BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 64, No. 2-3, 2016

PARTIAL DIFFERENTIAL EQUATIONS

On the Existence of a Non-trivial Non-negative Global Radial Weak Solution to a Fractional Laplacian Problem with a Singular Potential

by

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Summary. We prove the existence of a non-trivial non-negative radial weak solution to the problem

$$\begin{cases} (-\Delta)^\alpha u + bu = \lambda \dfrac{u}{|x|^{2\alpha}} + |u|^{p-1}u + \mu |u|^{r-1}u \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

Here $N > 2\alpha$, $\alpha \in (1/2, 1)$, $1 < r < p < \frac{N+2\alpha}{N-2\alpha}$ and $\mu \in \mathbb{R}$. We also assume that $b > 0$ and $0 < \lambda < 4^\alpha \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})}$.

1. Introduction. In this article we show the existence of a non-trivial non-negative radial weak solution to the problem

$$\begin{cases} (-\Delta)^\alpha u + bu = \lambda \dfrac{u}{|x|^{2\alpha}} + |u|^{p-1}u + \mu |u|^{r-1}u \quad \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \\ u \in H^\alpha(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N > 2\alpha$, $\alpha \in (1/2, 1)$, $\mu \in \mathbb{R}$ and $1 < r < p < \frac{N+2\alpha}{N-2\alpha}$. We also assume that $b > 0$ and $0 < \lambda < 4^\alpha \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})}$. The function space $H^\alpha(\mathbb{R}^N)$ will be introduced later in Section 2.

2010 *Mathematics Subject Classification*: Primary 35R11; Secondary 35B25.
Key words and phrases: fractional Laplacian, Hardy-type potentials, global solutions.
Received 25 June 2016; revised 24 September 2016.
Published online 10 October 2016.

In the case of bounded smooth domains containing the origin, and zero Dirichlet conditions, the above problem has been studied for $p = \frac{N+2\alpha}{N-2\alpha}$, $b = 0$ and $\mu = 0$ in [12].

In [7] the authors consider problem (1.1) for $b = 0$, $\mu = 0$ and with more singular terms. They show that a positive solution exists. For different kinds of problems related to (1.1) the readers can consult the references in [7, 12].

In Section 2 we briefly describe the natural functional framework for problem (1.1). In Section 3 we prove the existence of a non-trivial non-negative radial weak solution to problem (1.1) via variational methods.

2. Prerequisites. In this section we gather some tools that will be used in Section 3 to prove our existence theorem. First we define, for each $\alpha \geq 0$, the fractional Sobolev space

$$H^\alpha(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\xi|^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^N)\},$$

via the Fourier transform $\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\, u(x)\, dx$. For $\alpha \in (0,1)$, it is well-known that $H^\alpha(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} bu^2\, dx + \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}(\xi)|^2\, d\xi\Big)^{1/2}, \tag{2.1}$$

$$= \Big(\int_{\mathbb{R}^N} bu^2\, dx + \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2\, dx\Big)^{1/2},$$

which is induced by the scalar product

$$\langle u, v\rangle_{H^\alpha(\mathbb{R}^N)} = \int_{\mathbb{R}^N} buv\, dx + \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} u\, (-\Delta)^{\alpha/2} v\, dx.$$

Here $(-\Delta)^{\alpha/2}$ is the fractional Laplacian defined on the Schwartz class (of rapidly decaying C^∞ functions in $\mathbb{R}^N$) through the Fourier transform,

$$(-\Delta)^{\alpha/2} u = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}u).$$

See [5] and references therein for the basics on the fractional Laplacian.

The following embedding theorem will be used. The proof can be found in [3].

THEOREM 2.1. *Assume that $N > 2\alpha$ with $\alpha \in (0,1)$. Then there is a continuous embedding*

$$H^\alpha(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \quad \text{for } 2 \leq p \leq 2^*_\alpha = \frac{2N}{N-2\alpha}. \tag{2.2}$$

The embedding (2.2) is not compact, due to the invariance of the $H^\alpha(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ norms under translations (see [4]). A natural attempt to overcome this problem is to guess that translation invariance is the only reason for the failure of compactness, and to try to work in the space of radial functions where translations are not allowed. We define

$$H^\alpha_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in H^\alpha(\mathbb{R}^N) : u \text{ is radial}\}.$$

Now we recall the following two theorems that characterize some properties of the function space $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$.

THEOREM 2.2 ([4, Theorem 7.1]). *Let $N > 2\alpha$ with $\alpha \in (0,1)$, and $p \in \big(2, \frac{2N}{N-2\alpha}\big)$. Then the embedding of $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is compact.*

THEOREM 2.3 ([4, Theorem 6.1]). *Let $N > 2\alpha$ with $\alpha \in (1/2,1)$, and $u \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$. Then u is almost everywhere equal to a continuous function in $\mathbb{R}^N - \{0\}$ that satisfies*

$$|u(x)| \le C|x|^{\alpha-N/2}\|u\|_{H^\alpha(\mathbb{R}^N)}.$$

For $N > 2\alpha$, $\alpha \in (0,1)$ and $u \in H^\alpha(\mathbb{R}^N)$ we also recall the following fractional Hardy inequality from [13] in which $\lambda_{\alpha,N} = 4^\alpha \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})}$ is the optimal constant:

$$\lambda_{\alpha,N} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2\alpha}}\,dx \le \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u(x)|^2\,dx. \tag{2.3}$$

If $0 < \lambda < \lambda_{\alpha,N}$, then by using (2.3) we can verify that the following norm is equivalent to (2.1):

$$|u|_{H^\alpha(\mathbb{R}^N)} := \bigg(\|(-\Delta)^{\alpha/2}u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} bu^2\,dx - \lambda \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2\alpha}}\,dx\bigg)^{1/2}. \tag{2.4}$$

Notice that the norm (2.4) is induced by the scalar product

$$(u|v)_{H^\alpha(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2}u\,(-\Delta)^{\alpha/2}v\,dx + \int_{\mathbb{R}^N} buv\,dx - \lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^{2\alpha}}\,dx. \tag{2.5}$$

We need a method to pass from functions in $H^\alpha(\mathbb{R}^N)$ to functions in $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$. One way to do this is called Schwarz symmetrization [2, 8, 9]. Let $N > 2\alpha$ with $\alpha \in (0,1)$. By using [8, Theorem 3.4] and [2, 9] we deduce that for every non-negative $u \in H^\alpha(\mathbb{R}^N)$ there exists $u^* \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$, $u^* \ge 0$, such that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u^*|^2\,dx \le \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u|^2\,dx, \tag{2.6}$$

$$\int_{\mathbb{R}^N} |u^*|^p\,dx = \int_{\mathbb{R}^N} |u|^p\,dx, \qquad \forall p > 1, \tag{2.7}$$

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^{2\alpha}}\,dx \le \int_{\mathbb{R}^N} \frac{{u^*}^2}{|x|^{2\alpha}}\,dx. \tag{2.8}$$

u^* is called the *Schwarz symmetrization* of u.

Also by [10, (A.11)], for every $u \in H^\alpha(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \big|(-\Delta)^{\alpha/2}|u|\big|^2\,dx \le \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u|^2\,dx. \tag{2.9}$$

Let us start with a simple observation. If H is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, then $J(u) = \|u\|^2 = \langle u,u\rangle$ on H is Fréchet differentiable at $u \in H$ and $J'(u)v = 2\langle u,v\rangle$ for all $v \in H$.

The following lemma will be used in Section 3. The proof is essentially the same as in [1, Examples 1.3.20, 1.3.21].

LEMMA 2.4. *Let $N > 2\alpha$ with $\alpha \in (0,1)$, and $1 \le p \le \frac{N+2\alpha}{N-2\alpha}$. Consider the functional $J : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$ given by*

$$J(u) = \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx.$$

Then J is Fréchet differentiable on $H^\alpha(\mathbb{R}^N)$ and

$$J'(u)v = \int_{\mathbb{R}^N} u|u|^{p-1}v\,dx \quad \text{for all } u,v \in H^\alpha(\mathbb{R}^N).$$

3. Existence result. In this section we prove the existence of a non-trivial non-negative radial weak solution to problem (1.1). First we give the following definition.

DEFINITION 3.1. We say $u \in H^\alpha(\mathbb{R}^N)$ is a *non-trivial non-negative weak solution* to problem (1.1) if $u \not\equiv 0$ a.e. in $\mathbb{R}^N$, $u \ge 0$ a.e. in $\mathbb{R}^N$, $\lim_{|x|\to\infty} u(x) = 0$, and

$$(u|v)_{H^\alpha(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u|u|^{p-1}v\,dx + \mu\int_{\mathbb{R}^N} u|u|^{r-1}v\,dx$$

for all $v \in H^\alpha(\mathbb{R}^N)$.

Notice that by the definition of the scalar product (2.5) the left hand side of the above equality is well defined. Moreover by considering the assumption $1 < r < p < \frac{N+2\alpha}{N-2\alpha}$ and using the Hölder inequality and the embedding theorem 2.1, we see that the right hand side is well defined too.

Now we have our existence theorem.

THEOREM 3.2. *Let $N > 2\alpha$, $\alpha \in (1/2,1)$, $1 < r < p < \frac{N+2\alpha}{N-2\alpha}$, $\mu \in \mathbb{R}$, $b > 0$ and $0 < \lambda < 4^\alpha \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})}$. Then problem (1.1) has a non-trivial non-negative weak solution in $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$.*

Proof. Consider the functional $I : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}(u|u)_{H^\alpha(\mathbb{R}^N)} - \int_{\mathbb{R}^N} \frac{|u|^{p+1}}{p+1}\,dx - \mu\int_{\mathbb{R}^N} \frac{|u|^{r+1}}{r+1}\,dx.$$

By Lemma 2.4 and Fréchet differentiability of the first term in the above expression it follows that I is Fréchet differentiable on $H^\alpha(\mathbb{R}^N)$. Hence

$$I'(u)v = (u|v)_{H^\alpha(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N} u|u|^{p-1}v\,dx - \mu \int\limits_{\mathbb{R}^N} u|u|^{r-1}v\,dx$$

for all $u, v \in H^\alpha(\mathbb{R}^N)$. Thus weak solutions to problem (1.1) can be found among the critical points of I. So we look for a solution of (1.1) as a minimizer of the associated functional I constrained on the Nehari manifold

$$\begin{aligned}\mathcal{N} &= \{u \in H^\alpha(\mathbb{R}^N) : u \neq 0,\, I'(u)u = 0\}\\ &= \{u \in H^\alpha(\mathbb{R}^N) : u \neq 0,\, |u|^2_{H^\alpha(\mathbb{R}^N)} = \|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \mu\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^N)}\}.\end{aligned}$$

The Nehari manifold $\mathcal{N}$ is not empty: to see this, fix $u \neq 0$ in $H^\alpha(\mathbb{R}^N)$ and define

$$h(t) = I'(tu)tu = t^2|u|^2_{H^\alpha(\mathbb{R}^N)} - t^{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} - \mu t^{r+1}\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^N)};$$

then $h(t)$ is positive for small $t > 0$ and $\lim_{t\to\infty} h(t) = -\infty$. Continuity of $h(t)$ implies that there exists $t_0 > 0$ such that $I'(t_0u)t_0u = 0$, so $t_0u \in \mathcal{N}$. Notice that if $u \in \mathcal{N}$, then

$$I(u) = \left(\frac{1}{2} - \frac{1}{r+1}\right)|u|^2_{H^\alpha(\mathbb{R}^N)} + \left(\frac{1}{r+1} - \frac{1}{p+1}\right)\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}. \tag{3.1}$$

Now define

$$m = \inf_{u\in\mathcal{N}} I(u).$$

If $u \in \mathcal{N}$, then by the embedding theorem 2.1 there exists $K > 0$ such that

$$|u|^2_{H^\alpha(\mathbb{R}^N)} = \|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \mu\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^N)} \le K\big(|u|^{p+1}_{H^\alpha(\mathbb{R}^N)} + |u|^{r+1}_{H^\alpha(\mathbb{R}^N)}\big).$$

Hence

$$1 \le K\big(|u|^{p-1}_{H^\alpha(\mathbb{R}^N)} + |u|^{r-1}_{H^\alpha(\mathbb{R}^N)}\big).$$

If $|u|_{H^\alpha(\mathbb{R}^N)} \le 1$, then the above inequality implies $1 \le 2K|u|^{r-1}_{H^\alpha(\mathbb{R}^N)}$. So we have obtained, for all $u \in \mathcal{N}$,

$$|u|_{H^\alpha(\mathbb{R}^N)} \ge \min\{1, (2K)^{-1/(r-1)}\} = C > 0. \tag{3.2}$$

On the other hand, by invoking the assumption $1 < r < p$ and using (3.1) and (3.2) we get

$$I(u) > \left(\frac{1}{2} - \frac{1}{r+1}\right)|u|^2_{H^\alpha(\mathbb{R}^N)} \ge \left(\frac{1}{2} - \frac{1}{r+1}\right)C^2 \tag{3.3}$$

for all $u \in \mathcal{N}$. Therefore m is positive.

Now we want to show that there exists $u \in \mathcal{N}$ such that $I(u) = m$. First we show that a minimizing sequence for m can be chosen from $\mathcal{N} \cap H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$. To see this, let $\{v_k\} \subset \mathcal{N}$ be any minimizing sequence, that is,

$$I(v_k) \to m \quad k \to \infty.$$

Because of (2.9) we can assume $v_k \geq 0$. Now let $w_k = v_k^* \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$ be the Schwarz symmetrization of v_k. By applying (2.6)–(2.8) we can write

$$\begin{aligned}
|w_k|^2_{H^\alpha(\mathbb{R}^N)} &= \|(-\Delta)^{\alpha/2} v_k^*\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} b v_k^{*2}\,dx - \lambda \int_{\mathbb{R}^N} \frac{|v_k^*(x)|^2}{|x|^{2\alpha}}\,dx \\
&\leq \|(-\Delta)^{\alpha/2} v_k\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} b v_k^2\,dx - \lambda \int_{\mathbb{R}^N} \frac{|v_k(x)|^2}{|x|^{2\alpha}}\,dx \\
&= |v_k|^2_{H^\alpha(\mathbb{R}^N)},
\end{aligned}$$

so for every non-negative $v_k \in H^\alpha(\mathbb{R}^N)$ we have

$$|v_k^*|^2_{H^\alpha(\mathbb{R}^N)} \leq |v_k|^2_{H^\alpha(\mathbb{R}^N)}. \tag{3.4}$$

By using (3.4) and (2.7) we get

$$\begin{aligned}
|w_k|^2_{H^\alpha(\mathbb{R}^N)} \leq |v_k|^2_{H^\alpha(\mathbb{R}^N)} &= \|v_k\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \mu\|v_k\|^{r+1}_{L^{r+1}(\mathbb{R}^N)} \\
&= \|w_k\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \mu\|w_k\|^{r+1}_{L^{r+1}(\mathbb{R}^N)}.
\end{aligned}$$

Hence if we set

$$gl(t) = t^2|w_k|^2_{H^\alpha(\mathbb{R}^N)} - \big(t^{p+1}\|w_k\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \mu t^{r+1}\|w_k\|^{r+1}_{L^{r+1}(\mathbb{R}^N)}\big),$$

we obtain $gl(1) \leq 0$, while $gl(t) > 0$ for all t positive and small. Therefore there exists $0 < t_k \leq 1$ such that $gl(t_k) = 0$, that is, $t_k w_k \in \mathcal{N}$. Thus from (3.1), (2.7) and (3.4) we obtain

$$\begin{aligned}
m &\leq I(t_k w_k) \\
&= t_k^2\Big(\frac{1}{2} - \frac{1}{r+1}\Big)|w_k|^2_{H^\alpha(\mathbb{R}^N)} + t_k^{p+1}\Big(\frac{1}{r+1} - \frac{1}{p+1}\Big)\|w_k\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\
&\leq t_k^2\Big(\frac{1}{2} - \frac{1}{r+1}\Big)|v_k|^2_{H^\alpha(\mathbb{R}^N)} + t_k^{p+1}\Big(\frac{1}{r+1} - \frac{1}{p+1}\Big)\|v_k\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\
&\leq I(v_k).
\end{aligned}$$

This implies that $t_k w_k \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$ is also a minimizing sequence for m. From (3.3) we see that $\{t_k w_k\}$ is a bounded sequence in $H^\alpha(\mathbb{R}^N)$. We set $u_k = t_k w_k$. Of course, $u_k \geq 0$, and we can assume that, up to subsequences, $u_k \rightharpoonup u$ in $H^\alpha(\mathbb{R}^N)$ weakly. By the compactness of the embedding $H^\alpha_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$ for $p \in (1, 2^*_\alpha - 1)$ (Theorem 2.2), we can deduce

$$u_k \to u \quad \text{in } L^{p+1}(\mathbb{R}^N) \text{ and } L^{r+1}(\mathbb{R}^N).$$

Again up to subsequences, $u_k(x) \to u(x)$ almost everywhere, so that $u \geq 0$ a.e. in $\mathbb{R}^N$ and $u \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$.

We now prove that the weak limit u belongs to $\mathcal{N}$ and $I(u) = m$. From (3.2) and the definition of the Nehari manifold $\mathcal{N}$ we have

$$0 < C^2 \leq |u_k|^2_{H^\alpha(\mathbb{R}^N)} = \|u_k\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \mu\|u_k\|^{r+1}_{L^{r+1}(\mathbb{R}^N)}, \tag{3.5}$$

and hence, passing to the limit,

$$0 < C^2 \leq \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \mu\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1}.$$

This implies $u \not\equiv 0$ a.e. in $\mathbb{R}^N$. Now from (3.5), we also get

$$|u|_{H^\alpha(\mathbb{R}^N)}^2 \leq \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \mu\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1}.$$

If

$$|u|_{H^\alpha(\mathbb{R}^N)}^2 = \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \mu\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1},$$

then $u \in \mathcal{N}$. Towards a contradiction, assume that

$$|u|_{H^\alpha(\mathbb{R}^N)}^2 < \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \mu\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1}.$$

For $t > 0$, let

$$h(t) = I'(tu)tu = t^2|u|_{H^\alpha(\mathbb{R}^N)}^2 - t^{p+1}\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} - \mu t^{r+1}\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1}.$$

Then $h(t) > 0$ for small $t > 0$, while

$$h(1) = I'(u)u = |u|_{H^\alpha(\mathbb{R}^N)}^2 - \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} - \mu\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1} < 0.$$

Thus, there is $0 < t < 1$ such that $tu \in \mathcal{N}$. Hence by the weak lower semicontinuity of the norm (2.4), that is, $|u|_{H^\alpha(\mathbb{R}^N)} \leq \liminf_{n\to\infty} |u_n|_{H^\alpha(\mathbb{R}^N)}$ for every sequence $u_n \in H^\alpha(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^N)$ weakly, we deduce that

$$\begin{aligned}
I(tu) &= t^2\left(\frac{1}{2} - \frac{1}{r+1}\right)|u|_{H^\alpha(\mathbb{R}^N)}^2 + t^{p+1}\left(\frac{1}{r+1} - \frac{1}{p+1}\right)\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\
&< \left(\frac{1}{2} - \frac{1}{r+1}\right)|u|_{H^\alpha(\mathbb{R}^N)}^2 + \left(\frac{1}{r+1} - \frac{1}{p+1}\right)\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\
&\leq \liminf_{k\to\infty}\left(\frac{1}{2} - \frac{1}{r+1}\right)|u_k|_{H^\alpha(\mathbb{R}^N)}^2 \\
&\quad + \lim_{k\to\infty}\left(\frac{1}{r+1} - \frac{1}{p+1}\right)\|u_k\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \\
&\leq \liminf_{k\to\infty}\left(\left(\frac{1}{2} - \frac{1}{r+1}\right)|u_k|_{H^\alpha(\mathbb{R}^N)}^2 + \left(\frac{1}{r+1} - \frac{1}{p+1}\right)\|u_k\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}\right) \\
&= \liminf_{k\to\infty} I(u_k) = m.
\end{aligned}$$

This contradiction proves that $|u|_{H^\alpha(\mathbb{R}^N)}^2 = \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \mu\|u\|_{L^{r+1}(\mathbb{R}^N)}^{r+1}$. Therefore $u \in \mathcal{N}$. Again, by the weak lower semicontinuity of the norm it is easy to check that $I(u) \leq \liminf_{k\to\infty} I(u_k) = m$, and therefore $I(u) = m$.

Now we claim that the minimum u obtained above is a critical point of I in $H^\alpha(\mathbb{R}^N)$. To see this, fix $v \in H^\alpha(\mathbb{R}^N)$ and $\epsilon > 0$ such that $u + sv \neq 0$ for all $s \in (-\epsilon, \epsilon)$. Define $\phi : (-\epsilon, \epsilon) \times (0, \infty) \to \mathbb{R}$ by

$$\phi(s, t) = I'(t(u + sv))t(u + sv).$$

Then $\phi(0,1) = |u|^2_{H^\alpha(\mathbb{R}^N)} - \|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} - \mu\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^N)} = 0$, and

$$\begin{aligned}\frac{\partial\phi}{\partial t}(0,1) &= 2|u|^2_{H^\alpha(\mathbb{R}^N)} - (p+1)\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} - (r+1)\mu\|u\|^{r+1}_{L^{r+1}(\mathbb{R}^N)} \\ &= (1-r)|u|^2_{H^\alpha(\mathbb{R}^N)} + (r-p)\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} < 0.\end{aligned}$$

Therefore, by the Implicit Function Theorem there exists a C^1 function $t : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ such that $t(0) = 1$ and $\phi(s, t(s)) = 0$ for all $s \in (-\epsilon_0, \epsilon_0)$. Defining

$$\gamma(s) = I(t(s)(u + sv)),$$

we see that γ is differentiable and has a minimum point at $s = 0$, therefore

$$0 = \gamma'(0) = I'(t(0)u)(t'(0)u + t(0)v) = I'(u)v.$$

Since this holds for all $v \in H^\alpha(\mathbb{R}^N)$, we have $I'(u) = 0$. So $u \in H^\alpha_{\mathrm{rad}}(\mathbb{R}^N)$ is a critical point of I. Also by using Theorem 2.3, we obtain

$$|u(x)| \le C_0|x|^{\alpha - N/2}$$

for some $C_0 > 0$ and any $x \neq 0$. The above estimate and the assumption $N > 2\alpha$ imply that $\lim_{|x|\to\infty} u(x) = 0$. Therefore u is a non-trivial non-negative radial weak solution of problem (1.1). ■

Remark 3.3. It can be proved that the non-negative solution obtained above is actually everywhere positive in $\mathbb{R}^N$. One way to prove this is the following. We construct recursively a sequence $\{u_n\}$ starting with

$$\begin{cases} (-\Delta)^\alpha u_1 + bu_1 = T_1(u^p + \mu u^r) & \text{in } \mathbb{R}^N, \\ u_1 \ge 0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u_1(x) = 0, \end{cases}$$

where u is the weak solution of (1.1) obtained above and

$$T_k(s) = \begin{cases} s, & |s| \le k, \\ k\,\mathrm{sign}(s), & |s| \ge k, \end{cases}$$

is the usual truncation operator. By iteration we define, for $n > 1$,

$$\begin{cases} (-\Delta)^\alpha u_n + bu_n = \lambda\dfrac{u_{n-1}}{|x|^{2\alpha} + 1/n} + T_{n-1}(u^p + \mu u^r) & \text{in } \mathbb{R}^N, \\ u_n \ge 0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u_n(x) = 0. \end{cases}$$

By [6, Proposition 5.1.1] one can show that $u_n \in L^\infty(\mathbb{R}^N)$. By [11, Theorem 1] one can show that u_n is also a solution in the viscosity sense. Then the Strong Maximum Principle [11, Proposition 5.2.1] implies that u_n is everywhere positive in $\mathbb{R}^N$. On the other hand, by using [11, Lemma 6], we can deduce that $u_1 \le \cdots \le u_n \le u$ in $\mathbb{R}^N$. Therefore, $u > 0$ in $\mathbb{R}^N$.

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