)$). We work with $d = 1$. We have

$$\mathscr{P}_1(V) = \operatorname{span}\{1, x, y\} \ \Rightarrow \ \mathscr{P}_1^{\mathscr{L}} = \operatorname{span}\{1, x, x^3\}.$$

Since 1, x and x^3 are three obviously linearly independent elements in $\mathscr{P}_{1\downarrow}^{\mathscr{L}}$, which is a vector space of the same dimension as $\mathscr{P}_1^{\mathscr{L}}$, we actually have

$$\mathscr{P}_1^{\mathscr{L}} = \mathscr{P}_{1\downarrow}^{\mathscr{L}},$$

so that

$$\mathbf{pow}(0, 1) = \{0, 1, 3\}.$$

It follows that 0 is not 1-Taylorian.

Now, if $X = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \subset V$ with $\mathbf{x}_j = (x_j, y_j) = (x_j, x_j^3)$, $j = 0, 1, 2$, then

$$\mathrm{VDM}(\mathscr{B}; X) = \begin{vmatrix} 1 & x_0 & x_0^3 \\ 1 & x_1 & x_1^3 \\ 1 & x_2 & x_2^3 \end{vmatrix} = (x_0 + x_1 + x_2) \prod_{0 \le i < j \le 2} (x_j - x_i).$$

In any neighbourhood Ω of 0 in V we can choose $X = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \subset \Omega$ such that $\mathbf{x}_i \neq \mathbf{x}_j$ for $i \neq j$ and $\mathrm{VDM}(\mathscr{B}; X) = 0$.

4. Existence and characterization of the limit

4.1. Existence. As a consequence of Theorem 3.2, the polynomial $\mathbf{L}_V[A; f]$ is well defined for every choice of $m_d(V)$ points sufficiently close to $\mathbf{a}$, and it is therefore meaningful to study $\lim_{A \to \mathbf{a}} \mathbf{L}_V[A; f](\mathbf{x})$ where $A \to \mathbf{a}$ retains its ordinary meaning. We now prove the existence of the limit.

THEOREM 4.1. *Let V be an irreducible algebraic curve in $\mathbb{C}^2$ and $\mathbf{a} \in V^0$ a d-Taylorian point. Let $X_n \subset V$, $n \in \mathbb{N}$, be a sequence of unisolvent sets for $\mathscr{P}_d(V)$ whose points tend to $\mathbf{a}$, i.e.* $\max\{|\mathbf{x} - \mathbf{a}| : \mathbf{x} \in X_n\} \to 0$ *as* $n \to \infty$.

Then

$$\lim_{n\to\infty} \mathbf{L}_V[X_n; f] \quad \textit{exists} \tag{4.1}$$

for every function f holomorphic on a neighbourhood of $\mathbf{a}$ in $\mathbb{C}^2$.

Equivalently, since the limit must be independent of the sequence (X_n), the theorem states that for every d-Taylorian point $\mathbf{a}$ in V^0,

$$\lim_{A\to\mathbf{a}} \mathbf{L}_V[A; f] \quad \text{exists.}$$

Proof. Let $\mathscr{L} = (0, U, R)$ be a local parametrization of V^0 at $\mathbf{a}$ and let $\mathscr{B} = \{p_0, \ldots, p_{m-1}\}$ be any basis for $\mathscr{P}_d(V)$, where $m = m_d(V)$. For n large enough we have $X_n \subset R(U)$, so that

$$X_n = \{R(t_0^n), \ldots, R(t_{m-1}^n)\} \quad \text{with } t_j^n \to 0 \text{ as } n\to\infty,\ j = 0, \ldots, m-1.$$

In view of (2.1), we have

$$\mathbf{L}_V[X_n; f](\mathbf{x}) = \sum_{i=0}^{m-1} \frac{\mathrm{VDM}(\mathscr{B}[p_i \leftarrow f]; X_n)}{\mathrm{VDM}(\mathscr{B}; X_n)} p_i(\mathbf{x}). \tag{4.2}$$

Now, setting $\mathscr{F} = \mathscr{B} \circ R$ and $T_n = \{t_0^n, \ldots, t_{m-1}^n\}$, we have $\mathrm{VDM}(\mathscr{B}; X_n) = \mathrm{VDM}(\mathscr{F}; T_n)$, and likewise for $\mathrm{VDM}(\mathscr{B}[p_i \leftarrow f]; X_n)$, so that, passing to the limit in (4.2) with the help of (2.9), we get

$$\begin{aligned} \lim_{n\to\infty} \mathbf{L}_V[X_n; f](\mathbf{x}) &= \lim_{n\to\infty} \left\{ \sum_{i=0}^{m-1} \frac{\mathrm{VDM}(\mathscr{F}[f_i \leftarrow f\circ R]; T_n)}{\mathrm{VDM}(\mathscr{F}; T_n)} p_i(\mathbf{x}) \right\} \\ &= \sum_{i=0}^{m-1} \frac{W(\mathscr{F}[f_i \leftarrow f\circ R]\,, 0)}{W(\mathscr{F}\,, 0)} p_i(\mathbf{x}), \end{aligned} \tag{4.3}$$

where $f_i = p_i \circ R$. The limit is well defined. Indeed, in view of Theorem 3.1, since $\mathbf{a}$ is d-Taylorian, $W(\mathscr{F}, 0)$ does not vanish. ∎

The theorem explains the negative conclusion in Example 1.1. Indeed, when $V = \{y = x^3\}$, the point $\mathbf{a} = (0, 0)$ is not 1-Taylorian. This fact was proved in Example 3.3.

4.2. Characterization of the limit: Taylor interpolation on curves. The right hand side of (4.3) actually gives an explicit expression for the limit, and according to the discussion in the introduction, it provides a suitable definition for a Taylor polynomial of f at $\mathbf{a}$. The formula however apparently depends on R, and more importantly it is desirable to know whether such a Taylor polynomial furnishes a local approximant of f in the sense that generalizes the classical one. In fact, it turns out that this limiting operator coincides with a Hermitian interpolation procedure studied in [5].

THEOREM 4.2 (Bos–Calvi). *Let V be an irreducible algebraic curve in $\mathbb{C}^2$ and $\mathbf{a} \in V^0$ a d-Taylorian point, $d \geq 1$. Then, for every function f holomorphic on a neighbourhood of $\mathbf{a}$ in $\mathbb{C}^2$, there exists a unique polynomial $P \in \mathscr{P}_d(V)$ such that, for every local parametrization $\mathscr{L} = (0, U, R)$ of V at $\mathbf{a}$ with $R(0) = \mathbf{a}$,*

$$(P \circ R)^{(i)}(0) = (f \circ R)^{(i)}(0), \qquad i = 0, \dots, m_d(V) - 1. \tag{4.4}$$

The above interpolation polynomial is called the *d-Taylor polynomial* of f at $\mathbf{a}$ and is denoted by $\mathbf{T}^d_{\mathbf{a}}(f)$. The map $f \mapsto \mathbf{T}^d_{\mathbf{a}}(f)$ is the *Taylor projector* of degree d at $\mathbf{a}$ on V.

Corresponding to (1.1), we have

$$(f - \mathbf{T}^d_{\mathbf{a}}(f))(R(t)) = o(t^{m-1}) \qquad (t \to 0).$$

Observe that (4.4) implies that a polynomial P computed with one local parametrization works as well for any other parametrization. In other words, to check that a given polynomial $p \in \mathscr{P}_d(V)$ equals $\mathbf{T}^d_{\mathbf{a}}(f)$, it suffices to check (4.4) for only one parametrization. One says that the linear forms $f \mapsto (f \circ R)^{(i)}(0)$ are the *interpolation conditions* for $\mathbf{T}^d_{\mathbf{a}}$.

We will use the following multiplicative property of Taylor interpolation on curves (see [5, Corollary 3.6]). For any suitably defined functions g_1 and g_2, we have

$$\mathbf{T}^d_{\mathbf{a}}(g_1 g_2) = \mathbf{T}^d_{\mathbf{a}}\big(\mathbf{T}^d_{\mathbf{a}}(g_1)\mathbf{T}^d_{\mathbf{a}}(g_2)\big). \tag{4.5}$$

THEOREM 4.3. *The limit whose existence is proved in Theorem 4.1 is $\mathbf{T}^d_{\mathbf{a}}(f)$. Thus, for every d-Taylorian point $\mathbf{a}$ on V,*

$$\lim_{A \to \mathbf{a}} \mathbf{L}_V[A; f] = \mathbf{T}^d_{\mathbf{a}}(f).$$

Proof. We just need to prove that the limit given by the right hand side of (4.3), and which we will now denote by P,

$$P(\mathbf{x}) = \sum_{i=0}^{m-1} \frac{W(\mathscr{F}[f_i \leftarrow f \circ R], 0)}{W(\mathscr{F}, 0)} p_i(\mathbf{x}),$$

is exactly $\mathbf{T}^d_{\mathbf{a}}(f)$. To do so, since $P \in \mathscr{P}_d(V)$, according to the remark following Theorem 4.2, it suffices to check that $(P \circ R)^{(i)}(0) = (f \circ R)^{(i)}(0)$ for $i = 0, \dots, m-1$. We use the notation introduced in the proof of Theorem 4.1. Observe that since the map $f \to W(\mathscr{F}[f_i \leftarrow f \circ R], 0)$ is linear and

$$W(\mathscr{F}[f_i \leftarrow f_j], 0) = \delta_{ij}\, W(\mathscr{F}, 0), \qquad i, j = 0, \dots, m-1,$$

we have

$$\begin{aligned} W(\mathscr{F}[f_j \leftarrow P \circ R], 0) &= \sum_{i=0}^{m-1} \frac{W(\mathscr{F}[f_i \leftarrow f \circ R], 0)}{W(\mathscr{F}, 0)} \cdot \delta_{ij} W(\mathscr{F}, 0) \\ &= W(\mathscr{F}[f_j \leftarrow f \circ R], 0), \qquad j = 0, \dots, m-1. \end{aligned}$$

By expanding the above relations, we obtain the differential identities

$$(4.6)\qquad \sum_{k=0}^{m-1} C_{jk}\frac{(P\circ R)^{(k)}(0)}{k!} = \sum_{k=0}^{m-1} C_{jk}\frac{(f\circ R)^{(k)}(0)}{k!},\qquad j=0,\dots,m-1,$$

where C_{jk} is the cofactor of the matrix giving $W(\mathscr{F},0)$. Since this matrix is invertible, so is its co-matrix; hence, relations (4.6) imply that $(P\circ R)^{(j)}(0) = (f\circ R)^{(j)}(0)$ for $j=0,\dots,m-1$, and this concludes the proof of the theorem. ■

Using (2.10) instead of (2.9), we obtain the following corollary which we will need in Section 6.3.

COROLLARY 4.4. *Under the assumptions of Theorem* 4.1, *if* g_n *converges to* g *in the standard topology of* $\mathscr{A}(\Omega)$ *where* Ω *is an open neighbourhood of* $\mathbf{a}$ *in* V^0 *then*

$$(4.7)\qquad \lim_{n\to\infty} \mathbf{L}_V[X_n; g_n] = \mathbf{T}^d_{\mathbf{a}}(g).$$

5. A new characterization of Taylorian points on curves. The main objective, in this part, is to prove a converse to Theorem 4.1: if $\mathbf{L}_V[X_n; f]$ converge for a sole sequence (X_n) of interpolation arrays tending to a limit point $\mathbf{a}$, and for all holomorphic functions on a neighbourhood of $\mathbf{a}$, then $\mathbf{a}$ must be d-Taylorian (and the limit is the Taylor polynomial defined above).

5.1. Analytic functionals. Recall that an analytic functional μ belongs to $\mathscr{A}'(0)$ if its restriction to all $\mathscr{A}(\Omega)$ is continuous, where Ω runs over all open neighbourhoods of the origin. In particular, $\mu\in\mathscr{A}'(0)$ if and only if for every $\rho>0$, there exists a positive constant M_ρ such that

$$|\mu(f)| \le M_\rho \|f\|_{\overline{D}(0,\rho)},\qquad f\in\mathscr{A}(\overline{D}(0,\rho)).$$

In particular, taking $f(z)=z^m$ we obtain

$$|\mu(z^m)| \le M_\rho \rho^m,\qquad m\ge 0.$$

Observe that if $f(z)=\sum_{m=0}^\infty a_m z^m$ for $|z|\le\rho$, we have

$$(5.1)\qquad \mu(f) = \sum_{m=0}^{\infty} a_m \mu(z^m) = \sum_{m=0}^{\infty} b_m f^{(m)}(0),\qquad b_m = \mu(z^m)/m!.$$

Note that

$$b_m = \mu(z^m)/m! = O(\rho^m/m!),$$

and since this is true for every positive ρ, the function $F(z)=\sum_{m=0}^\infty b_m z^m$ is entire. The relation $\mu(f)=\sum_{m=0}^\infty b_m D^m(f)(0)$ will be written as $\mu = F(D)$.

Likewise, given an irreducible algebraic curve $V=\{q=0\}$ and $\mathbf{a}\in V^0$, we may consider $\nu\in\mathscr{A}'(\mathbf{a})$. In fact, given such a ν and a local parametrization $\mathscr{L} = (0, U, R)$ of V^0 at $\mathbf{a}$, there exists $\mu_R \in \mathscr{A}'(0)$ such that $\nu(f) = \mu_R(f \circ R)$ for every $f \in \mathscr{A}(\mathbf{a})$ (since the map $f \in \mathscr{A}(\mathbf{a}) \mapsto f \circ R \in \mathscr{A}(0)$ is onto, the relation $\nu(f) = \mu_R(f \circ R)$ defines μ_R on $\mathscr{A}(0)$ and it is readily seen that it is continuous). Thus any $\nu \in \mathscr{A}'(\mathbf{a})$ is defined by a relation of the form

$$\nu(f) = F(D)(f \circ R), \quad f \in \mathscr{A}(\mathbf{a}), \tag{5.2}$$

where F is a univariate entire function. We will write $\nu = F_R(D)$. If F is a *polynomial*, we say that ν is a *Hermitian functional* at $\mathbf{a}$.

5.2. Polynomial projectors whose coefficients are Hermitian functionals. If (p_i) is a basis for $\mathscr{P}_d(V)$ and $\pi : \mathscr{A}(\mathbf{a}) \to \mathscr{P}_d(V)$ is a polynomial projector then for every f,

$$\pi(f) = \sum_{i=1}^{m_d(V)} c_i(f) p_i, \tag{5.3}$$

where the c_i are analytic functionals which we refer to as the *coefficients* of π (with respect to the basis (p_i)).

THEOREM 5.1. *Let $V = V(q)$ be an irreducible curve in $\mathbb{C}^2$ and $\mathbf{a} \in V^0$. If there exists a polynomial projector $\pi : \mathscr{A}(\mathbf{a}) \to \mathscr{P}_d(V)$ such that*

(1) $\ker \pi$ *is an ideal, and*
(2) *the coefficients of π are Hermitian functionals at $\mathbf{a}$,*

then $\mathbf{a}$ is d-Taylorian and $\pi = \mathbf{T}^d_{\mathbf{a}}$.

Clearly, the second assumption does not depend on the basis we choose for $\mathscr{P}_d(V)$. This assumption is given in a more intrinsic form below.

Let $\mathscr{L} = (0, U, R)$ be a parametrization of V at $\mathbf{a} = (a_1, a_2)$ such that $R(U) \subset V^0$. We have $R'(0) \neq 0$. We will always assume that the first coordinate of $R'(0)$ is not null. Otherwise, we may work with the second coordinate R_2 with obvious changes. This assumption implies:

(P1) R_1 defines a complex diffeomorphism between a neighbourhood of 0 and a neighbourhood of a_1, which implies the following.
(P2) The map

$$\Theta : \phi \in \mathscr{A}(a_1) \mapsto \phi \circ R_1 \in \mathscr{A}(0) \tag{5.4}$$

defines an isomorphism from the vector space $\mathscr{A}(a_1)$ onto $\mathscr{A}(0)$.

We denote by $S(\pi)$ the subspace of $\mathscr{A}'(\mathbf{a})$ spanned by the functionals μ such that $\mu(f) = \mu(\pi(f))$ for every $f \in \mathscr{A}(\mathbf{a})$. Equivalently, $S(\pi)$ is the ($m_d(V)$-dimensional) subspace of $\mathscr{A}'(\mathbf{a})$ spanned by the coordinates of π in any basis of $\mathscr{P}_d(V)$; for instance, in view of (5.3),

$$S(\pi) = \operatorname{span}\{c_i : i = 1, \dots, m_d(V)\}.$$

In view of the definition of a Hermitian functional given above, assumption (2) of Theorem 5.1 ensures that every c_i, hence every element μ of $S(\pi)$, is defined by a relation of the form

$$\mu(f) = Q_R(D)(f) = Q(D)(f \circ R) = \sum_{j \in J_\mu} c_j D^j (f \circ R)(0),$$

where Q is the polynomial $Q(z) = \sum_{j \in J_\mu} c_j z^j$ (J_μ is finite).

LEMMA 5.2. *With the assumptions of the theorem, if $Q_R(D) \in S(\pi)$ then $Q_R^{(i)}(D) \in S(\pi)$ for $i = 1, \dots, \deg Q$.*

Proof. Suppose that, for some $j \in \{1, \dots, s\}$ where $s = \deg Q$, $\mu = Q_R^{(j)}(D)$ does not belong to $S(\pi)$. First, we can find $f \in \ker \pi$ such that $Q_R^{(j)}(D)(f) \neq 0$. Indeed, since $\mu \notin S(\pi)$, there exists $h \in \mathscr{A}(\mathbf{a})$ such that $\mu(h - \pi(h)) \neq 0$ and it suffices to take $f = h - \pi(h)$, the fact that π is a projector ensuring $f \in \ker \pi$. The assumption now gives $fg \in \ker \pi$, hence in particular $Q_R(D)(fg) = 0$ for all $g \in \mathscr{A}(\mathbf{a})$. The use of a general Leibniz formula (which is readily derived from the usual one) now yields

$$Q_R(D)(fg) = \sum_{i=0}^{s} \frac{1}{i!} (g \circ R)^{(i)}(0) Q_R^{(i)}(D)(f).$$

Omitting the term for $i = 0$ (which vanishes), we have

$$0 = \sum_{i=1}^{s} \frac{1}{i!} (g \circ R)^{(i)}(0) Q_R^{(i)}(D)(f), \qquad g \in \mathscr{A}(\mathbf{a}). \tag{5.5}$$

Yet, since the vector $(Q_R^{(1)}(D)(f), \dots, Q_R^{(s)}(D)(f))$ is not null (because its jth coordinate is nonzero), a nonorthogonal vector can be found, that is, a vector $(v_1, \dots, v_s)$ such that $\sum_{i=1}^{s} v_i Q_R^{(i)}(D)(f) \neq 0$. Hence, to obtain a contradiction, it suffices to show that there exists $g \in \mathscr{A}(\mathbf{a})$ with

$$\frac{1}{i!} (g \circ R)^{(i)}(0) = v_i, \qquad i = 1, \dots, s.$$

This can be done as follows. The automorphism Θ in (5.4) ensures that there exists $h \in \mathscr{A}(a_1)$ such that $(1/i!)(h \circ R_1)^{(i)}(0) = v_i$, $i = 1, \dots, s$ (take the preimage under Θ of $\sum_{i=1}^{s} v_i x^i$). Then it suffices to define $g(x, y) = h(x)$. ∎

Proof of Theorem 5.1. An application of the previous lemma readily gives

$$S(\pi) = \operatorname{span}\{\Gamma_i : i = 0, \dots, M\}, \qquad \Gamma_i : f \mapsto (f \circ R)^{(i)}(0), \tag{5.6}$$

where M is the supremum of the degrees of the polynomials Q for which $Q_R(D) \in S(\pi)$. Indeed, if Q_M is a polynomial of maximal degree, then $\{Q_M^{(i)} : i = 0, \dots, M\}$ forms a basis of $\mathscr{P}_M(\mathbb{C})$ so that each z^j is a linear combination of the $Q_M^{(i)}$ for $j = 0, \dots, M$. Hence, in view of Lemma 5.2 the right hand side of (5.6) is included in $S(\pi)$. The converse inclusion follows from the maximality of M.

We construct a basis for $\sigma(\pi)$, the space of restrictions of elements of $S(\pi)$ to $\mathscr{P}_d(V)$, as follows. Recall that

$$\dim S(\pi) = \dim \sigma(\pi) = m, \qquad \text{where } m = m_d(V).$$

We let

$$\gamma_i = \mathit{\Gamma}_{i|\mathscr{P}_d(V)}, \qquad i = 0, \dots, M.$$

If the indices $i_0 < \cdots < i_k$ have been selected, we choose i_{k+1} as the smallest index such that

$$\gamma_{i_{k+1}} \notin \sigma_k = \operatorname{span}\{\gamma_{i_j} : j = 0, \dots, k\}.$$

Observe that $i_0 = 0$ and, for $i_k < j < i_{k+1}$, $\gamma_j \in \sigma_k$. In this way, we obtain a basis γ_{i_j}, $j = 0, \dots, m-1$, for $\sigma(\pi)$.

We claim that the theorem will be proved if we show that $i_{m-1} = m-1$, so that $i_j = j$ for every j. Indeed, in that case $\mathbf{a}$ will be d-Taylorian (and $\pi = \mathbf{T}^d_{\mathbf{a}}$ since the interpolation conditions of both projectors will coincide). To see that, observe that if $\sigma(\pi) = \operatorname{span}\{\gamma_i : i = 0, \dots, m-1\}$ then the γ_i form a basis for $\sigma(\pi)$ and we may construct a dual basis (p_j) for $\mathscr{P}_d(V)$, that is,

$$\gamma_i(p_j) = \delta_{ij}, \qquad i, j = 0, \dots, m-1.$$

So, using the notation introduced in Subsection 3.1, we have $(p_j \circ R) = z^j/j! + (\text{terms of higher order})$, which means $(p_j \circ R)_\downarrow = t^j/j!$, so that $t^j \in \mathscr{P}^{\mathscr{L}}_{d\downarrow}$. Since this is true for every $j \in \{0, \dots, m-1\}$ and $\dim \mathscr{P}^{\mathscr{L}}_{d\downarrow} = m$, we have $\mathbf{pow}(\mathbf{a}, d) = \{0, \dots, m-1\}$ and $\mathbf{a}$ is Taylorian.

We now turn to the proof of the above claim. The reasoning is inspired from [5, proof of Theorem 3.2] .

We consider the sequence of integers

$$I = (i_0, i_1, \dots, i_{m-1}).$$

We assume that $i_{m-1} > m-1$ and look for a contradiction.

Since $i_{m-1} > m-1$, there must be at least one gap in the sequence I and we may define i_s as the greatest integer in I so that $i_s + 1 \notin I$. Note that $s \le m-2$ and I contains all integers from i_{s+1} to i_{m-1}, that is, $i_{s+k} = i_{s+1} + k - 1$ for $k = 1, \dots, m-1-s$. Consider the function f defined by $f(x, y) = (x - a_1)^\tau$ where $\tau = i_{s+1} - 1 \ge i_s + 1$ so that τ does not belong to I. We have

$$\begin{aligned}(f \circ R)(t) &= R_1'(0)t^\tau + o(t^{\tau+1}) \\ &= R_1'(0)t^\tau + \alpha_1 t^{i_{s+1}} + \alpha_2 t^{i_{s+2}} + \cdots + \alpha_{m-1-s} t^{i_{m-1}} + o(t^{i_{m-1}+1}).\end{aligned}$$

Now, for $r = 1, \ldots, m-1-s$, we may find $p_r \in \mathscr{P}_d(V)$ such that

$$\gamma_{i_j}(p_r) = \delta_{j,\,s+r}, \qquad j = 0, \ldots, m-1.$$

Since $\operatorname{span}\{\gamma_i : i < i_{s+1}\} = \sigma_s$, this implies further that $\gamma_i(p_r) = 0$ for $i < i_{s+1}$, hence

$$(p_r \circ R)(t) = \frac{1}{i_{s+r}!} t^{i_{s+r}} + o(t^{i_{m-1}+1}).$$

It follows that, setting

$$F = f - \sum_{r=1,\ \alpha_r \neq 0}^{m-1-s} \frac{i_{s+r}!}{\alpha_r} p_r,$$

we have

$$(F \circ R)(t) = R_1'(0) t^{\tau} + o(t^{i_{m-1}+1}).$$

This implies that $\Gamma_{i_k}(F) = 0$ for all $k \in I$, hence $F \in \ker \pi$. Yet xF does not belong to $\ker \pi$ since $\gamma_{i_{s+1}}(xF) = R_1'(0)(i_{s+1}!) \neq 0$. This contradicts the fact that $\ker \pi$ is an ideal and concludes the proof the theorem. ∎

5.3. A converse to Theorem 4.1

THEOREM 5.3. *Let V be an irreducible curve in $\mathbb{C}^2$ and $\mathbf{a} \in V^0$. If there exists a sequence $X_n \subset V$, $n \in \mathbb{N}$, of unisolvent sets for $\mathscr{P}_d(V)$ whose points tend to $\mathbf{a}$ such that for every $f \in \mathscr{A}(\mathbf{a})$ the limit*

$$\lim_{n\to\infty} \mathbf{L}_V[X_n; f] = L(f) \tag{5.7}$$

exists (as an element of $\mathscr{P}_d(V)$), then $\mathbf{a}$ is d-Taylorian and $L = \mathbf{T}_{\mathbf{a}}^d$.

Proof. We will prove that the projector L satisfies the assumptions of Theorem 5.1.

STEP 1. *L is a continuous linear projector on $\mathscr{A}(\mathbf{a})$.* Let Ω be an open neighbourhood of $\mathbf{a}$ in V^0. For n large enough, the coefficients $c_i^n(f)$ of $\mathbf{L}_V[X_n; f]$ relative to any fixed basis $(p_1, \ldots, p_{m_d(V)})$ of $\mathscr{P}_d(V)$, which are linear combinations of point-evaluations at elements of X_n, can be regarded as elements of $\mathscr{A}'(\Omega)$. The assumption ensures that for every $f \in \mathscr{A}(\Omega)$, $c_i^n(f)$ converges to $c_i(f)$, the corresponding coefficient of $L(f)$. By the uniform boundedness principle, the convergence is uniform. In particular, for every $\rho > 0$, there exists M_ρ such that

$$|c_i^n(f)| \le M_\rho \|f\|_{B_\rho}, \qquad n \ge n_0,\ f \in \mathscr{A}(\Omega),$$

where $B_\rho = V^0 \cap \overline{B}(\mathbf{a}, \rho) \subset \Omega$. Letting $n \to \infty$, we obtain an inequality which shows that since ρ can be taken arbitrarily small, each c_i is an element of $\mathscr{A}'(\mathbf{a})$. Hence L is a continuous linear map on $\mathscr{A}(\mathbf{a})$, and it is immediate that it is a projector (with values in $\mathscr{P}_d(V)$).

Step 2. *The kernel of L is an ideal.* We start from the relation (cf. (4.5))

$$\mathbf{L}_V[X_n; g_1 g_2] = \mathbf{L}_V\big[X_n; \mathbf{L}_V[X_n; g_1]\mathbf{L}_V[X_n; g_2]\big],$$

(both sides are elements of $\mathscr{P}_d(V)$ that match $g_1 g_2$ on X_n). Passing to the limit, we readily get $L(g_1 g_2) = L(L(g_1)L(g_2))$ so that if $g_1 \in \ker L$ then $0 = L(g_1) \Rightarrow L(L(g_1)L(g_2)) = L(0) = 0 \Rightarrow L(g_1 g_2) = 0$, so that $g_1 g_2 \in \ker L$ and $\ker L$ is an ideal.

Step 3. *The functionals c_i are Hermitian.* Given a parametrization $\mathscr{L} = (0, U, R)$, since, as shown in the first step, $c_i \in \mathscr{A}'(\mathbf{a})$, according to the discussion in Subsection 5.1 we have

$$c_i(f) = (F_i)_R(D)(f) = F_i(D)(f \circ R) = \sum_{m=0}^{\infty} b_m^i D^m (f \circ R)(0),$$

where $F_i(z) = \sum_{m=0}^{\infty} b_m^i z^m$ is entire. For $n > n_0$, we have $X_n \subset R(U)$. Consider the function

$$f_n(x, y) = (g \circ R_1^{-1})(x) \prod_{\mathbf{b}=(b_1,b_2)\in X_n} (R_1^{-1}(x) - R_1^{-1}(b_1)),$$

where $g(x)$ is any univariate entire function. (Recall that we have assumed $R_1'(0) \neq 0$, see (P2) following Theorem 5.1, so that $f_n \in \mathscr{A}(\mathbf{a})$.) Since $f_n(\mathbf{b}) = 0$ for $\mathbf{b} \in X_n$, we have $\mathbf{L}_V[X_n; f_n] = 0$, and passing to the limit yields $L(f) = 0$ where

$$f(x, y) = (g \circ R_1^{-1})(x)(R_1^{-1}(x))^{m_d}.$$

Here we have used the uniform convergence of $\mathbf{L}_V[X_n; \cdot]$ to L (see the first step). Hence, $c_i(f) = 0$ and

$$\begin{aligned} 0 &= \sum_{m=0}^{\infty} b_m^i D^m (f \circ R)(0) = \sum_{m=0}^{\infty} b_m^i D^m (x^{m_d} g(x))(0) \\ &= \sum_{m=m_d}^{\infty} b_m^i D^m (x^{m_d} g(x))(0). \end{aligned}$$

In particular, if g is the entire function

$$g(x) = \sum_{m=m_d}^{\infty} \overline{b_m^i} x^{m-m_d},$$

then the above identity reduces to

$$0 = \sum_{m=m_d}^{\infty} |b_m^i|^2 m!,$$

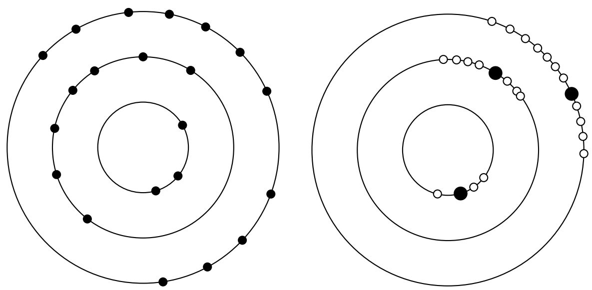
so that $b_m^i = 0$ for $m \geq m_d$. This proves that c_i is Hermitian and concludes the proof of the theorem. ∎

6. Application. Limit of bivariate Lagrange projectors

6.1. Introduction. So far, we have considered problems for which the interpolation points all tend to the same limit point. Relatively little is known, it seems, in the more general case for which the limiting configuration is not reduced to a single point. An interesting result was obtained by Shekhtman [10] who specified a (computable) set of conditions enabling one to decide whether a given projector is the limit of Lagrange projectors.

Here, we apply the results obtained in the previous sections to address the problem of limits of sequences of Lagrange interpolation polynomials at natural bivariate configurations of points. A typical question that is answered in this section is illustrated below.

Lagrange interpolation — Hermite interpolation



(A) (B)

In (A), we show a Bos configuration formed by $21 = 11 + 7 + 3$ points distributed on three circles in $\mathbb{R}^2 \subset \mathbb{C}^2$ (the number of points on each circle is fixed) for interpolation by polynomials of degree at most five in $\mathbb{C}^2$. The construction is recalled below. Now, assume that, as indicated in (B), on each circle, the points move towards one limit point, drawn in larger size, one limit point for each circle. The natural questions are the following.

(1) Does the sequence of Lagrange operators obtained by moving the points converge?
(2) If yes, how is the limiting operator defined? Can it be understood as a Hermitian interpolation operator at the three limiting points?

We will show that the answers to the above questions are (in general) positive and that the limiting operator is given by a Hermitian interpolation procedure previously introduced by Bos and Calvi [4, 5] whose construction will be recalled below. Note that Phung [9] also studied the limit of Lagrange projectors when the interpolation points are Bos configurations on circles. But the coalescence of points is different: the centres of the circles are fixed and the radii tend to 0.

6.2. Bos configurations. In the whole section, we work with the following objects:

(1) A family of $\nu \geq 2$ curves $V_i \subset \mathbb{C}^2$, where
(2) for each i, $V_i = \{q_i = 0\}$ with q_i an irreducible polynomial of degree $r_i \geq 1$ in $\mathbb{C}^2$.
(3) To each $d \in \mathbb{N}$ such that

$$r_1 + \cdots + r_{\nu-1} < d \leq r_1 + \cdots + r_\nu, \tag{6.1}$$

we associate a finite sequence of integers s_i defined as

$$s_1 = d \quad \text{and} \quad s_i = d - r_1 - \cdots - r_{i-1}, \quad i = 2, \ldots, \nu. \tag{6.2}$$

A difference occurs according to whether equality holds (*case* 2) in the upper bound of (6.1), i.e. $d = \sum_{i=1}^{\nu} r_i$, or not (*case* 1). The latter leads to somewhat shorter statements. The above definitions are motivated by the following theorem due to Bos [3] which proposes a way to construct suitable configurations of points for interpolation in $\mathbb{C}^2$ by collecting unisolvent sets on the curves V_i (see [7] for the treatment of the complex case).

THEOREM 6.1. *For $i = 1, \ldots, \nu$, let $X_i \subset V_i$ be unisolvent for $\mathscr{P}_{s_i}$ with $X_i \cap V_j = \emptyset$ for $i > j$.*

(1) *In case 1, $X = \bigcup_{i=1}^{\nu} X_i$ is unisolvent for $\mathscr{P}_d(\mathbb{C}^2)$.*
(2) *In case 2, $X \cup \{\mathbf{b}\}$, with $\mathbf{b}$ outside $\bigcup_{i=1}^{\nu} V_i$, is unisolvent for $\mathscr{P}_d(\mathbb{C}^2)$.*

Such a configuration will be called a *Bos configuration*. The distribution of points shown in Figure (A) is an example of such a configuration (case 1). It corresponds to $d = 5$, $\nu = 3$ and $V_i = \{x^2 + y^2 = \rho_i^2\}$.

Within the same setting, Bos and Calvi [4, 5] later proved the following result.

THEOREM 6.2. *For $i = 1, \ldots, \nu$, let $\mathbf{a}_i$ be an s_i-Taylorian point on V_i such that $\mathbf{a}_i \notin V_j$ for $i > j$ and a suitably defined function f.*

(1) *In case 1, there exists a unique polynomial*

$$\mathbf{H} = \mathbf{H}[(\mathbf{a}_1, s_1), \ldots, (\mathbf{a}_\nu, s_\nu); f] \in \mathscr{P}_d(\mathbb{C}^2)$$

such that

$$\mathbf{T}_{\mathbf{a}_i}^{s_i}(f - \mathbf{H}) = 0, \quad i = 1, \ldots, \nu. \tag{6.3}$$

(2) *In case 2, if $\mathbf{a}_{\nu+1}$ is a supplementary point lying outside the V_i, there exists a unique polynomial $\mathbf{H} = \mathbf{H}[(\mathbf{a}_1, s_1), \ldots, (\mathbf{a}_\nu, s_\nu), \mathbf{a}_{\nu+1}; f] \in \mathscr{P}_d(\mathbb{C}^2)$ such that together with (6.3), we have $\mathbf{H}(\mathbf{a}_{\nu+1}) = f(\mathbf{a}_{\nu+1})$.*

Here and below, by a 'suitably defined function f', we mean a function which is holomorphic on a neighbourhood of each point $\mathbf{a}_i$.

6.3. The limit theorem. Now, the natural conjecture is to expect that when the points of each X_i in a Bos configuration converge to $\mathbf{a}_i$, the corresponding Lagrange interpolation operator will converge to $\mathbf{H}[(\mathbf{a}_1, s_1), \dots, (\mathbf{a}_\nu, s_\nu); \cdot]$ (in case 1). This is the next theorem.

THEOREM 6.3. *For $i = 1, \dots, \nu$ and $n \in \mathbb{N}$, let $X_i^n \subset V_i$ be unisolvent for $\mathscr{P}_{s_i}$ with $X_i^n \cap V_j = \emptyset$ for $i > j$ such that*

$$\max\{|\mathbf{x} - \mathbf{a}_i| : \mathbf{x} \in X_i^n\} \to 0 \quad \text{as } n \to \infty \tag{6.4}$$

where $\mathbf{a}_i$ is an s_i-Taylorian point for V_i.

(1) *In case 1, setting $X^n = \bigcup_{i=1}^{\nu} X_i^n$, for any suitably defined function f, we have*

$$\lim_{n\to\infty} \mathbf{L}[X^n; f] = \mathbf{H}[(\mathbf{a}_1, s_1), \dots, (\mathbf{a}_\nu, s_\nu); f].$$

(2) *In case 2, if the above set X^n is replaced by $X^n \cup \{\mathbf{b}^n\}$ with $\mathbf{b}^n$ outside $\bigcup_{i=1}^{\nu} V_i$ and converging to a point $\mathbf{a}_{\nu+1}$, then*

$$\lim_{n\to\infty} \mathbf{L}[X^n \cup \{\mathbf{b}^n\}; f] = \mathbf{H}[(\mathbf{a}_1, s_1), \dots, (\mathbf{a}_\nu, s_\nu), \mathbf{a}_{\nu+1}; f].$$

Proof. We only prove the first statement. The proof of the second one is similar. It is a consequence of Theorem 4.1 (and of Theorem 6.2) once we observe that $L^n = \mathbf{L}[X^n; f]$ is given by

(6.5)
$$L^n = \sum_{i=1}^{\nu} L_i^n, \qquad L_k^n = q_1 \cdots q_{k-1} \mathbf{L}_{V_k}\left[X_k^n\,;\left(\frac{f - L_1^n - \cdots - L_{k-1}^n}{q_1 \cdots q_{k-1}}\right)_{|V_k}\right],$$

$k = 1, \dots, \nu$, where the empty product in the definition of L_1^n is taken as 1 (and the empty sum is taken as 0).

The formula requires some explanation since L_k^n must be an element of $\mathscr{P}_d(\mathbb{C}^2)$ while $\mathbf{L}_{V_k}[X_k^n; \cdot]$ produces an element of $\mathscr{P}_d(V_k)$. Actually, it is readily seen that choosing different polynomials in $\mathscr{P}_d(\mathbb{C}^2)$ whose restrictions to V_k coincide would merely result in adding a zero polynomial to the final sum. A more rigorous formula is obtained by using a linear isomorphism between $\mathscr{P}_d(V_k)$ and a (fixed) complementary space of the kernel of $p \mapsto p_{|V_k}$ in $\mathscr{P}_d(\mathbb{C}^2)$. In that case, the notation in (6.5) confuses an element of $\mathscr{P}_d(V_k)$ with its preimage in $\mathscr{P}_d(\mathbb{C}^2)$. The continuity of such a map is used below.

Note that the interpolated function in (6.5) is well defined on X_k^n since, by assumption, $q_1 \cdots q_{k-1}$ does not vanish on X_k^n.

This being said, one readily verifies that the right hand side of (6.5) provides a polynomial of degree at most d which matches f on X^n. Indeed, on X_1^n, $L^n = L_1^n = f$ and, for $j > 1$, on X_j^n, we have

$$L^n = \sum_{k=1}^{j} L_k^n = L_j^n + \sum_{k=1}^{j-1} L_k^n = q_1 \cdots q_{k-1} \left(\frac{f - \sum_{k=1}^{j-1} L_k^n}{q_1 \cdots q_{k-1}} \right) + \sum_{k=1}^{j-1} L_k^n = f.$$

Now, an application of Theorem 4.3 and its corollary shows that L^n has a limit, given by

$$\lim_{n\to\infty} L^n = \sum_{k=1}^{\nu} \mathbf{T}_{\mathbf{a_k}}^{s_k} \left(\frac{f - \Lambda_1 - \cdots - \Lambda_{k-1}}{q_1 \cdots q_{k-1}} \right) \prod_{i=1}^{k-1} q_i \tag{6.6}$$

with

$$\Lambda_k = q_1 \cdots q_{k-1} \mathbf{T}_{\mathbf{a}_k}^{s_k} \left(\frac{f - \Lambda_1 - \cdots - \Lambda_{k-1}}{q_1 \cdots q_{k-1}} \right), \qquad k = 1, \dots, \nu. \tag{6.7}$$

In fact, Theorem 4.3 implies $L_1^n \to \Lambda_1$, and then Corollary 4.4 may be used to prove $L_k^n \to \Lambda_k$ for $k = 2, \dots, \nu$.

To conclude the proof, it remains to show that the polynomial on the right hand side of (6.6) is indeed $\mathbf{H}[(\mathbf{a}_1, s_1), \dots, (\mathbf{a}_\nu, s_\nu); f]$. To do so, we will use the multiplicative property (4.5). We denote by Q the right hand side of (6.6) and prove that

$$\mathbf{T}_{\mathbf{a}_j}^{s_j}(Q) = \mathbf{T}_{\mathbf{a}_j}^{s_j}(f), \qquad j = 1, \dots, \nu.$$

For $j = 1$, the claim is easy, for on V_1, $Q = \mathbf{T}_{\mathbf{a}_1}^{s_1}(f)$. For $j > 1$, we first observe that on V_j, all the terms for $k > j$ vanish, so that

$$\mathbf{T}_{\mathbf{a}_j}^{s_j}(Q) = \sum_{k=1}^{j} \mathbf{T}_{\mathbf{a}_j}^{s_j} \left(\mathbf{T}_{\mathbf{a}_k}^{s_k} \left(\frac{f - \Lambda_1 - \cdots - \Lambda_{k-1}}{q_1 \cdots q_{k-1}} \right) q_1 \cdots q_{k-1} \right) = \sum_{k=1}^{j} \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Lambda_k),$$

where we have used (6.7). We first consider the term for $k = j$. Writing $\Pi_j = q_1 \cdots q_{k-1}$ and $\Sigma_j = \sum_{k=1}^{j-1} \Lambda_k$, using (4.5) we obtain

$$\begin{aligned} \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Lambda_j) &= \mathbf{T}_{\mathbf{a}_j}^{s_j} \left\{ \mathbf{T}_{\mathbf{a}_j}^{s_j} \left(\frac{f - \Sigma_j}{\Pi_j} \right) \Pi_j \right\} \\ &= \mathbf{T}_{\mathbf{a}_j}^{s_j} \left\{ \mathbf{T}_{\mathbf{a}_j}^{s_j} \left(\mathbf{T}_{\mathbf{a}_j}^{s_j} \left(\frac{f - \Sigma_j}{\Pi_j} \right) \right) \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Pi_j) \right\}. \end{aligned}$$

Now, since $\mathbf{T}_{\mathbf{a}_j}^{s_j} \circ \mathbf{T}_{\mathbf{a}_j}^{s_j} = \mathbf{T}_{\mathbf{a}_j}^{s_j}$, using again (4.5) on the second line we get

$$\begin{aligned} \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Lambda_j) &= \mathbf{T}_{\mathbf{a}_j}^{s_j} \left\{ \mathbf{T}_{\mathbf{a}_j}^{s_j} \left(\frac{f - \Sigma_j}{\Pi_j} \right) \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Pi_j) \right\} = \mathbf{T}_{\mathbf{a}_j}^{s_j} \left\{ \left(\frac{f - \Sigma_j}{\Pi_j} \right) \cdot \Pi_j \right\} \\ &= \mathbf{T}_{\mathbf{a}_j}^{s_j}(f - \Sigma_j) = \mathbf{T}_{\mathbf{a}_j}^{s_j}(f) - \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Sigma_j). \end{aligned}$$

Hence,

$$\mathbf{T}_{\mathbf{a}_j}^{s_j}(f) = \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Sigma_j) + \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Lambda_j) = \sum_{k=1}^{j} \mathbf{T}_{\mathbf{a}_j}^{s_j}(\Lambda_k) = \mathbf{T}_{\mathbf{a}_j}^{s_j}(Q),$$

and this concludes the proof of the theorem. ■

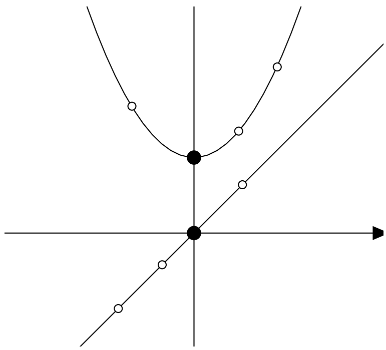
6.4. A few examples

EXAMPLE 6.4. We are now able to answer the problem raised in Subsection 6.1. When the interpolation points in Figure (A) tend to the points $\mathbf{a_i} = \rho_i(\cos(\theta_i), \sin(\theta_i))$ where ρ_i is the radius of the ith circle, $i = 1, 2, 3$, as in Figure (B), the corresponding sequence of Lagrange interpolation polynomials converges to the operator $\mathbf{H}[(\mathbf{a}_1, 5), (\mathbf{a}_2, 3), (\mathbf{a}_3, 1); \cdot]$ which is the Hermitian projector defined by the interpolation conditions

$$f \mapsto \frac{d^k}{d\theta^k} f\big((\rho_i \cos\theta, \rho_i \sin\theta)\big)\Big|_{\theta=\theta_i}, \qquad k = 0, \dots, m_i,\ i = 1, 2, 3,$$

where $m_1 = 10$, $m_2 = 6$ and $m_3 = 2$.

EXAMPLE 6.5. Let $V_1 = \{y = x\}$ and $V_2 = \{y = x^2 + 1\}$. We obtain a Bos configuration for interpolation by polynomials of degree at most 2 by taking three points on V_1 ($s_1 = 2$) and three points V_2 ($s_2 = 1$) as in the figure below.



We want to study the limit of the Lagrange interpolation projector constructed with these six points when the three points on the line tend to $\mathbf{a}_1 = (0, 0)$ (which is 2-Taylorian) and the three points on the parabola tend to $\mathbf{a}_2 = (1, 0)$ (which is 1-Taylorian). According to Theorem 6.3, the limit is given by the projector $\mathbf{H}[(\mathbf{a}_1, 2), (\mathbf{a}_2, 1); \cdot]$ defined by the following six interpolation conditions:

$$f \mapsto f(0,0),\ f \mapsto f(0,1),\ f \mapsto \partial_2 f(0,0) + \partial_1 f(0,0),\ f \mapsto \partial_1 f(0,1),$$
$$f \mapsto \partial_2^2 f(0,0) + \partial_1^2 f(0,0) + 2\partial_{12}^2 f(0,0),\ f \mapsto 2\partial_2 f(0,1) + \partial_1^2 f(0,1).$$

EXAMPLE 6.6. Since the projectors $\mathbf{H}[(\mathbf{a}_1, s_1), \dots, (\mathbf{a}_m, s_m); \cdot]$ are limits of Lagrange projectors, it is natural to further ask what happens when the points $\mathbf{a}_i$ in turn tend to a unique limit point, a situation that may occur when the intersection of the curves V_i is not empty. Here, we present a two-point example $\mathbf{H}[(\mathbf{a}_1, 3), (\mathbf{a}_2, 1); \cdot]$ for which the limit does not exist. Since, in view of Theorem 6.3, there exist Lagrange projectors (at Bos configurations for degree 3) arbitrarily close to $\mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);\cdot]$, this example shows that a sequence of Lagrange interpolation projectors at Bos configurations whose points tend to a single limit point may fail to converge. Of course, this motivates the question to find conditions upon which convergence would hold.

We consider the irreducible polynomials $q_1(\mathbf{x}) = y - x^2$ and $q_2(\mathbf{x}) = y - x^2 - x^8$ and the corresponding curves $V_1 = \{\mathbf{x} : q_1(\mathbf{x}) = 0\}$ and $V_2 = \{\mathbf{x} : q_2(\mathbf{x}) = 0\}$. Every point is 3-Taylorian for V_1. We consider the points $\mathbf{a}_1 = (0,0)$ and $\mathbf{a}_2 = (\varepsilon, \varepsilon^2 + \varepsilon^8)$ which is easily shown to be 1-Taylorian and show that, for $f(x,y) = xy^3$, $\mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);f]$ has no limit as $\varepsilon \to 0$. Clearly, the failed convergence is associated to the fact that q_1 and q_2 locally coincide up to order 7 in a neighbourhood of the origin. We have $\mathbf{T}^3_{\mathbf{a_1}}(f) = 0$ (because the jth derivative of $t \mapsto f(t,t^2)$ at $t = 0$ vanishes for $j = 0,\dots,6$). Hence, in view of (6.7), we have

$$\mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);f] = q_1 \mathbf{T}^1_{\mathbf{a}_2}(f/q_1).$$

Since $q_1(x,y) = y - x^2$, the coefficient of y in $\mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);f]$ is given by

$$\mathbf{T}^1_{\mathbf{a}_2}\left(\frac{f}{q_1}\right)(\mathbf{a}_2) = \frac{f(\mathbf{a}_2)}{q_1(\mathbf{a}_2)} = \frac{\varepsilon(\varepsilon^2+\varepsilon^8)^3}{\varepsilon^8} \sim \frac{1}{\varepsilon},$$

so that $\mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);f]$ has no limit as $\varepsilon \to 0$.

Interestingly enough, if we change q_2 to $q_2(\mathbf{x}) = y - x^2 - x^k$ with $5 \le k < 8$, then for the same points $\mathbf{a}_1$ and $\mathbf{a}_2$ and the same function f, it can be shown that $\mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);f]$ has a limit as $\varepsilon \to 0$ but the limit depends on $k = 5,6,7$ and never coincides with the Taylor polynomial of f at the origin, namely we have

$$\lim_{\varepsilon\to 0} \mathbf{H}[(\mathbf{a}_1,3),(\mathbf{a}_2,1);f] = \begin{cases} y(y-x^2) & (k=5), \\ x(y-x^2) & (k=6), \\ y-x^2 & (k=7). \end{cases}$$

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