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On the Ritt property and weak type maximal inequalities for convolution powers on $\ell^1(\mathbb{Z})$

by

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Abstract. We study the behaviour of convolution powers of probability measures μ on $\mathbb{Z}$ such that $(\mu(n))_{n\in\mathbb{N}}$ is completely monotone or such that μ is centred with a second moment. In particular we exhibit many new examples of probability measures on $\mathbb{Z}$ having the so-called Ritt property and whose convolution powers satisfy weak type maximal inequalities in $\ell^1(\mathbb{Z})$.

1. Introduction. Let μ be a probability on $\mathbb{Z}$. Given an invertible bimeasurable transformation τ on a measure space $(\mathbb{S}, \mathcal{S}, \lambda)$ we define a *positive contraction* of every $L^p(\lambda)$, $1 \le p \le \infty$, by setting

$$P_\mu(\tau)(f) := \sum_{k\in\mathbb{Z}} \mu(k) f\circ\tau^k \qquad \forall f \in L^p(\lambda).$$

Several authors (see for instance [4], [5], [22], [6], [3], [16], [20], [26], [24], [23] or [7]) studied the almost everywhere behaviour of the iterates of $\mu(\tau)$, i.e. of $(\mu^{*n}(\tau))_{n\ge1}$, acting on $L^p(\lambda)$, $1 \le p < \infty$.

When $p > 1$, the almost everywhere behaviour has been characterized by Losert [20] thanks to the so-called *bounded angular ratio* property, introduced by Bellow–Jones–Rosenblatt [6] and which is equivalent to the so-called *Ritt property* on $\ell^p(\mathbb{Z})$, $p > 1$. Let us recall the definition of those properties.

Definition 1.1. Let μ be a probability measure on $\mathbb{Z}$. We say that μ is *strictly aperiodic* if $|\hat\mu(\theta)| < 1$ for every $\theta \in (0, 2\pi)$. We say that μ has *bounded angular ratio* (BAR) if moreover

$$\sup_{\theta\in(0,2\pi)} \frac{|1-\hat\mu(\theta)|}{1-|\hat\mu(\theta)|} < \infty. \tag{1.1}$$

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The strict aperiodicity is equivalent to the support of μ not being contained in a coset of a proper subgroup of $\mathbb{Z}$. In particular, it holds whenever the support of μ contains two consecutive integers.

Definition 1.2. We say that a probability measure μ on $\mathbb{Z}$ is *Ritt on* $\ell^p(\mathbb{Z})$, for some $p \ge 1$, if

$$\sup_{n\ge 1} n\|\mu^{*n} - \mu^{*(n+1)}\|_{\ell^p(\mathbb{Z})} < \infty.$$

When $p = 1$ we say simply that μ is *Ritt*, because then it is Ritt on all $\ell^r(\mathbb{Z})$, $r \ge 1$. Denote by $\mathcal{R}$ the set of Ritt probability measures on $\mathbb{Z}$.

A version of the next theorem may be found for instance in Cohen, Cuny and Lin [7, Theorem 4.3]. Their result is not formulated exactly as below but the proof of Theorem 1.3 may be done similarly. The equivalence of (vi) with the other items follows from [7, Proposition 6.4]. Throughout we use the notation $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Theorem 1.3. *Let μ be a strictly aperiodic probability on $\mathbb{Z}$. The following are equivalent:*

(i) *μ has BAR.*

(ii) *There exist $p > 1$ and $C_p > 0$ such that for every invertible bi-measurable transformation τ on a measure space $(\mathbb{S}, \mathcal{S}, \lambda)$,*

$$\Big\|\sup_{n\ge 1} |(P_\mu(\tau))^n f|\Big\|_{p,\lambda} \le C_p\|f\|_{p,\lambda} \quad \forall f \in L^p(\lambda). \tag{1.2}$$

(iii) *There exists $p > 1$ such that for every invertible bi-measurable transformation τ on a probability space $(\mathbb{S}, \mathcal{S}, \lambda)$ and every $f \in L^p(\lambda)$, $((P_\mu(\tau))^n f)_{n\in\mathbb{N}}$ converges λ-a.e.*

(iv) *There exist $p > 1$ and $C_p > 0$ such that*

$$\Big\|\sup_{n\ge 1} |(P_\mu(R))^n f|\Big\|_{\ell^p(\mathbb{Z})} \le C_p\|f\|_{\ell^p(\mathbb{Z})}, \qquad f \in \ell^p(\mathbb{Z}), \tag{1.3}$$

where R is the right shift on $\mathbb{Z}$.

(v) *There exists $p > 1$ such that μ is Ritt on $\ell^p(\mathbb{Z})$.*

(vi) *There exists $p > 1$ such that, for every $m \in \mathbb{N}$, there exists $C_{m,p} > 0$ such that*

$$\Big\|\sup_{n\ge 1} n^m |(I - P_\mu)^m (P_\mu(R))^n f|\Big\|_{\ell^p(\mathbb{Z})} \le C_{m,p}\|f\|_{\ell^p(\mathbb{Z})}, \qquad f \in \ell^p(\mathbb{Z}).$$

Actually, if any of the above properties holds, then the conclusions of (ii), (iii), (iv) and (v) hold for all $p > 1$.

The proof of the above theorem follows from recent work of Le Merdy and Xu [17], [18], who studied positive Ritt contractions T of $L^p(\mathbb{S}, \mathcal{S}, \lambda)$ (p being fixed). Recall that a contraction T on a Banach space X is *Ritt* if $\sup_{n\in\mathbb{N}} n\|T^n - T^{n+1}\|_X < \infty$; this is compatible with our Definition 1.2 which just says that the operator of convolution by μ is Ritt on $X = \ell^p(\mathbb{Z})$.

Le Merdy and Xu proved that any positive Ritt contraction satisfies maximal inequalities in the spirit of (1.2). They also obtained square function estimates, oscillation inequalities and variation inequalities. See also [7] for related results.

In this paper we are concerned with the case when $p = 1$, and we address the following two questions.

Question 1. For what probability measures μ on $\mathbb{Z}$ does one have a weak type $(1,1)$-maximal inequality:

$$\sup_{\lambda>0} \lambda \#\Big\{k \in \mathbb{Z} : \sup_{n\geq 1} |\mu^{*n} * f| > \lambda\Big\} \leq C\|f\|_{\ell^1(\mathbb{Z})} \qquad \forall f \in \ell^1(\mathbb{Z})\ ? \tag{1.4}$$

More generally, given $m \in \mathbb{N}$, does there exist $C_m > 0$ such that (with the convention $(\delta_0 - \mu)^{*0} = \delta_0$)

$$\begin{aligned} \sup_{\lambda>0} \lambda \#\Big\{k \in \mathbb{Z} : \sup_{n\geq 1} n^m |\mu^{*n} * (\delta_0 - \mu)^{*m} * f(k)| \geq \lambda\Big\} \\ \leq C_m\|f\|_{\ell^1(\mathbb{Z})} \qquad \forall f \in \ell^1(\mathbb{Z}). \end{aligned} \tag{1.5}$$

Question 2. For what probability measures μ on $\mathbb{Z}$ does one have the Ritt property in $\ell^1(\mathbb{Z})$:

$$\sup_{n\geq 1} n\|\mu^{*n} - \mu^{*(n+1)}\|_{\ell^1(\mathbb{Z})} < \infty\ ? \tag{1.6}$$

Notice that if μ satisfies (1.4) then, by the Marcinkiewicz interpolation theorem (between weak L^1 and L^∞), it does satisfy (1.2), hence μ has BAR. Notice also that if μ satisfies (1.6) then by Theorem 1.3 it has BAR as well. Hence, the questions we intend to answer are: what extra conditions, in addition to the BAR property, are sufficient to have (1.4), (1.5) or (1.6)?

Let us discuss the known results concerning those questions, before presenting our results. As far as we know, when $m \geq 1$, (1.5) has not been investigated before.

The simplest examples of probability measures having BAR are the symmetric ones. Bellow, Jones and Rosenblatt [6] proved that if μ is symmetric such that $(\mu(n))_{n\geq 0}$ is non-increasing then (1.4) holds. We do not know whether (1.6) holds as well in this case, but we provide sufficient conditions in Section 5.

Another case where (1.4) holds is when $\sum_{k\in\mathbb{Z}} k^2\mu(k) < \infty$ (i.e. μ has a second moment) and $\sum_{k\in\mathbb{Z}} k\mu(k) = 0$. This has been proved by Bellow and Calderón [3]. Again the Ritt property is not known in that case. The proof of Bellow and Calderón is based on general intermediary results that have recently been extended by Wedrychowicz [26]. He proved that (1.4) holds for centred probability measures (hence with a first moment) having BAR and satisfying some extra conditions. Examples without second moment are also presented in [26].

Several examples of probabilities having the Ritt property in $\ell^1(\mathbb{Z})$ may be found in Dungey [10, Sections 4 and 5].

Let us now present our results. As mentioned above, the method of Bellow and Calderón is fairly general. Actually, if one follows their paper carefully, the following definition comes naturally into play.

Definition 1.4. We say that a probability measure μ on $\mathbb{Z}$ satisfies *hypothesis* $(\mathbf{H})$ if $\hat{\mu}$ is twice continuously differentiable on $[-\pi,\pi]-\{0\}$ and there exists an even and continuous function ψ on $[-\pi,\pi]$, vanishing at 0 and continuously differentiable on $[-\pi,\pi]-\{0\}$, and some constants $c, C>0$ such that for every $\theta\in(0,\pi]$:

$(\mathbf{H})$(i) $|\hat{\mu}(\theta)|\le 1-c\psi(\theta)$;
$(\mathbf{H})$(ii) $|\theta\hat{\mu}'(\theta)|\le C\psi(\theta)$;
$(\mathbf{H})$(iii) $|\hat{\mu}'(\theta)|\le C\psi'(\theta)$;
$(\mathbf{H})$(iv) $|\theta\hat{\mu}''(\theta)|\le C\psi'(\theta)$.

Denote by $\mathcal{H}$ the set of probability measures satisfying hypothesis $(\mathbf{H})$.

The relevance of hypothesis $(\mathbf{H})$ lies in the following, where we also give stability properties of $\mathcal{H}$ as well as of $\mathcal{R}$. We say that a set of probability measures on $\mathbb{Z}$ is *stable by symmetrization* if whenever $\mu=(\mu(n))_{n\in\mathbb{Z}}$ belongs to that set, so does $\check{\mu}=(\mu(-n))_{n\in\mathbb{Z}}$.

Theorem 1.5.

(i) *The set $\mathcal{H}$ is convex and stable by convolution and by symmetrization.*
(ii) *The set $\mathcal{R}$ is convex and stable by convolution and by symmetrization.*
(iii) *Let $\mu\in\mathcal{H}$. Then μ satisfies* (1.4).
(iv) *Let $\mu\in\mathcal{H}\cap\mathcal{R}$. Then, for every $m\in\mathbb{N}$, there exists $C_m>0$ such that μ satisfies* (1.5).

Theorem 1.5 follows from several results: item (i) follows from Proposition 2.4, item (ii) may be proved just as Proposition 3.15, and items (iii) and (iv) follow from Proposition 2.3.

Our goal is to provide many examples of elements of $\mathcal{H}\cap\mathcal{R}$. The first ones are those already considered by Bellow and Calderón; in particular the fact that μ as in the next theorem satisfies (1.4) is not new, while the Ritt property is new. The proof of Theorem 1.6 is given in Section 2.3.

Theorem 1.6. *Let μ be a centred and strictly aperiodic probability measure on $\mathbb{Z}$ with finite second moment. Then $\mu\in\mathcal{H}\cap\mathcal{R}$.*

Next, we shall consider probability measures μ such that $(\mu(n))_{n\geq 0}$ is completely monotone (see the next section for the definition). In this context we are able to characterize the BAR property. The idea of considering completely monotone sequences was motivated by Gomilko–Haase–Tomilov [12] and Cohen–Cuny–Lin [7].

THEOREM 1.7. *Let μ be a probability measure on $\mathbb{Z}$ supported on $\mathbb{N}$, such that $(\mu(n))_{n\in\mathbb{N}}$ is completely monotone. Then:*

(i) *μ has BAR if and only if there exists $C > 0$ such that*

$$\sum_{k=1}^{n} k\mu(k) \leq Cn \sum_{k\geq n} \mu(k) \quad \forall n \geq 1. \tag{1.7}$$

(ii) *Assume that μ has BAR. Let σ be a probability measure on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} n^2\sigma(n) < \infty$. Then $\mu*\sigma \in \mathcal{H}\cap\mathcal{R}$ and for every $\alpha \in (0,1]$, $\alpha\mu + (1-\alpha)\sigma \in \mathcal{H}\cap\mathcal{R}$. In particular (take $\sigma = \delta_0$), $\mu \in \mathcal{H}\cap\mathcal{R}$.*

REMARKS. Notice that we do not assume σ to be centred. The conclusion of (ii) actually holds for σ such that $\hat{\sigma}$ is twice continuously differentiable on $[-\pi,\pi]-\{0\}$ with $\hat{\sigma}'$ and $\theta \mapsto \theta\hat{\sigma}''(\theta)$ bounded. Moreover (see Proposition 3.15), it is possible to relax the conditions on $\hat{\sigma}$ if one is only concerned with the Ritt property. We have not been able to provide a perturbation result in the spirit of Theorem 1.8 below.

Item (i) of Theorem 1.7 follows from Propositions 3.7 and 3.9. Item (ii) is proved in Sections 3.2 and 3.3.

The proof of the Ritt property in Theorem 1.7 is based on a recent result of Gomilko and Tomilov [14]. The fact that if μ has BAR then $\delta_1 * \mu$ is Ritt has been proven by Gomilko and Tomilov [15, Theorem 7.1]. Their proof is also based on [14].

THEOREM 1.8. *Let μ be a* centred *probability measure on $\mathbb{Z}$ supported on $\{-1\}\cup\mathbb{N}$ such that $(\mu(n))_{n\in\mathbb{N}}$ is completely monotone. Then:*

(i) *μ has BAR if and only if there exists $C > 0$ such that*

$$n\sum_{k\geq n} k\mu(k) \leq C\sum_{k=1}^{n} k^2\mu(k) \quad \forall n \geq 1. \tag{1.8}$$

(ii) *Assume that μ has BAR. Let σ be a centred probability measure on $\mathbb{Z}$ such that there exists $a > 0$ with $\sum_{n\in\mathbb{Z}} n^2|\sigma(n) - a\mu(n)| < \infty$. Then $\sigma \in \mathcal{H}\cap\mathcal{R}$. In particular (take $a = 1$ and $\sigma = \mu$), $\mu \in \mathcal{H}\cap\mathcal{R}$.*

Moreover (see Section 5), we also study symmetric probability measures with completely monotone coefficients.

In the above theorems, we obtain weak type maximal inequalities in $\ell^1(\mathbb{Z})$. Of course, by means of transference principles (see e.g. [2] or [25,

p. 164], one may derive similar results for the operator $P_\mu(\tau)$ in the spirit of (1.2) as well as some almost everywhere convergence results. We leave that "standard" task to the reader.

The paper is organized as follows. In Section 2, we prove Theorem 1.5 and prove the Ritt property under a slightly stronger assumption than hypothesis (**H**). In Section 3, we consider probability measures as in Theorem 1.7 and prove Theorem 1.7. In Section 4, we consider probability measures as in Theorem 1.8 and prove Theorem 1.8. In Section 5 we deal with symmetric probability measures. Finally, in Section 6 we discuss several open questions.

Before going into the proofs, we mention that the above theorems provide new situations to which the results of Cuny and Lin [9] apply (see Examples 1 and 2 there).

2. General criteria for maximal inequalities and for the Ritt property. In this section we give general conditions ensuring weak type maximal inequalities associated with sequences of probabilities on $\mathbb{Z}$ and conditions ensuring the Ritt property.

In the case of weak type maximal inequalities, the conditions obtained are derived from slight modifications of known results (see e.g. [3] and [26]).

2.1. Sufficient conditions for weak type maximal inequalities. We start with the following result of Bellow and Calderón [3] (see also Zo [28] for a related result). Actually, Bellow and Calderón only considered the case of probability measures, but their proof extends to the situation below.

THEOREM 2.1 (Bellow–Calderón). *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite* signed *measures on $\mathbb{Z}$ such that $\sup_{n\in\mathbb{N}}\|\sigma_n\|_{\ell^1}<\infty$. Assume that there exists $C>0$ such that for any $k,\ell\in\mathbb{Z}$ with $0<2|k|\le\ell$,*

$$|\sigma_n(k+\ell)-\sigma_n(\ell)|\le C\frac{k}{\ell^2}\qquad\forall n\in\mathbb{N}. \tag{2.1}$$

Then there exists $C'>0$ such that for every $f\in\ell^1(\mathbb{Z})$,

$$\#\Big\{k\in\mathbb{Z}:\sup_{n\in\mathbb{N}}|\sigma_n*f(k)|\ge\lambda\Big\}\le\frac{C'}{\lambda}\|f\|_{\ell^1}.$$

In order to apply Theorem 2.1 we shall need the following version of [3, Corollary 3.4].

LEMMA 2.2. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite signed measures such that for every $n\in\mathbb{N}$, $\hat\sigma_n$ is twice continuously differentiable on $\mathbb{R}-2\pi\mathbb{Z}$. If moreover*

$$\sup_{n\in\mathbb{N}}\int_{-\pi}^{\pi}|\theta\hat\sigma_n''(\theta)|\,d\theta<\infty \tag{2.2}$$

and

$$\lim_{\theta\to 0,\,\theta\neq 0} \theta\hat\sigma_n'(\theta) = 0 \qquad \forall n \in \mathbb{N}, \tag{2.3}$$

then (2.1) *holds.*

REMARK. It follows from (2.2) (and the continuity of $\hat\sigma_n$ at 0) that the limit in (2.3) exists, hence condition (2.3) is just that the limit is 0.

Proof of Lemma 2.2. For every $k \in \mathbb{Z} - \{0\}$, we have

$$\sigma_n(k) = \int_{-\pi}^{\pi} \hat\sigma_n(\theta) e^{-ik\theta}\, d\theta.$$

Let $\pi > \varepsilon > 0$. Performing two integrations by parts as in [3] to evaluate $\int_\varepsilon^\pi \hat\sigma_n(\theta)e^{-ik\theta}\, d\theta$ and $\int_{-\pi}^{-\varepsilon} \hat\sigma_n(\theta)e^{-ik\theta}\, d\theta$, using our assumptions and letting $\varepsilon \to 0$, we see that

$$\sigma_n(k) = \int_{-\pi}^{\pi} \hat\sigma_n''(\theta) \frac{1-e^{-ik\theta}}{k^2}\, d\theta.$$

Then we conclude the proof as in [3]. ∎

PROPOSITION 2.3. *Let μ be a probability measure on $\mathbb{Z}$ satisfying hypothesis* (**H**). *Then, for every $m \in \mathbb{N}$,*

$$\sup_{n\geq 1} n^m \int_{-\pi}^{\pi} |\theta|\, |(\hat\mu^n(1-\hat\mu)^m)''(\theta)|\, d\theta < \infty \tag{2.4}$$

and

$$\lim_{\theta\to 0,\,\theta\neq 0} \theta(\hat\mu^n(1-\hat\mu)^m)'(\theta) = 0 \qquad \forall n \in \mathbb{N}, \tag{2.5}$$

In particular, there exists $C > 0$ such that for every $f \in \ell^1(\mathbb{Z})$,

$$\sup_{\lambda>0} \lambda \#\Big\{k \in \mathbb{Z} : \sup_{n\geq 1} |\mu^{*n} * f(k)| \geq \lambda\Big\} \leq C\|f\|_{\ell^1}. \tag{2.6}$$

If moreover μ is Ritt then, for every $m \geq 1$, there exists $C_m > 0$ such that for every $f \in \ell^1(\mathbb{Z})$,

$$\sup_{\lambda>0} \lambda \#\Big\{k \in \mathbb{Z} : \sup_{n\geq 1} n^m|\mu^{*n} * (\delta_0 - \mu)^{*m} * f(k)| \geq \lambda\Big\} \leq C_m\|f\|_{\ell^1}. \tag{2.7}$$

REMARKS. The proposition is related to Theorem 2.10 of Wedrychowicz [26]. Notice that, by (**H**)(ii) of Definition 1.4, ψ is non-negative and by (**H**)(iii) it is non-decreasing. We shall see in Proposition 2.5 that if there exists $C > 0$ such that $\psi(\theta) \leq C\theta\psi'(\theta)$ for every $\theta \in (0, \pi]$, then μ is automatically Ritt.

Proof of Proposition 2.3. If $\mu = \delta_0$ the result is trivial. Hence we assume that $\mu \neq \delta_0$. In particular, by (**H**)(ii), ψ cannot vanish in a neighbourhood of 0, hence is positive on $(0,\pi]$. Then $|\hat{\mu}|<1$ on $(0,\pi]$ (so μ is strictly aperiodic).

We have, on $(0,\pi]$,

$$
\begin{aligned}
(2.8)\qquad & n^m|(\hat{\mu}^n(1-\hat{\mu})^m)''| \\
&\le n^{m+2}|\hat{\mu}|^{n-2}|\hat{\mu}'|^2|1-\hat{\mu}|^m+2mn^{m+1}|\hat{\mu}|^{n-1}|\hat{\mu}'|^2|1-\hat{\mu}|^{m-1}\\
&\quad+n^{m+1}|\hat{\mu}|^{n-1}|\hat{\mu}''|\,|1-\hat{\mu}|^{m-1}+mn^m|\hat{\mu}|^n|\hat{\mu}''|\,|1-\hat{\mu}|^{m-1}\\
&\quad+m(m-1)n^m|\hat{\mu}|^n|\hat{\mu}'|^2|1-\hat{\mu}|^{m-2}.
\end{aligned}
$$

Using $(\mathbf{H})$(i) and the fact that ψ is continuous with $\psi(0)=0$, we deduce that there exist $\eta,c>0$ such that for every $\theta\in[0,\eta]$,

$$
|\hat{\mu}(\theta)|\le e^{-c\psi(\theta)}. \tag{2.9}
$$

Since $\sup_{\theta\in[\eta,\pi]}|\hat{\mu}(\theta)|<1$, taking c smaller if necessary, we may assume that (2.9) holds for every $\theta\in[0,\pi]$.

Using $(\mathbf{H})$(iii), as ψ is continuous at 0, we see that ψ' and $\hat{\mu}'$ are in L^1, hence

$$
|1-\hat{\mu}(\theta)|\le\psi(\theta)\qquad\forall\theta\in(0,\pi]. \tag{2.10}
$$

Using $(\mathbf{H})$(ii) and $(\mathbf{H})$(iii) we find that for every $\theta\in(0,\pi]$,

$$
\theta|\hat{\mu}'(\theta)|^2\le C^2\psi(\theta)\psi'(\theta). \tag{2.11}
$$

Combining (2.8)–(2.11) and $(\mathbf{H})$(iv) (be careful with the cases $m=0$ and $m=1$), we see that there exists $C_m>0$ such that for all $\theta\in(0,\pi]$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
&n^m|\theta(\hat{\mu}^n(1-\hat{\mu})^m)''(\theta)|\\
&\qquad\le C_m\big(n^{m+1}\psi^{m+1}(\theta)+n^m\psi^m(\theta)+n^{m-1}\psi^{m-1}(\theta)\big)e^{-c(n-2)\psi(\theta)}n\psi'(\theta).
\end{aligned}
$$

Using the fact that the integrand below is even and the change of variable $u=(n-2)\psi(\theta)$, we infer that

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}} n^m\int_{-\pi}^{\pi}|\theta|\,|(\hat{\mu}^n(1-\hat{\mu})^m)''(\theta)|\,d\theta\\
&\qquad\qquad\le\tilde{C}_m\int_0^\infty(u^{m+1}+u^m+(m-1)u^{m-1})\mathrm{e}^{-cu}\,du<\infty,
\end{aligned}
$$

and (2.4) holds.

Formula (2.5) follows from $(\mathbf{H})$(ii), by using the facts that ψ is continuous at 0, with $\psi(0)=0$, and that $\hat{\mu}$ is bounded. Notice that (2.5) implies (2.3) with $(\sigma_n)_{n\in\mathbb{N}}=(n^m\mu^{*n}*(\delta_0-\mu)^{*m})_{n\in\mathbb{N}}$.

Let $m\in\mathbb{N}$ and set $\sigma_n=\sigma_{n,m}:=n^m\mu^{*n}(\delta_0-\mu)^{*m}$ for every $n\in\mathbb{N}$.

It follows from Theorem 2.1 and Lemma 2.2 that (2.6) holds.

When $m \geq 1$, it follows from Theorem 2.1 and Lemma 2.2 that (2.7) holds provided that

$$\sup_{n\in\mathbb{N}} \|\sigma_{n,m}\|_{\ell^1(\mathbb{Z})} < \infty. \tag{2.12}$$

When $m = 1$, (2.12) is just the definition of the Ritt property.

Let $m \geq 2$, write $n = m\ell + k$ with $\ell \in \mathbb{N}$ and $0 \leq k \leq m-1$. We have

$$\|n^m \mu^{*n} * (\delta_0 - \mu)^{*m}\|_{\ell^1(\mathbb{Z})} \leq m^m \|(\ell+1)\mu^{*\ell} * (\delta_0 - \mu)\|^m_{\ell^1(\mathbb{Z})},$$

and the latter is bounded uniformly with respect to $\ell \in \mathbb{N}$, as μ is Ritt. ∎

To conclude this subsection we shall study stability properties of sets of probabilities satisfying weak type maximal inequalities.

It is well-known (see e.g. [10, Proposition 3.2]) that the set of Ritt probability measures on $\mathbb{Z}$ is convex and stable by convolution. Actually, [10] deals with probability measures supported on $\mathbb{N}$, but the proof is the same.

Let $p > 1$. It is not difficult to see that the set of probability measures μ on $\mathbb{Z}$ such that there exists $C_p > 0$ such that (1.3) holds is also convex and stable by convolution.

However, it is unclear (and probably not true) whether the set of probability measures μ on $\mathbb{Z}$ satisfying (2.7) for every $m \in \mathbb{N}$ (or for some $m \in \mathbb{N}$) is convex and stable by convolution. Nevertheless, we have the following.

Proposition 2.4. *Let μ_1 and μ_2 be probability measures satisfying hypothesis* (**H**). *Let $\alpha \in (0,1)$. Then $\check{\mu}_1$, $\mu_1 * \mu_2$ and $\alpha\mu_1 + (1-\alpha)\mu_2$ satisfy hypothesis* (**H**).

Remark. Recall that $\check{\mu}_1$ is the probability measure defined by $\check{\mu}_1(n) = \mu_1(-n)$ for every $n \in \mathbb{Z}$.

Proof of Proposition 2.4. The fact that $\check{\mu}_1$ satisfies hypothesis (**H**) is obvious.

Let ψ_i, c_i, C_i be the terms associated with μ_i ($i \in \{1,2\}$) such that items (i)–(iv) of hypothesis (**H**) are satisfied.

Define $\mu := \mu_1 * \mu_2$ and $\psi := c_1\psi_1 + c_2\psi_2$. Let $\theta \in (0,\pi]$. We have

$$|\hat{\mu}(\theta)| = |\hat{\mu}_1(\theta)|\,|\hat{\mu}_2(\theta)| \leq 1 - \psi(\theta) + c_1c_2\psi_1(\theta)\psi_2(\theta).$$

Since ψ_1 and ψ_2 are continuous with $\psi_1(0) = \psi_2(0) = 0$, there exist $c \in (0,1)$ and $\eta \in (0,\pi)$ such that $c_1c_2\psi_1(\theta)\psi_2(\theta) \leq (1-c)\psi(\theta)$ for every $\theta \in (0,\eta)$.

Hence, $|\hat{\mu}| \leq 1 - c\psi$ on $(0,\eta)$. Arguing as in the previous proof, we see that on taking c smaller if necessary, the inequality holds on $(0,\pi]$ too.

Using $\hat{\mu}' = \hat{\mu}_1'\hat{\mu}_2 + \hat{\mu}_1\hat{\mu}_2'$, we infer that (**H**)(ii) and (**H**)(iii) hold.

We have $\hat{\mu}'' = \hat{\mu}_1''\hat{\mu}_2 + 2\hat{\mu}_1'\hat{\mu}_2' + \hat{\mu}_1\hat{\mu}_2''$. Hence, for every $\theta \in (0,\pi]$,

$$|\theta\hat{\mu}''(\theta)| \leq C_1\psi_1'(\theta) + 2C_1\psi_1(\theta)C_2\psi_2'(\theta) + C_2\psi_2'(\theta),$$

and we see that (**H**)(iv) holds, since ψ_1 is bounded.

Let $\alpha \in (0,1)$. Let $\mu := \alpha\mu_1 + (1-\alpha)\mu_2$. One can see that $(\mathbf{H})$(i)–(iv) hold with $\psi := \alpha\psi_1 + (1-\alpha)\psi_2$. ■

2.2. A sufficient condition for the Ritt property. In this subsection we derive a condition ensuring that a probability measure is Ritt. This condition will be used for centred probability measures with either a second moment, or a first moment and completely monotone coefficients. For non-centred probability measures another argument will be needed.

We start with a general result.

PROPOSITION 2.5. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite signed measures on $\mathbb{Z}$ such that for every $n \in \mathbb{N}$, $\hat{\sigma}_n$ is twice differentiable on $\mathbb{R} - 2\pi\mathbb{Z}$. Assume moreover that*

(i) $\displaystyle \sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} \frac{|\hat{\sigma}_n(\theta)|}{|\theta|}\, d\theta < \infty$;

(ii) $\displaystyle \sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\hat{\sigma}_n'(\theta)|\, d\theta < \infty$;

(iii) $\displaystyle \sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\theta|\, |\hat{\sigma}_n''(\theta)|\, d\theta < \infty$.

Then $\sup_{n\in\mathbb{N}} \|\sigma_n\|_{\ell^1(\mathbb{Z})} < \infty$.

Proof. We first notice that, by (i),

$$\sup_{n\in\mathbb{N}} |\sigma_n(0)| \le \sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)|\, d\theta < \infty.$$

Let $k \in \mathbb{Z} - \{0\}$. We have

$$\begin{aligned}
\sigma_n(k) &= \int_{-\pi}^{\pi} \hat{\sigma}_n(\theta) e^{-ik\theta}\, d\theta \\
&= \int_{-\pi/|k|}^{\pi/|k|} \hat{\sigma}_n(\theta) e^{-ik\theta}\, d\theta + \int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n(\theta) e^{-ik\theta}\, d\theta.
\end{aligned}$$

Integrating by parts and using the fact that $\hat{\sigma}_n$ is 2π-periodic, we obtain

$$\begin{aligned}
\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n(\theta) e^{-ik\theta}\, d\theta &= -\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n'(\theta) \frac{e^{-ik\theta}}{-ik}\, d\theta \\
&\quad + \frac{\sigma_n(-\pi/|k|) - \sigma_n(\pi/|k|)}{-ik},
\end{aligned}$$

and

$$\int\limits_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n'(\theta)\frac{e^{-ik\theta}}{-ik}\,d\theta = -\int\limits_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n''(\theta)\frac{e^{-ik\theta}}{-k^2}\,d\theta + \frac{\sigma_n'(-\pi/|k|)-\sigma_n'(\pi/|k|)}{-k^2}.$$

Now,

$$\sum_{|k|\ge 1}\Big|\int\limits_{-\pi/|k|}^{\pi/|k|} \hat{\sigma}_n(\theta)e^{-ik\theta}\,d\theta\Big| \le \int\limits_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)| \sum_{1\le|k|\le\pi/|\theta|} 1\,d\theta \le 2\pi\int\limits_{-\pi}^{\pi}\frac{|\hat{\sigma}_n(\theta)|}{|\theta|}\,d\theta$$

and

$$\sum_{|k|\ge 1}\Big|\int\limits_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n''(\theta)\frac{e^{-ik\theta}}{-k^2}\,d\theta\Big| \le \int\limits_{-\pi}^{\pi}|\hat{\sigma}_n''(\theta)|\sum_{|k|\ge\pi/|\theta|}\frac{1}{k^2} \le C\int\limits_{-\pi}^{\pi}|\theta|\,|\hat{\sigma}_n''(\theta)|\,d\theta.$$

Hence, it remains to show that

$$\sup_{n\in\mathbb{N}}\sum_{|k|\ge 1}\frac{|\hat{\sigma}_n(\pi/k)|}{|k|}<\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}}\sum_{|k|\ge 1}\frac{|\sigma_n'(\pi/k)|}{k^2}<\infty.$$

Let $f_n(\theta) := \theta\hat{\sigma}_n(\theta)$ for every $\theta\in\mathbb{R}-2\pi\mathbb{Z}$. Then f_n is differentiable on $\mathbb{R}-2\pi\mathbb{Z}$ and, by (i) and (ii), $\hat{\sigma}_n', f_n'\in L^1([0,2\pi])$. Hence, $\hat{\sigma}_n$ and f_n can be continuously extended to $\mathbb{R}$ with $f_n(0)=0$. Then, for every $k\ge 1$,

$$\frac{\pi}{k}|\hat{\sigma}_n(\pi/k)| = \Big|\int\limits_0^{\pi/k} f_n'(\theta)\,d\theta\Big| \le \int\limits_0^{\pi/k}|\hat{\sigma}_n(\theta)|\,d\theta + \int\limits_0^{\pi/k}\theta|\hat{\sigma}_n'(\theta)|\,d\theta.$$

Dealing similarly with $k\le -1$ we infer that

$$\sum_{|k|\ge 1}\frac{|\hat{\sigma}_n(\pi/k)|}{|k|} \le \sum_{k\ge 1}\Big(\int\limits_{-\pi/k}^{\pi/k}|\hat{\sigma}_n(\theta)|\,d\theta + \int\limits_{-\pi/k}^{\pi/k}\theta|\hat{\sigma}_n'(\theta)|\,d\theta\Big) \le \pi\int\limits_{-\pi}^{\pi}\frac{|\hat{\sigma}_n(\theta)|}{|\theta|}\,d\theta + \pi\int\limits_{-\pi}^{\pi}|\hat{\sigma}_n'(\theta)|\,d\theta,$$

which is bounded uniformly with respect to n.

Proceeding as above with $g_n(\theta) := \theta^2\hat{\sigma}_n'(\theta)$ in place of $f_n(\theta)$ we see that, by (ii) and (iii), $\sup_{n\in\mathbb{N}}\sum_{|k|\ge 1}|\sigma_n'(\pi/k)|/k^2<\infty$. ∎

Let μ be a probability measure on $\mathbb{Z}$. We say that μ satisfies *hypothesis* $(\tilde{\mathbf{H}})$ if it satisfies hypothesis $(\mathbf{H})$ with a function ψ such that there exists

$D > 0$ such that for every $\theta \in (0, \pi]$,

$$\psi(\theta) \le D\theta\psi'(\theta). \tag{2.13}$$

PROPOSITION 2.6. *Let μ be a probability measure on $\mathbb{Z}$ satisfying hypothesis* ($\tilde{\mathbf{H}}$). *Then $(\sigma_n)_{n\in\mathbb{N}} := (n(\mu^{*n} - \mu^{*(n+1)}))_{n\in\mathbb{N}}$ satisfies items* (i)–(iii) *of Proposition* 2.5. *In particular,* $\sup_{n\in\mathbb{N}} n\|\mu^{*n} - \mu^{*(n+1)}\|_{\ell^1(\mathbb{Z})} < \infty$, *i.e. μ is Ritt.*

Proof. By Proposition 2.3 we already know that (iii) of Proposition 2.5 holds. It follows from the proof of Proposition 2.3 and from (2.13) that there exist $C, c > 0$ such that for every $\theta \in (0, \pi]$,

$$\begin{aligned} |\hat{\sigma}_n(\theta)|/\theta &\le Cne^{-cn\psi(\theta)}\psi(\theta)/\theta \le CDne^{-cn\psi(\theta)}\psi'(\theta), \\ |\hat{\sigma}_n'(\theta)| &\le Cne^{-cn\psi(\theta)}\psi'(\theta)(n\psi(\theta) + 1). \end{aligned}$$

Then we proceed as in the proof of Proposition 2.3. ∎

We now provide a sufficient condition for sequences of finite signed measures on $\mathbb{Z}$ to be bounded in $\ell^1(\mathbb{Z})$; it will be needed later on.

PROPOSITION 2.7. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite signed measures on $\mathbb{Z}$ such that for every $n \in \mathbb{N}$, $\hat{\sigma}_n$ is continuously differentiable on $[-\pi, \pi] - \{0\}$. Assume that*

(i) $\displaystyle\sup_{n\in\mathbb{N}} n \int_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)|\, d\theta < \infty$;

(ii) $\displaystyle\sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} \frac{|\hat{\sigma}_n'(\theta)|^2}{n+1}\, d\theta < \infty$.

Then $\sup_{n\in\mathbb{N}} \|\sigma_n\|_{\ell^1(\mathbb{Z})} < \infty$.

Proof. Let $n \ge 1$. Let $k \in \mathbb{Z}$. We have

$$\sigma_n(k) = \int_{-\pi}^{\pi} \hat{\sigma}_n(\theta)e^{-ik\theta}\, d\theta, \tag{2.14}$$

and if $k \ne 0$,

$$\sigma_n(k) = \int_{-\pi}^{\pi} \hat{\sigma}_n'(\theta)\frac{e^{-ik\theta}}{ik}\, d\theta. \tag{2.15}$$

Using (2.14), we infer that $\sum_{0\le|k|\le n} |\sigma_n(k)| \le (2n+1)\int_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)|\, d\theta$. Using (2.15), Cauchy–Schwarz and Parseval, we infer that

$$\Big(\sum_{|k|>n} |\sigma_n(k)|\Big)^2 \le \Big(\int_{-\pi}^{\pi} |\hat{\sigma}_n'(\theta)|^2\, d\theta\Big) \sum_{|k|>n} \frac{1}{k^2} \le \frac{C}{n} \int_{-\pi}^{\pi} |\hat{\sigma}_n'(\theta)|^2\, d\theta.$$

Then the assertion follows thanks to (i) and (ii). ∎

2.3. Centred probability measures with a second moment. It is known (see [3]) that a centred and strictly aperiodic probability measure μ on $\mathbb{Z}$ with a second moment satisfies (2.6). As an application of the previous subsections we add here that μ is moreover Ritt and satisfies (2.7). Indeed, we shall prove Theorem 1.6.

By Propositions 2.3 and 2.6, it suffices to prove that a centred and strictly aperiodic probability measure μ with a second moment satisfies ($\tilde{\mathbf{H}}$) for some function ψ.

We shall take $\psi(\theta) = \theta^2$ for every $\theta \in [-\pi, \pi]$. Then ψ satisfies (2.13), hence we just have to prove that μ satisfies ($\mathbf{H}$).

Since μ has a second moment and is centred, it is twice continuously differentiable on $[-\pi, \pi]$ and we have

$$\lim_{\theta\to 0,\, \theta\neq 0} (1 - \operatorname{Re} \hat{\mu}(\theta))/\theta^2 = \hat{\mu}''(0)/2 > 0 \quad \text{and} \quad \lim_{\theta\to 0,\, \theta\neq 0} \operatorname{Im} \hat{\mu}(\theta))/\theta^2 = 0.$$

It follows that ($\mathbf{H}$)(i) is satisfied for θ close enough to 0. Then, on taking c smaller if necessary, it holds on $(0, \pi]$ by strict aperiodicity.

Using again the fact that μ has a second moment and is centred, we see that $|\hat{\mu}'(\theta)| \leq \|\hat{\mu}''\|_\infty |\theta|$ for every $\theta \in [-\pi, \pi]$. Hence ($\mathbf{H}$)(ii) and ($\mathbf{H}$)(iii) hold. Similarly, ($\mathbf{H}$)(iv) holds.

3. Probability measures without first moment. In this section, as well as in Sections 4 and 5, we shall consider probability measures μ on $\mathbb{Z}$ such that $(\mu(n))_{n\in\mathbb{N}}$ is a *completely monotone sequence*. Let us recall some definitions and facts.

Definition 3.1. Let Δ be the operator defined for every sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers by $(\Delta t_n)_{n\in\mathbb{N}} = (t_n - t_{n+1})_{n\in\mathbb{N}}$. We say that $(t_n)_{n\geq 0}$ is *completely monotone* if for every $m \geq 0$ (with the convention $\Delta^0 = \mathrm{Id}$), the sequence $(\Delta^m t_n)_{n\geq 0}$ is non-negative.

Definition 3.2. We say that an infinitely differentiable function $f : [s, \infty) \to [0, \infty)$ is *completely monotone* if $(-1)^m f^{(m)} \geq 0$ for every $m \geq 0$.

The following characterization of completely monotone sequences is due to Hausdorff and may be found in Widder [27, p. 108].

Proposition 3.3 (Hausdorff). *A sequence $(\mu_n)_{n\in\mathbb{N}}$ is completely monotone if and only if there exists a finite positive measure ν on $[0,1]$ such that $\mu_n = \int_0^1 t^n \,\nu(dt)$ for every $n \in \mathbb{N}$.*

A way to generate completely monotone sequences is the following (see [27, Theorem 11d, p. 158].

Proposition 3.4. *Let f be a completely monotone function. Then the sequence $(f(n+1))_{n\in\mathbb{N}}$ is completely monotone.*

Definition 3.5. We say that a probability measure μ on $\mathbb{Z}$ is *CM* if $\mu \neq \delta_0$, μ is supported on $\mathbb{N}$ and there exists a finite (positive) measure ν on $[0,1]$ such that

$$\int_0^1 \frac{\nu(dt)}{1-t} = 1, \ (^1) \tag{3.1}$$

$$\mu(n) = \int_0^1 t^n \, \nu(dt) \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

To emphasize ν we shall say that μ is a CM probability measure on $\mathbb{Z}$ *with representing measure* ν.

Since $\mu \neq \delta_0$, we have $\nu \neq \delta_0$ and $\mu(n) > 0$ for every $n \in \mathbb{N}$. In particular μ is strictly aperiodic.

3.1. Characterization of the BAR property. We first give an equivalent formulation of the BAR property that will be more convenient.

Definition 3.6. We say that a subset of $\mathbb{C}$ is a *Stolz region* if it is the convex hull of 1 and a circle centred at 0 with radius $0 < r < 1$.

It is known that μ is strictly aperiodic and has BAR if and only if the range of $\hat{\mu}$ is included in a Stolz region.

If μ is strictly aperiodic, then for every $\varepsilon \in (0,\pi)$, $\hat{\mu}([\varepsilon, 2\pi-\varepsilon])$ is included in a disk centred at 0 with radius strictly smaller than 1. Hence, a strictly aperiodic μ has BAR if and only if

$$\sup_{\theta\in(0,2\pi)} \frac{|\mathrm{Im}\,\hat{\mu}(\theta)|}{1-\mathrm{Re}\,\hat{\mu}(\theta)} < \infty. \tag{3.3}$$

We shall consider the following condition on ν: there exists $L > 0$ such that for every $x \in [0,1)$,

$$\int_0^x \frac{t}{(1-t)^2}\,\nu(dt) \le \frac{L}{1-x}\int_x^1 \frac{t}{1-t}\,\nu(dt). \tag{3.4}$$

Let us notice that this condition implies that $\int_0^1 \frac{\nu(dt)}{(1-t)^2} = \infty$, or equivalently $\sum_{n\in\mathbb{N}} n\mu(n) = \infty$, i.e. μ does not have a first moment. Indeed, if $\int_0^1 \frac{\nu(dt)}{(1-t)^2} < \infty$ then $\frac{1}{1-x}\int_x^1 \frac{t}{1-t}\,\nu(dt) \to 0$ as $x \to 1$. Hence, if ν satisfies (3.4), then $\nu = \delta_0 = \mu$, which is excluded.

Proposition 3.7. *Let μ be a CM probability measure on $\mathbb{Z}$ with representing measure ν. Then μ has BAR if and only if there exists $L > 0$ such*

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(1) All along the paper (for esthetical reasons) we shall adopt the convention $\int_a^b \varphi\,d\nu = \int_{[a,b]} \varphi\,d\nu$. Hence, for non-negative φ we will have $\int_a^c \varphi\,d\nu \le \int_a^b \varphi\,d\nu + \int_b^c \varphi\,d\nu$ with equality if $\nu(\{b\}) = 0$.

that ν satisfies (3.4)*. Moreover, in this case*

$$\frac{1-\operatorname{Re}\hat{\mu}(\theta)}{|\theta|} \xrightarrow[\theta\to 0]{} \infty. \tag{3.5}$$

We deduce the following corollary, in the spirit of Dungey [10, Theorem 4.1].

COROLLARY 3.8. *Let μ be a CM probability measure on $\mathbb{Z}$ with representing measure ν satisfying* (3.4) *for some $L > 0$. Let τ be a probability measure on $\mathbb{Z}$ such that there exists $a > 0$ with*

$$\sum_{n\in\mathbb{Z}} n|\tau(n) - a\mu(n)| < \infty.$$

Then τ has BAR.

REMARK. Notice that in Corollary 3.8 we do not assume that τ is supported on $\mathbb{N}$.

Throughout the paper we will make use of the following easy inequalities:

$$|\sin\theta| \le |\theta|, \qquad 1-\cos\theta \le \theta^2/2 \quad \forall\theta\in\mathbb{R}, \tag{3.6}$$

$$|\sin\theta| \ge 2|\theta|/\pi, \quad 1-\cos\theta \ge \theta^2/4 \quad \forall\theta\in[-1,1]. \tag{3.7}$$

Proof of Proposition 3.7. Assume first that ν satisfies (3.4). Since $\nu \neq \delta_0$ and is not the null measure, the support of μ is $\mathbb{N}$ and μ is strictly aperiodic.

Hence, we just have to prove that there exists $K > 0$ such that

$$|\operatorname{Im}\hat{\mu}(\theta)| \le K(1-\operatorname{Re}\hat{\mu}(\theta)) \quad \forall\theta\in[-\pi,\pi]. \tag{3.8}$$

We have, for every $\theta\in[-\pi,\pi]$,

$$\hat{\mu}(\theta) = \int_0^1 \frac{\nu(dt)}{1-te^{i\theta}}.$$

Notice that $|1-te^{i\theta}|^2 = 1+t^2-2t\cos\theta = (1-t)^2+2t(1-\cos\theta)$ and

$$\begin{aligned}\operatorname{Re}\Big(\frac{1}{1-t} - \frac{(1-te^{-i\theta})}{|1-te^{i\theta}|^2}\Big) &= \frac{(1-t)^2+2t(1-\cos\theta)-(1-t)(1-t\cos\theta)}{(1-t)|1-te^{i\theta}|^2}\\ &= \frac{t(1+t)(1-\cos\theta)}{(1-t)|1-te^{i\theta}|^2}.\end{aligned}$$

Consequently, using (3.1), we obtain

$$1-\operatorname{Re}\hat{\mu}(\theta) = \int_0^1 \frac{t(1+t)(1-\cos\theta)}{(1-t)((1-t)^2+2t(1-\cos\theta))}\,\nu(dt). \tag{3.9}$$

Moreover,

$$\operatorname{Im}\hat{\mu}(\theta) = \int_0^1 \frac{t\sin\theta}{|1-te^{i\theta}|^2}\,\nu(dt) = \int_0^1 \frac{t\sin\theta}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt). \tag{3.10}$$

Since, $\hat\mu$ is continuous and $1-\operatorname{Re}\hat\mu$ vanishes in $[-\pi,\pi]$ only at 0, it is enough to prove (3.8) for $\theta\in[-1/2,1/2]$. Moreover, (3.8) is clear for $\theta=0$. So, let $\theta\in[-1/2,1/2]-\{0\}$.

Let us first estimate $1-\operatorname{Re}\hat\mu(\theta)$. Because $(1-t)^2+2t(1-\cos\theta)\le(1-t)^2+\theta^2\le 2\max((1-t)^2,\theta^2)$, we obtain

$$1-\operatorname{Re}\hat\mu(\theta)\ge\frac12\int_0^{1-|\theta|}\frac{t(1-\cos\theta)}{(1-t)^3}\,\nu(dt)+\frac18\int_{1-|\theta|}^1\frac{t}{1-t}\,\nu(dt). \tag{3.11}$$

Now, we estimate $\operatorname{Im}\hat\mu$. We have

$$\begin{aligned}\int_{1-|\theta|}^1&\frac{t|\sin\theta|}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt)\\&\le\int_{1-|\theta|}^1\frac{t\theta^2}{(1-t)((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\\&\le 4\int_{1-|\theta|}^1\frac{t(1-\cos\theta)}{(1-t)((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\\&\le 4(1-\operatorname{Re}\hat\mu(\theta)),\end{aligned}$$

Using our assumption on ν and (3.11), we obtain

$$\begin{aligned}\int_0^{1-|\theta|}&\frac{t|\sin\theta|}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt)\\&\le\int_0^{1-|\theta|}\frac{t|\sin\theta|}{(1-t)^2}\,\nu(dt)\\&\le\frac{L|\sin\theta|}{|\theta|}\int_{1-|\theta|}^1\frac{t}{1-t}\,\nu(dt)\le 8L(1-\operatorname{Re}\hat\mu(\theta)),\end{aligned}$$

and we see that (3.8) holds.

Let us prove the converse. Assume that (3.8) holds for some $K>0$. Notice that it is enough to prove (3.4) for, say, every $x\in[1/2,1)$. If $\nu([1/2,1))=0$, then μ has a first moment and it is well-known (see [6, Proposition 1.9]) that μ must be centred, contradicting the fact that μ is supported on $\mathbb{N}$. Hence, $\nu([1/2,1))>0$ and we have a bound (3.4) for any $x\in[0,1/2]$.

Let $S\ge1$ be fixed for the moment. Let $\theta\in[-1/(2S),1/(2S)]-\{0\}$.

Using (3.6) we see that $(1-t)^2+2t(1-\cos\theta)\le(1-t)^2+t\theta^2\le(1+1/S^2)(1-t)^2$ whenever $0\le t\le 1-S|\theta|$. So, by (3.10), (3.7) and (3.8), we have

$$\int_0^{1-S|\theta|} \frac{t\,\nu(dt)}{(1-t)^2} \le \frac{1}{|\sin\theta|} \int_0^{1-S|\theta|} \frac{t|\sin\theta|}{(1-t)^2}\,\nu(dt) \le \frac{1+1/S^2}{2|\theta|/\pi} |\mathrm{Im}\,\hat{\mu}(\theta)| \tag{3.12}$$
$$\le K\frac{1+1/S^2}{2|\theta|/\pi}(1-\mathrm{Re}\,\hat{\mu}(\theta)) \le \frac{K\pi}{|\theta|}(1-\mathrm{Re}\,\hat{\mu}(\theta)).$$

Now, using (3.9), we observe that (recall $1/2 \le 1 - S|\theta|$)

$$1-\mathrm{Re}\,\hat{\mu}(\theta) \le 2(1-\cos\theta) \int_0^{1-S|\theta|} \frac{t\,\nu(dt)}{(1-t)^3} + \int_{1-S|\theta|}^1 \frac{\nu(dt)}{1-t}$$
$$\le \frac{2(1-\cos\theta)}{S|\theta|} \int_0^{1-S|\theta|} \frac{t\,\nu(dt)}{(1-t)^2} + \int_{1-S|\theta|}^1 \frac{2t\,\nu(dt)}{1-t}.$$

Combining this bound with (3.12) and (3.6), we infer that

$$\int_0^{1-S|\theta|} \frac{t\,\nu(dt)}{(1-t)^2} \le \frac{K\pi}{S} \int_0^{1-S|\theta|} \frac{t\,\nu(dt)}{(1-t)^2} + \frac{K\pi}{|\theta|} \int_{1-S|\theta|}^1 \frac{t\,\nu(dt)}{1-t}.$$

Taking $S = 1 + 2K\pi \ge 1$, we derive that for every $x \in [1/2, 1)$ (setting $\theta = (1-x)/S$) we have

$$\int_0^x \frac{t\,\nu(dt)}{(1-t)^2} \le \frac{2K(1+2K\pi)\pi}{1-x} \int_x^1 \frac{t\,\nu(dt)}{1-t},$$

which proves the desired bound.

It remains to prove (3.5). Using (3.4), we see that

$$\frac{1-\mathrm{Re}\,\hat{\mu}(\theta)}{|\theta|} \ge \frac{1-\cos\theta}{2|\theta|^3} \int_{1-|\theta|}^1 \frac{t}{(1-t)}\,\nu(dt)$$
$$\ge \frac{1-\cos\theta}{2|\theta|^2} \int_0^{1-|\theta|} \frac{t}{(1-t)^2}\,\nu(dt) \xrightarrow[\theta\to 0]{} \infty,$$

hence the result. ∎

Proof of Corollary 3.8. By assumption and Proposition 3.7, there exists $K > 0$ such that $\sum_{n\ge 1} n|\tau(n) - a\mu(n)| \le K$ and for every $\theta \in [-\pi, \pi]$, $|\mathrm{Im}\,\hat{\mu}(\theta)| \le K(1-\mathrm{Re}\,\hat{\mu}(\theta))$.

Let us prove that τ is strictly aperiodic. If it were not, there would exist $\ell \ge 2$ and $0 \le k \le \ell - 1$ such that the support of τ would be contained in $k + \ell\mathbb{Z}$. In particular $\tau(k+1+\ell m) = 0$ for every $m \in \mathbb{Z}$. Hence,

$$\sum_{m\in\mathbb{Z}} |m|\mu(k+1+\ell m) < \infty$$

and (because $(\mu(n))_{n\geq 1}$ is non-increasing) μ must have a first moment, contradicting (3.4) (see the remark after (3.4)).

We now show that there exists $C>0$ such that for every $\theta\in[-\pi,\pi]$,

$$\mathrm{Re}(1-\hat{\tau}(\theta))\geq C|\theta|. \tag{3.13}$$

Since τ is strictly aperiodic, it is enough to prove (3.13) for θ's small enough. By Proposition 3.7, there exists $\delta\in(0,\pi)$ such that for every $\theta\in[-\delta,\delta]$, $|\theta|\leq(1-\mathrm{Re}\,\hat{\mu}(\theta)/(2K)$. Then, since $1-\cos u\leq|u|$ for every $u\in\mathbb{R}$,

$$\begin{aligned}|\theta|&\leq\frac{1-\mathrm{Re}\,\hat{\tau}(\theta)}{2K}+\frac{1}{2K}\sum_{n\geq 1}|\tau(n)-a\mu(n)|(1-\cos(n\theta))\\&\leq(1-\mathrm{Re}\,\hat{\tau}(\theta))/(2K)+|\theta|/2,\end{aligned}$$

and (3.13) follows.

Let $\theta\in[-\pi,\pi]$. Since $|\sin u|\leq u$ for every $u\in\mathbb{R}$, we have

$$\begin{aligned}|\mathrm{Im}\,\hat{\tau}(\theta)|&\leq a|\mathrm{Im}\,\hat{\mu}(\theta)|+\sum_{n\geq 1}|\tau(n)-a\mu(n)|\,|\sin(n\theta)|\\&\leq aK(1-\mathrm{Re}\,\hat{\mu}(\theta))+K|\theta|\\&\leq K(1-\mathrm{Re}\,\hat{\tau}(\theta))+\sum_{n\geq 1}|\tau(n)-a\mu(n)|\,|1-\cos(n\theta)|+K|\theta|\\&\leq K(1-\mathrm{Re}\,\hat{\tau}(\theta))+2K|\theta|\leq K(1+2C)(1-\mathrm{Re}\,\hat{\tau}(\theta)),\end{aligned}$$

and the corollary is proved. ∎

From a practical point of view it is better to have a condition on $(\mu(n))_{n\in\mathbb{Z}}$. Indeed, we may consider completely monotone sequences given thanks to Proposition 3.4, in which case we do not know ν.

Proposition 3.9. *Let μ be a CM probability measure on $\mathbb{Z}$ with representing measure ν. Then ν satisfies* (3.4) *for some $L>0$ if and only if there exists $D>0$ such that for every $n\geq 1$,*

$$\sum_{k=1}^{n}k\mu(k)\leq Dn\sum_{k\geq n}\mu(k). \tag{3.14}$$

Proof. Assume that (3.4) holds for some $L>0$. Let $n\geq 1$. We have

$$\begin{aligned}\sum_{k=1}^{n}k\mu(k)&\leq\int_0^{1-1/n}\frac{t}{(1-t)^2}\,\nu(dt)+n\int_{1-1/n}^{1}\frac{t}{1-t}\,\nu(dt)\\&\leq(1+L)n\int_{1-1/n}^{1}\frac{t}{1-t}\,\nu(dt).\end{aligned}$$

Notice that $\sum_{k\geq n}\mu(k)=\int_0^1\frac{t^n}{1-t}\,\nu(dt)$ and that $x\mapsto(1-1/x)^{x-1}$ is non-decreasing on $(1,\infty)$ with limit e^{-1} as $x\to\infty$. Hence,

$$\sum_{k=1}^{n}k\mu(k)\leq(1+L)en\int_{1-1/n}^{1}\frac{t\,\nu(dt)}{1-t}\leq(1+L)en\sum_{k\geq n}\mu(k),$$

and (3.14) holds with $D=(1+L)e$.

Assume now that (3.14) holds for some $D>0$. Let $A\geq 1$ be a positive integer fixed for the moment. Let $n\geq 2$.

Let $1\leq m\leq n-1$ be an integer and let $t\in[1-1/m,1-1/(m+1)]$. Using again the behaviour of $x\mapsto(1-1/x)^{x-1}$, we deduce that (with the convention $0^0=1$)

$$\sum_{k=1}^{An}kt^k\geq t\sum_{k=0}^{m-1}(k+1)(1-1/m)^{m-1}\geq\frac{tm(m+1)}{2e}\geq\frac{t}{e(1-t)^2}.$$

Hence,

$$\int_0^{1-1/n}\frac{t}{(1-t)^2}\,\nu(dt)\leq e\sum_{k=1}^{An}k\mu(k)\leq eDAn\sum_{k\geq An}\mu(k)=eDAn\int_0^1\sum_{k\geq An}t^k\,\nu(dt).$$

Now notice that for $t\in[0,1-1/n]$,

$$\sum_{k\geq An}t^{k-1}\leq\frac{1}{A^2n^2}\sum_{k\geq An}k(k+1)t^{k-1}\leq\frac{1}{A^2n^2(1-t)^3}\leq\frac{1}{A^2n(1-t)^2},$$

and that for $t\in[1-1/n,1]$, $\sum_{k\geq An}t^{k-1}\leq 1/(1-t)$. Therefore

$$\int_0^{1-1/n}\frac{t}{(1-t)^2}\,\nu(dt)\leq\frac{eD}{A}\int_0^{1-1/n}\frac{t}{(1-t)^2}\,\nu(dt)+eDAn\int_{1-1/n}^{1}\frac{t}{1-t}\,\nu(dt).$$

Taking $A=1+2eD\geq 1$, we derive that

$$\int_0^{1-1/n}\frac{t}{(1-t)^2}\,\nu(dt)\leq 2eD(1+2eD)n\int_{1-1/n}^{1}\frac{t}{1-t}\,\nu(dt).$$

Let $x\in[0,1)$. There exists $n\geq 2$ such that $1-1/(n-1)\leq x<1-1/n$. Then $n\leq 1+1/(1-x)\leq 2/(1-x)$ and

$$\begin{aligned}\int_0^{x}\frac{t}{(1-t)^2}\,\nu(dt)&\leq\int_0^{1-1/n}\frac{t}{(1-t)^2}\,\nu(dt)\\&\leq 2eD(1+2eD)n\int_{1-1/n}^{1}\frac{t}{1-t}\,\nu(dt)\leq\frac{4eD(1+2eD)}{1-x}n\int_x^1\frac{t}{1-t}\,\nu(dt).\end{aligned}$$

Hence (3.4) holds with $L=4eD(1+2eD)$. ∎

3.2. Hypothesis (H) for CM probability measures. We shall prove that the conditions imposed in the previous subsection ensure hypothesis (**H**).

PROPOSITION 3.10. *Let μ be a CM probability measure on $\mathbb{Z}$ satisfying* (3.14). *Then μ satisfies hypothesis* (**H**).

Proof. To check conditions (**H**)(i)–(iv) with a suitable function ψ we must first estimate $\hat{\mu}$ and its derivatives.

Let us first compute the derivatives of $\hat{\mu}$. Recall that for every $\theta \in [-\pi, \pi]$,

$$1 - \operatorname{Re}\hat{\mu}(\theta) = \int_0^1 \frac{t(1+t)(1-\cos\theta)}{(1-t)\big((1-t)^2 + 2t(1-\cos\theta)\big)}\,\nu(dt)$$

$$= \frac{1}{2}\int_0^1 \frac{(1+t)\,\nu(dt)}{1-t} - \frac{1}{2}\int_0^1 \frac{(1-t^2)}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt),$$

$$\operatorname{Im}\hat{\mu}(\theta) = \int_0^1 \frac{t\sin\theta}{|1-te^{i\theta}|^2}\,\nu(dt) = \int_0^1 \frac{t\sin\theta}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt).$$

Hence, for every $\theta \in [-\pi, \pi] - \{0\}$,

$$\operatorname{Re}\hat{\mu}'(\theta) = -\sin\theta \cdot \int_0^1 \frac{t(1-t^2)}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt), \tag{3.15}$$

$$\operatorname{Re}\hat{\mu}''(\theta) = -\cos\theta \cdot \int_0^1 \frac{t(1-t^2)}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt) + 2\sin^2\theta \cdot \int_0^1 \frac{t(1-t^2)}{((1-t)^2 + 2t(1-\cos\theta))^3}\,\nu(dt), \tag{3.16}$$

$$\operatorname{Im}\hat{\mu}'(\theta) = \int_0^1 \frac{t\cos\theta}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt) - \int_0^1 \frac{2t^2\sin^2\theta}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt), \tag{3.17}$$

$$\operatorname{Im}\hat{\mu}''(\theta) = \int_0^1 \frac{-t\sin\theta}{(1-t)^2 + 2t(1-\cos\theta)} - \int_0^1 \frac{8t^2\sin\theta\cos\theta}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt) + \int_0^1 \frac{4t^3\sin^3\theta}{((1-t)^2 + 2t(1-\cos\theta))^3}\,\nu(dt). \tag{3.18}$$

For $\theta \in [-\pi, \pi]$, define

$$\psi(\theta) = \int_0^1 \frac{t|\theta|}{(1-t)(1-t+t|\theta|)}\,\nu(dt) = 1 - \int_0^1 \frac{\nu(dt)}{(1-t+t|\theta|)}. \tag{3.19}$$

Then, for every $\theta \in (0, \pi]$,

$$\psi'(\theta) = \int_0^1 \frac{t\,\nu(dt)}{(1-t+t\theta)^2}. \tag{3.20}$$

Notice that, for every $\theta \in (0, 1/2]$,

$$\begin{aligned} &\frac{\theta}{2}\int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^2} + \frac{1}{2}\int_{1-\theta}^1 \frac{t\,\nu(dt)}{1-t} \\ &\qquad\qquad \le \psi(\theta) \le \theta \int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^2} + \int_{1-\theta}^1 \frac{\nu(dt)}{1-t}; \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\frac{\theta}{4}\int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^2} + \frac{1}{4\theta}\int_{1-\theta}^1 t\,\nu(dt) \\ &\qquad\qquad \le \theta\psi'(\theta) \le \theta \int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^2} + \frac{2}{|\theta|}\int_{1-\theta}^1 \nu(dt). \end{aligned} \tag{3.22}$$

Claim 1. *There exists $C > 0$ such that for every $\theta \in [0, \pi]$,*

$$1 - \operatorname{Re}\hat{\mu}(\theta) \ge C\psi(\theta).$$

Proof. It suffices to prove the inequality for $\theta \in (0, 1/2]$. Using (3.11), (3.4) and (3.21) we obtain

$$1 - \operatorname{Re}\hat{\mu}(\theta) \ge \int_{1-|\theta|}^1 \frac{t}{8(1-t)}\,\nu(dt) \ge \frac{1}{8(L+2)}\psi(\theta),$$

and the assertion follows. ∎

Claim 2. *There exists $C > 0$ such that $|\hat{\mu}'(\theta)| \le C\psi'(\theta)$ for every $\theta \in (0, \pi]$.*

Proof. Again, we only consider the case when $\theta \in (0, 1/2]$. We deal separately with the real and imaginary parts of μ'. Using (3.15) and (3.22) we have

$$|\operatorname{Re}\hat{\mu}'(\theta)| \le 2\theta \int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^3} + \frac{16}{\theta^3}\int_{1-\theta}^1 t(1-t)\,\nu(dt) \le 64\psi'(\theta).$$

Similarly,

$$\begin{aligned}|\mathrm{Im}\,\hat\mu'(\theta)| &\le \int_0^{1-\theta}\Big(\frac{t}{(1-t)^2}+\frac{t\theta^2}{(1-t)^4}\Big)\,\nu(dt)\\ &\quad+\Big(\frac{1}{1-\cos\theta}+\frac{2\theta^2}{(1-\cos\theta)^2}\Big)\int_{1-\theta}^{1} t\,\nu(dt)\\ &\le C\psi'(\theta). \ \blacksquare\end{aligned}$$

Claim 3. *There exists $C>0$ such that $|\theta\hat\mu'(\theta)|\le C\psi(\theta)$ for every $\theta\in(0,\pi]$.*

Proof. Combine Claim 2 and (3.22). ■

Claim 4. *There exists $C>0$ such that $|\theta\hat\mu''(\theta)|\le C\psi'(\theta)$ for every $\theta\in(0,\pi]$.*

Proof. We assume that $\theta\in(0,1/2]$. By (3.16) and (3.22), we have

$$\begin{aligned}|\mathrm{Re}\,\hat\mu''(\theta)| &\le \int_0^{1-\theta}\Big(\frac{2t}{(1-t)^3}+\frac{4t\theta^2}{(1-t)^5}\Big)\,\nu(dt)\\ &+\frac{2}{(1-\cos\theta)^2}\int_{1-\theta}^{1}\frac{t}{1-t}\,\nu(dt)+\frac{4\theta^2}{(1-\cos\theta)^3}\int_{1-\theta}^{1} t(1-t)\,\nu(dt)\le C\psi'(\theta)/\theta.\end{aligned}$$

Similar computations based on (3.15) and (3.22) yield

$$|\mathrm{Im}\,\hat\mu''(\theta)|\le C\psi'(\theta)/\theta. \ \blacksquare$$

Now, $(\mathbf{H})$(ii)–(iv) follow from the combination of Claims 2, 3 and 4.

Let us prove $(\mathbf{H})$(i). Let $\theta\in[0,1/2]$. Recall that we have $|\mathrm{Im}\,\hat\mu(\theta)|\le C(1-\mathrm{Re}\,\hat\mu(\theta))$. Hence,

$$\begin{aligned}|\hat\mu(\theta)|^2 &= |\mathrm{Im}\,\hat\mu(\theta)|^2+1-2(1-\mathrm{Re}\,\hat\mu(\theta))+(1-\mathrm{Re}\,\hat\mu(\theta))^2\\ &=1-(1-\mathrm{Re}\,\hat\mu(\theta))\big(2-(C+1)(1-\mathrm{Re}\,\hat\mu(\theta))\big).\end{aligned}$$

Since $1-\mathrm{Re}\,\hat\mu(\theta)\xrightarrow[\theta\to 0]{}0$, using Claim 1, we infer that for θ small enough

$$|\hat\mu(\theta)|^2\le 1-\tilde C\psi(\theta).$$

Hence $(\mathbf{H})$(i) holds for, say, $\theta\in[0,\eta]$ with η small enough. Since $\sup_{\theta\in[\eta,\pi]}|\hat\mu(\theta)|<1$, we see that $(\mathbf{H})$(i) holds for every $\theta\in[0,\pi]$, upon taking c smaller if necessary. ■

Corollary 3.11. *Let μ be a CM probability measure on $\mathbb{Z}$. Let σ be a probability measure on $\mathbb{Z}$ such that $\hat\sigma$ is twice continuously differentiable on $[-\pi,\pi]-\{0\}$, and $\hat\sigma'$ and $\theta\mapsto\theta\hat\sigma''(\theta)$ are bounded. Then $\sigma*\mu$ satisfies hypothesis $(\mathbf{H})$. Moreover, if μ satisfies hypothesis $(\tilde{\mathbf{H}})$, so does $\sigma*\mu$.*

REMARK. The assumption on ν holds, for instance, when $\sum_{n\in\mathbb{Z}} n^2\sigma(n) < \infty$.

Proof of Corollary 3.11. Let ψ be the function defined in (3.19). Since $\int_0^1 \frac{\nu(dt)}{(1-t)^2} = \infty$, one easily infers from (3.20) that $\liminf_{\theta\to 0,\, \theta>0} \psi'(\theta) = \infty$. In particular, there exists $K > 0$ such that $\psi'(\theta) \ge K$ for every $\theta \in (0, \pi]$, and consequently $\psi(\theta) \ge K\theta$. Then the fact that $\sigma * \mu$ satisfies hypothesis $(\mathbf{H})$, with the same function ψ as for μ, may be proved exactly as Proposition 2.4. Since we use the same function ψ for $\sigma * \mu$ and μ, we see that $\sigma * \mu$ satisfies hypothesis $(\tilde{\mathbf{H}})$ as soon as μ does. ■

COROLLARY 3.12. *Let τ be a probability measure on $\mathbb{Z}$. Assume there exists a CM probability measure μ and $a > 0$ with $\sum_{n\in\mathbb{Z}} n^2|\tau(n) - a\mu(n)| < \infty$. Then τ satisfies hypothesis $(\mathbf{H})$. If moreover μ satisfies hypothesis $(\tilde{\mathbf{H}})$, so does τ.*

REMARK. It follows from the proof that we only need that $\hat{\sigma}$ be twice continuously differentiable on $[-\pi, \pi] - \{0\}$ and that $\hat{\tau}'$ and $\theta \mapsto \theta\hat{\tau}''(\theta)$ be bounded.

Proof of Corollary 3.12. Define a *signed* measure by setting $\sigma := \tau - a\mu$. Then $\hat{\sigma}$ is twice continuously differentiable on $[-\pi, \pi]$, $\hat{\sigma}(0) = 1 - c$ and there exists $C > 0$ such that $|\hat{\sigma}(\theta) - (1 - a)| \le C\theta$ for every $\theta \in [0, \pi]$. The proof may be finished using the same arguments as in the proof of Corollary 3.11. ■

3.3. The Ritt property on $\ell^1(\mathbb{Z})$. In this section, we prove the Ritt property for probability measures as in Theorem 1.7. We first deal with CM probability measures, which corresponds to the case where $\sigma = \delta_0$.

Let μ be a probability measure on $\mathbb{Z}$. Notice that μ being Ritt is equivalent to

$$\sup_{n\ge 1} n\|\pi_\mu^n - \pi_\mu^{n+1}\|_{\ell^1(\mathbb{Z})} < \infty,$$

where π_μ stands for the operator of convolution by μ.

Let Γ be the open unit disk in the complex plane. By Theorem 1.5 of Dungey [10], μ is Ritt if and only if the spectrum $\sigma(\pi_\mu)$ is contained in $\Gamma \cup \{1\}$ and the semigroup $(e^{-t(I-\pi_\mu)})_{t\ge 0}$ is bounded analytic. The latter means that

$$\sup_{t>0} (\|e^{-t(\delta_0-\mu)}\|_{\ell^1(\mathbb{Z})} + t\|(I - T)e^{-t(\delta_0-\pi_\mu)}\|_{\ell^1(\mathbb{Z})}) < \infty.$$

REMARK. Notice that [10, Theorem 1.5] is valid for probabilities supported on $\mathbb{N}$.

PROPOSITION 3.13. *Let μ be a CM probability measure on $\mathbb{Z}$ with representing measure ν satisfying* (3.4). *Then μ is Ritt.*

We already saw that ν satisfies (3.4) if and only if μ has BAR. The fact that a CM probability measure on $\mathbb{Z}$ having BAR is Ritt has been proved very recently by Gomilko and Tomilov [15, Theorem 7.1] as a consequence of another recent result of theirs [14]. The latter paper deals with subordination semigroups, hence is written in a continuous setting. For the reader's convenience we explain below how to derive Proposition 3.13 from [14].

First of all, by [10, Theorem 2.1], we have $\sigma(\pi_\mu) \subset \hat{\mu}([-\pi,\pi]) \subset \mathit{\Gamma} \cup \{1\}$, where the right-hand inclusion follows from the fact that μ has BAR. Hence, Proposition 3.13 will be proved if we can show that $(e^{-t(I-\pi_\mu)})_{t\geq 0}$ is bounded analytic.

DEFINITION 3.14. An infinitely differentiable function $f:(0,\infty)\to[0,\infty)$ is called a *Bernstein function* if f' is completely monotone. If $\lim_{x\to 0^+} f(x)$ exists and if f admits a holomorphic extension to $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ such that $\operatorname{Im} f(z) \geq 0$, then f is called a *complete Bernstein function.*

Proof of Proposition 3.13. For every $x \geq 0$, define

$$\chi(x) := 1 - \int_0^1 \frac{\nu(dt)}{1-t+tx} = \int_0^1 \frac{\nu(dt)}{1-t} - \int_0^1 \frac{\nu(dt)}{1-t+tx}.$$

Then χ is non-decreasing with $\chi(0) = 0$, hence it is non-negative. It is not hard to see that it is infinitely differentiable and that χ' is completely monotone, hence χ is a Bernstein function and one can easily see that it is actually a complete Bernstein function.

Since χ is Bernstein, it is well-known (see e.g. [11, Theorem 1.2.4]) that there exists a convolution semigroup $(\sigma_t)_{t\geq 0}$ (of probability measures on $[0,\infty)$) such that for all $x,t \geq 0$,

$$\int_0^\infty e^{-xy}\,\sigma_t(dy) = e^{-t\chi(x)}.$$

Following Dungey [10, p. 1734], we consider the Poisson semigroup $(P_s)_{s\geq 0}$ acting by convolution on $\ell^1(\mathbb{N})$ and defined by

$$P_s := e^{-s(\delta_0-\delta_1)} = e^{-s}\sum_{k\geq 0}\frac{s^k}{k!}\delta_k \qquad \forall s \geq 0.$$

Consider also the associated subordinated semigroup $(Q_t)_{s\geq 0}$ defined by

$$Q_t := \int_0^\infty P_s\,\sigma_t(ds) \qquad \forall t \geq 0.$$

Let $t \geq 0$. Then Q_t is a probability measure on $\mathbb{N}$, whose generating function is given (on $[0,1]$) by

$$x \mapsto \int_0^\infty e^{-s(1-x)}\,\sigma_t(ds) = e^{-t\chi(1-x)}.$$

Let G_μ denote the generating function of μ, i.e.

$$G_\mu(x) = \sum_{n\geq 0} \mu(n)x^n = \int_0^1 \frac{\nu(dt)}{1-tx} = 1 - \chi(1-x)$$

for every $x \in [0,1]$. Then, for every $t \geq 0$, the generating function of the probability $e^{-t(I-\mu)} = e^{-t}\sum_{k\geq 0} t^k \mu^{*k}/k!$ is given by

$$e^{-t}\sum_{k\geq 0} \frac{t^k G_\mu^k}{k!} = e^{-t(1-G_\mu)}.$$

In particular, we see that the semigroups $(e^{-t(I-\pi_\mu)})_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ coincide. Hence, to prove that $(e^{-t(I-\pi_\mu)})_{t\geq 0}$ is bounded analytic, it is enough to show that any subordinated semigroup associated with $(\sigma_t)_{t\geq 0}$ is bounded analytic (see the introduction of [14] for more details). To prove the latter, since χ is complete Bernstein, in view of [14, Corollary 7.10] it is enough to show that χ sends the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ to a sector $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq C \operatorname{Re} z\}$ for some $C > 0$.

Let $z = a + ib$ be such that $a \geq 0$ and $|z|^2 = a^2 + b^2 \leq 1/4$. Using (3.4) we obtain

$$\begin{aligned}
|\operatorname{Im}\chi(z)| &= |b| \int_0^1 \frac{t}{(1-t+at)^2 + t^2b^2}\,\nu(dt)\\
&\leq |b| \int_0^{1-|z|} \frac{t}{(1-t)^2}\,\nu(dt) + \frac{4b}{|z|^2}\int_{1-|z|}^1 t\,\nu(dt) \leq K \int_{1-|z|}^1 \frac{t}{1-t}\,\nu(dt).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\operatorname{Re}\chi(z) &= \int_0^1 \frac{at(1-t) + |z|^2t^2}{(1-t)((1-t+at)^2 + t^2b^2)}\,\nu(dt)\\
&\geq \int_{1-|z|}^1 \frac{|z|^2t^2}{5|z|^2(1-t)}\,\nu(dt) \geq \frac{1}{10}\int_{1-|z|}^1 \frac{t}{1-t}\,\nu(dt).
\end{aligned}$$

This gives the desired bound when $|z|^2 \leq 1/4$.

Assume now that $|z|^2 \geq 1/4$. In particular, $4|z| \geq 2$. Hence,

$$|\mathrm{Im}\,\chi(z)| \leq \int_0^{(4|z|)^{-1}} \frac{t|z|}{(1-t)^2}\,\nu(dt) + \frac{|z|}{|z|^2} \int_{(4|z|)^{-1}}^1 t^{-1}\,\nu(dt)$$
$$\leq \frac{1}{4}\int_0^{1/2} \frac{\nu(dt)}{(1-t)^2} + 4\int_0^1 \nu(dt) < \infty.$$

Moreover, as the integrand below is non-decreasing with respect to $|z|$, we have

$$\mathrm{Re}\,\chi(z) \geq \int_0^1 \frac{|z|^2 t^2}{2(1-t)((1-t)^2 + t^2|z|^2)}\,\nu(dt)$$
$$\geq \frac{1}{8}\int_0^1 \frac{t^2}{(1-t)^2 + t^2/4}\,\nu(dt) > 0,$$

which finishes the proof of Proposition 3.13. ∎

Proposition 3.15. *Let μ be a CM probability measure on $\mathbb{Z}$ with representing measure ν satisfying* (3.4). *Let σ be a probability measure on $\mathbb{Z}$ such that $\hat\sigma$ is continuously differentiable on $[-\pi,\pi] - \{0\}$ and $\hat\sigma'$ is bounded on $[-\pi,\pi] - \{0\}$. Then*

$$\sup_{n\in\mathbb{N}} n\|(\delta_0 - \sigma) * \mu^{*n}\|_{\ell^1} < \infty. \tag{3.23}$$

In particular, $\sigma * \mu$ is Ritt, and for every $\alpha \in (0,1]$, $\alpha\mu + (1-\alpha)\sigma$ is Ritt.

Remark. In particular, the proposition applies when $\sum_{n\in\mathbb{Z}} |n|\mu(n) < \infty$.

Proof of Proposition 3.15. To prove (3.23), we check that $(\sigma_n)_{n\in\mathbb{N}} := (n(\delta_0 - \sigma) * \mu^{*n})_{n\in\mathbb{N}}$ satisfies items (i) and (ii) of Proposition 2.7.

By assumption there exists $L > 0$ such that $|\hat\sigma'| \leq L$ and it follows that $|1 - \hat\sigma(\theta)| \leq L|\theta|$ for every $\theta \in [-\pi,\pi]$.

Let ψ be the function given in (3.19). Recall that there exists $K > 0$ such that for every $\theta \in [-\pi,\pi] - \{0\}$, $\psi(\theta) \geq K\theta$ and $\psi'(\theta) \geq K$. Hence, for all $n \in \mathbb{N}$ and $\theta \in [-\pi,\pi] - \{0\}$,

$$n|\hat\sigma_n(\theta)| \leq \frac{Ln^2}{K^2}\psi(\theta)\psi'(\theta)(\hat\mu(\theta))^n.$$

So, arguing as in the proof of Proposition 2.6, we see that $(\sigma_n)_{n\in\mathbb{N}}$ satisfies item (i) in Proposition 2.7.

For every $\theta \in [-\pi,\pi] - \{0\}$, we have

$$\hat\sigma_n'(\theta) = -n\hat\sigma'(\theta)\hat\mu^n(\theta) + n^2(1 - \hat\sigma(\theta))\hat\mu'(\theta)\hat\mu^{n-1}(\theta).$$

We infer that

$$|\hat{\sigma}_n'(\theta)|^2 \le \frac{2n^2L^2}{K}\psi'(\theta)|\hat{\mu}^{2n}|(\theta) + \frac{2n^4L^2}{K^3}\psi^2(\theta)\psi'(\theta)|\hat{\mu}^{n-1}|(\theta).$$

Hence, arguing as in the proof of Proposition 2.6, we see that $(\sigma_n)_{n\in\mathbb{N}}$ satisfies item (ii) in Proposition 2.7.

It remains to prove the second part of Proposition 3.15.

Let $n \ge 1$. We have

$$(\delta_0 - \sigma * \mu) * (\sigma * \mu)^{*n} = \sigma^{*n} * [(\delta_0 - \mu) * \mu^{*n}] + [(\delta_0 - \sigma) * \mu^{*(n+1)}] * \sigma^{*n},$$

which proves that $\sigma * \mu$ is Ritt.

Let $\alpha \in (0,1]$ and $n \ge 1$, and $\tau := \alpha\mu + (1-\alpha)\sigma$. We have

$$\begin{aligned}(\delta_0 - \tau) * \tau^{*n} &= \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha^k(1-\alpha)^{n-k}}{k+1} \\ &\quad \times \big[\alpha(k+1)(\delta_0 - \mu) * \mu^{*k} + (1-\alpha)(k+1)(\delta_0 - \sigma) * \mu^{*k}\big] * \sigma^{*(n-k)}.\end{aligned}$$

Hence,

$$(n+1)\|(\delta_0 - \tau) * \tau^{*n}\|_{\ell^1} \le \frac{C}{\alpha} \sum_{k=0}^{n} \binom{n+1}{k+1} \alpha^{k+1}(1-\alpha)^{(n+1)-(k+1)} \le C,$$

and we see that τ is Ritt. ■

3.4. Proof of Theorem 1.7 and examples

Proof of Theorem 1.7. Let μ be a CM probability measure on $\mathbb{Z}$. Item (i) follows from Proposition 3.7 combined with Proposition 3.9.

Under the assumptions of (ii), by Corollary 3.11, $\mu * \sigma \in \mathcal{H}$, and by Proposition 3.15 (see the Remark after the proposition), $\mu * \sigma \in \mathcal{R}$.

Let $\alpha \in (0,1]$. Applying Corollary 3.12 with $\tau := \alpha\mu + (1-\alpha)\sigma$ and $a := \alpha$ we infer that $\tau \in \mathcal{H}$. The fact that $\tau \in \mathcal{R}$ follows from Proposition 3.15 again. ■

To give examples we will make use of Proposition 3.4. Hence we shall first exhibit completely monotone functions.

LEMMA 3.16 (Miller–Samko [21]). *Let $f, g : (0,\infty) \to (0,\infty)$ be infinitely differentiable functions such that g' is completely monotone.*

(i) *If f is completely monotone then so is $f \circ g$.*
(ii) *If f' is completely monotone then so is $(f \circ g)'$.*

Proof. Item (i) is just [21, Theorem 2]. For (ii), notice that $(f \circ g)' = f' \circ g \times g'$. By (i), $f' \circ g$ is completely monotone. Since the product of completely monotone functions is completely monotone (see e.g. [21, Theorem 1]) we infer that $(f \circ g)'$ is completely monotone. ■

Define by induction $L_1(x) = L(x) := \log(1+x)$ and $L_{k+1}(x) = L(L_k(x))$ for every $x > 0$.

Corollary 3.17. *For every integer $k \geq 1$ and any $\alpha_1, \dots, \alpha_k \in [0, \infty)$ and $\alpha \in [0, \infty)$ the function given by*

$$f_{\alpha,\alpha_1,\dots,\alpha_k}(x) = \frac{1}{x^\alpha L_1(x)^{\alpha_1} \dots L_k(x)^{\alpha_k}} \qquad \forall x \geq 0$$

is completely monotone.

Proof. Obviously, $x \mapsto x^{-\alpha}$ is completely monotone. By Lemma 3.16(ii), L_k has a completely monotone derivative and so $L_k^{-\alpha_k}$ is completely monotone by (i). The fact that $f_{\alpha,\alpha_1,\dots,\alpha_k}$ is also completely monotone then follows from [21, Theorem 1]. ■

Example 1. Let μ be a probability measure supported on $\mathbb{N}$ such that $\mu(n) = cf_{\alpha,\alpha_1,\dots,\alpha_k}(n+1)$ for every $n \in \mathbb{N}$, where $\alpha_1, \dots, \alpha_k \in [0, \infty)$, $\alpha \in (1, 2)$ and c is a normalizing constant. Then, by Theorem 1.7, $\mu \in \mathcal{H} \cap \mathcal{R}$. Of course one may take $\alpha = 1$ and $\alpha_1 > 1$, and so on. But for $\alpha = 2$, μ does not even have BAR.

It is more difficult to produce examples allowing negative α_k's. One way to handle this difficulty is to proceed as in the proof of [8, Proposition 5.11].

Example 2. Now we describe a basic example of Ritt probability measures already considered by Dungey [10] and Gomilko and Tomilov [15]. Let $\gamma \in (0, 1)$. We have a power series expansion $1 - (1-t)^\gamma = \sum_{n\geq 1} a_n(\gamma)t^n$, $0 \leq t \leq 1$. Notice that $\sum_{n\geq 1} a_n(\gamma) = 1$ and $a_n(\gamma) \geq 0$ for every $n \geq 1$. Define two probability measures τ and μ by setting $\mu(n) = a_{n+1}(\gamma) = \tau(n+1)$ for every $n \in \mathbb{N}$, so that $\tau = \delta_1 * \mu$. Then (see for instance [15, Example 3.10a]) τ is a CM probability measure which has BAR. In particular, by Theorem 1.7, $\tau \in \mathcal{H} \cap \mathcal{R}$ and $\mu \in \mathcal{H} \cap \mathcal{R}$.

4. Probability measures with a first moment.

When μ has a first moment, a necessary condition for the BAR property is that μ be centred, i.e. $\sum_{n\in\mathbb{Z}} n\mu(n) = 0$ (see [6, Proposition 1.9]). Hence we cannot consider probability measures μ supported on $\mathbb{N}$ anymore. We shall deal with the following situation.

Definition 4.1. We say that a probability measure μ on $\mathbb{Z}$ is *CCM* if $\mu \neq \delta_0$, μ is supported on $\{-1\} \cup \mathbb{N}$ and there exists a finite positive measure ν on $[0, 1]$ such that

$$\int_0^1 \frac{\nu(dt)}{(1-t)^2} = 1$$

and

$$\mu(n) := \int_0^1 t^n\,\nu(dt) \quad \forall n \in \mathbb{N}, \qquad \mu(-1) = 1 - \int_0^1 \frac{\nu(dt)}{1-t} = \int_0^1 \frac{t\,\nu(dt)}{(1-t)^2}.$$

It is not hard to see that μ is indeed a probability measure and that it is centred. Since $\mu \neq \delta_0$, we have $\nu \neq \delta_0$ and $\mu(n) > 0$ for every $n \in \mathbb{N}$. In particular, μ is strictly aperiodic.

4.1. Characterization of the BAR property. Let μ be a CCM probability measure on $\mathbb{Z}$ with representing measure ν. For every $\theta \in [-\pi, \pi]$, we have

$$\begin{aligned}\hat{\mu}(\theta) &= e^{-i\theta} \int_0^1 \frac{t\,\nu(dt)}{(1-t)^2} + \sum_{n\geq 0} e^{in\theta} \int_0^1 t^n\,\nu(dt)\\ &= \int_0^1 \frac{e^{-i\theta}t - t^2 + (1-t)^2}{(1-t)^2(1-te^{i\theta})}\,\nu(dt)\\ &= \int_0^1 \frac{1-2t+2t^2e^{-i\theta} - t^2e^{-2i\theta}}{(1-t)^2((1-t)^2 + 2t(1-\cos\theta))}\,\nu(dt).\end{aligned}$$

In particular, using $\cos\theta = 2\cos^2\theta - 1$, we infer that

$$\begin{aligned}1 - \operatorname{Re}\hat{\mu}(\theta) &= \int_0^1 \frac{\nu(dt}{(1-t)^2} - \int_0^1 \frac{1-2t+2t^2\cos\theta - t^2\cos(2\theta)}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\\ &= \int_0^1 \frac{2t(1-\cos\theta) + 2t^2(1-\cos\theta) + t^2(\cos(2\theta)-1)}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\\ &= (1-\cos\theta)\int_0^1 \frac{2t(1-t\cos\theta)}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\end{aligned}\tag{4.1}$$

and

$$\operatorname{Im}\hat{\mu}(\theta) = 2\sin\theta\cdot(1-\cos\theta)\int_0^1 \frac{t^2}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\,\nu(dt).\tag{4.2}$$

Consider the following condition on ν: there exist $L > 0$ and $\gamma \in [0, 1)$ such that

$$\frac{1}{1-x}\int_x^1 \frac{t\,\nu(dt)}{(1-t)^2} \leq L\int_0^x \frac{t\,\nu(dt)}{(1-t)^3} \qquad \forall x \in [\gamma, 1).\tag{4.3}$$

Notice that if $\int_0^1 \frac{\nu(dt)}{(1-t)^3} < \infty$ (i.e. μ has a moment of order 2), then $\frac{1}{1-x}\int_x^1 \frac{t\,\nu(dt)}{(1-t)^2}$ $\leq \int_x^1 \frac{t\,\nu(dt)}{(1-t)^3} \to 0$ as $x \to 1$ and condition (4.3) is automatically satisfied since $\nu \neq \delta_0$.

Proposition 4.2. *Let μ be a CCM probability measure on $\mathbb{Z}$ with representing measure ν. Then μ has BAR if and only if there exist $L > 0$ and $\gamma \in [0,1)$ such that ν satisfies* (4.3).

Proof. Assume (4.3) for some $L > 0$ and some $\gamma \in [0,1)$. Let us prove that μ satisfies (3.3). Since μ is strictly aperiodic and $\hat{\mu}$ is continuous, it is enough to prove that

$$\sup_{\theta \in [\gamma-1, 1-\gamma]} \frac{|\operatorname{Im}\hat{\mu}(\theta)|}{1-\operatorname{Re}\hat{\mu}(\theta)} < \infty. \tag{4.4}$$

From (3.6) and $(1-t)^2 + 2t(1-\cos\theta) \geq \max((1-t)^2, 2t(1-\cos\theta))$ we see that

$$|\operatorname{Im}\hat{\mu}(\theta)| \leq |\theta|^3 \int_0^{1-|\theta|} \frac{t}{(1-t)^4}\,\nu(dt) + |\theta| \int_{1-|\theta|}^1 \frac{t\,\nu(dt)}{(1-t)^2}.$$

Notice that $1 - t\cos\theta \geq 1-t$ and, if $0 \leq t \leq 1-|\theta|$, then (using (3.6)) $(1-t)^2 + 2t(1-\cos\theta) \leq 2(1-t)^2$. Hence, by (4.1) and (3.7),

$$1 - \operatorname{Re}\hat{\mu}(\theta) \geq \frac{\theta^2}{4} \int_0^{1-|\theta|} \frac{t\,\nu(dt)}{(1-t)^3}, \tag{4.5}$$

and (4.4) holds, by (4.3).

Let us prove that if μ has BAR, then (4.3) holds for some $L > 0$ and $\gamma \in [0,1)$. There exists $C > 0$ such that for every $\theta \in [-\pi, \pi]$,

$$|\operatorname{Im}\hat{\mu}(\theta)| \leq C(1 - \operatorname{Re}\hat{\mu}(\theta)). \tag{4.6}$$

Let $\alpha \in (0, 1/2]$ be fixed for the moment. Let $\theta \in [-\alpha, \alpha]$. Notice that if $1 - \alpha|\theta| \leq t \leq 1$, we have $(1-t)^2 + 2t(1-\cos\theta) \leq (1+\alpha^2)\theta^2$. Hence, using (3.7) and (4.2), we deduce

$$|\operatorname{Im}\hat{\mu}(\theta)| \geq \frac{(1-\alpha^2)|\theta|}{\pi(1+\alpha^2)} \int_{1-\alpha|\theta|}^1 \frac{t\,\nu(dt)}{(1-t)^2} \geq \frac{3|\theta|}{5\pi} \int_{1-\alpha|\theta|}^1 \frac{t\,\nu(dt)}{(1-t)^2}. \tag{4.7}$$

Notice that for every $t \in [0, 1-\alpha|\theta|]$,

$$1 - t\cos\theta \leq (1-t) + (1-\cos\theta) \leq \frac{2\alpha^2+1}{\alpha^2}(1-t),$$

and for every $t \in [1-\alpha|\theta|, 1]$ (recall that $|\theta| \leq \alpha$),

$$1 - t\cos\theta \leq 1 - (1-\alpha|\theta|)\cos\theta \leq 3\alpha|\theta|/2.$$

Hence, using (4.1), we infer that there exists $C_\alpha > 0$ such that

$$1 - \operatorname{Re}\hat{\mu}(\theta) \leq C_\alpha|\theta|^2 \int_0^{1-\alpha|\theta|} \frac{t\,\nu(dt)}{(1-t)^3} + \frac{3\alpha|\theta|}{2(1-\alpha|\theta|)} \int_{1-\alpha|\theta|}^1 \frac{t\,\nu(dt)}{(1-t)^2}. \tag{4.8}$$

Now, using (4.6)–(4.8) and the fact that $1-\alpha\theta \ge 3/4$, we obtain

$$\frac{3|\theta|}{5\pi}\int_{1-\alpha|\theta|}^{1}\frac{t\,\nu(dt)}{(1-t)^2} \le CC_\alpha|\theta|^2\int_0^{1-\alpha|\theta|}\frac{t\,\nu(dt)}{(1-t)^3} + 2C\alpha|\theta|\int_{1-\alpha|\theta|}^{1}\frac{t\,\nu(dt)}{(1-t)^2}.$$

Taking $\alpha = \frac{3}{20(C+1)\pi} \le 1/2$ gives the desired result. ■

As before, we shall now characterize the BAR property in terms of the coefficients of μ.

PROPOSITION 4.3. *Let μ be a CCM probability measure on $\mathbb{Z}$ with representing measure ν. Then ν satisfies* (4.3) *if and only if there exists $L > 0$ such that*

$$n\sum_{k\ge n} k\mu(k) \le L\sum_{k=1}^{n} k^2\mu(k) \qquad \forall n \ge 1. \tag{4.9}$$

Proof. Assume (4.3). Let $n \ge 2$. Then

$$n\sum_{k\ge n} k\mu(k) \le \int_0^{1-1/n}\sum_{k\ge n} k^2t^k\,\nu(dt) + n\int_{1-1/n}^{1}\frac{t\,\nu(dt)}{(1-t)^2} \le (1+L)\int_0^{1-1/n}\frac{t\,\nu(dt)}{(1-t)^3}.$$

Now, for every $1 \le \ell \le n-1$ and every $t \in [1-1/\ell, 1-1/(\ell+1)]$, we have $\sum_{k=1}^{n} k^2t^k \ge t\sum_{k=1}^{\ell} k^2e^{-1} \ge Ct/(1-t)^3$, where we have used the fact that $(1-1/m)^{m-1}$ decreases to e^{-1}. Hence, (4.9) holds.

Conversely, assume that (4.9) holds. Let $\gamma \in (0,1]$ and $n \ge 2$. For every $t \in [1-1/n, 1]$, since $\gamma \le 1$, we have

$$\sum_{k\ge\gamma n} kt^k = \sum_{k\ge 0}(k+n)t^{k+n}$$

Hence

$$n\int_{1-1/n}^{1}\frac{t\,\nu(dt)}{(1-t)^2} \le 2en\sum_{k\ge\gamma n} k\mu(k) \le \frac{2nLe}{[\gamma n]}\sum_{k=1}^{[\gamma n]} k^2\mu(k)$$
$$\le \frac{2nLe}{[\gamma n]}\bigg(\int_0^{1-1/n}\frac{t\,\nu(dt)}{(1-t)^3} + [\gamma n]^2\int_{1-1/n}^{1}\frac{t\,\nu(dt)}{1-t}\bigg),$$

and we conclude the proof by taking γ small enough. ■

THEOREM 4.4. *Let μ be a CCM probability measure on $\mathbb{Z}$ with representing measure ν. Assume that $(\mu(n))_{n\in\mathbb{N}}$ satisfies* (4.9)*. Then μ satisfies hypothesis* ($\tilde{\mathbf{H}}$)*. In particular $\mu \in \mathcal{R}$, and for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that* (1.5) *holds.*

Proof. We only have to check that μ satisfies hypothesis ($\tilde{\mathbf{H}}$). Indeed, then, by Proposition 2.6, $\mu \in \mathcal{R}$, and by Proposition 2.3, $\mu \in \mathcal{H}$ and μ satisfies (1.5).

To check the conditions we must estimate $\hat{\mu}$ and its derivatives. Define

$$\psi(\theta) = \theta^2 \int_0^1 \frac{t\,\nu(dt)}{(1-t)((1-t)^2+\theta^2)} = \int_0^1 \frac{t\,\nu(dt)}{1-t} - \int_0^1 \frac{t(1-t)\,\nu(dt)}{(1-t)^2+\theta^2}. \tag{4.10}$$

Then

$$\psi'(\theta) = 2\theta \int_0^1 \frac{t(1-t)\,\nu(dt)}{((1-t)^2+\theta^2)^2}. \tag{4.11}$$

Hence for every $\theta \in [0,1/2]$, we have

$$\begin{aligned} &\frac{\theta^2}{2}\int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^3} + \frac{1}{2}\int_{1-\theta}^1 \frac{t\,\nu(dt)}{1-t} \\ &\qquad\qquad \le \psi(\theta) \le \theta^2 \int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^3} + \int_{1-\theta}^1 \frac{t\,\nu(dt)}{1-t}, \\ &\theta \int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^3} + \frac{1}{\theta^3}\int_{1-\theta}^1 t(1-t)\,\nu(dt) \\ &\qquad\qquad \le \psi'(\theta) \le 2\theta \int_0^{1-\theta} \frac{t\,\nu(dt)}{(1-t)^3} + \frac{2}{\theta^3}\int_{1-\theta}^1 t(1-t)\,\nu(dt). \end{aligned} \tag{4.12}$$

Notice that $\int_{1-\theta}^1 \frac{t\,\nu(dt)}{1-t} \le \theta \int_{1-\theta}^1 \frac{t\,\nu(dt)}{(1-t)^2}$. In particular, using (4.3), we see that (2.13) holds.

Let us compute the derivatives of $\hat{\mu}$. We shall not give the full details here. Using (4.1), we infer that

$$1 - \operatorname{Re}\hat{\mu}(\theta) = \int_0^1 \frac{1-t\cos\theta}{(1-t)^2}\,\nu(dt) - \int_0^1 \frac{1-t\cos\theta}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt),$$

and

$$\begin{aligned} \operatorname{Re}\hat{\mu}'(\theta) &= 2\sin\theta\cdot(1-\cos\theta)\int_0^1 \frac{t^2}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\,\nu(dt) \\ &\qquad + \sin\theta\cdot\int_0^1 \frac{2t(1-t\cos\theta)}{((1-t)^2+2t(1-\cos\theta))^2}\,\nu(dt) \\ &= -\sin\theta\cdot\int_0^1 \frac{t}{(1-t)^2}\,\nu(dt) - \sin\theta\cdot\int_0^1 \frac{t(1-t^2)}{((1-t)^2+2t(1-\cos\theta))^2}\,\nu(dt) \end{aligned} \tag{4.13}$$

and

$$\operatorname{Re}\hat{\mu}''(\theta) = -\cos\theta \cdot \int_0^1 \frac{t}{(1-t)^2}\,\nu(dt) - \cos\theta \cdot \int_0^1 \frac{t(1-t^2)}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt) + 4\sin^2\theta \cdot \int_0^1 \frac{t^2(1-t^2)}{((1-t)^2 + 2t(1-\cos\theta))^3}\,\nu(dt). \tag{4.14}$$

Using (4.2), we infer that

$$\operatorname{Im}\hat{\mu}(\theta) = \sin\theta \cdot \int_0^1 \frac{t}{(1-t)^2}\,\nu(dt) - \sin\theta \cdot \int_0^1 \frac{t}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt),$$

and

$$\begin{aligned}\operatorname{Im}\hat{\mu}'(\theta) &= 2\cos\theta \cdot (1-\cos\theta) \int_0^1 \frac{t^2}{(1-t)^2((1-t)^2 + 2t(1-\cos\theta))}\,\nu(dt) \\ &= \cos\theta \cdot \int_0^1 \frac{t}{(1-t)^2}\,\nu(dt) + \int_0^1 \frac{2t^2 - t(1+t^2)\cos\theta}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt)\end{aligned} \tag{4.15}$$

and

$$\operatorname{Im}\hat{\mu}''(\theta) = -\sin\theta \cdot \int_0^1 \frac{t}{(1-t)^2}\,\nu(dt) + \sin\theta \cdot \int_0^1 \frac{t(1+t^2)}{((1-t)^2 + 2t(1-\cos\theta))^2}\,\nu(dt) - 4\sin\theta \cdot \int_0^1 \frac{t(2t^2 - t(1+t^2)\cos\theta)}{((1-t)^2 + 2t(1-\cos\theta))^3}\,\nu(dt). \tag{4.16}$$

We now derive the necessary estimates on $\hat{\mu}$ and its derivatives. Using (4.13), (4.15) and (4.14) we deduce, respectively, Claims 5–7:

CLAIM 5. *There exists $C > 0$ such that*

$$|\operatorname{Re}\hat{\mu}'(\theta)| \le C\theta \int_0^{1-\theta} \frac{t}{(1-t)^3}\,\nu(dt) + \frac{C}{\theta^2} \int_{1-\theta}^1 t\,\nu(dt) \quad \text{for every } \theta \in (0, 1/2].$$

CLAIM 6. *There exists $C > 0$ such that*

$$|\operatorname{Im}\hat{\mu}'(\theta)| \le C\theta^2 \int_0^{1-\theta} \frac{t}{(1-t)^4}\,\nu(dt) + C \int_{1-\theta}^1 \frac{t}{(1-t)^2}\,\nu(dt)$$

for every $\theta \in (0, 1/2]$.

CLAIM 7. *There exists $C > 0$ such that*

$$|\mathrm{Re}\, \hat{\mu}''(\theta)| \le C \int_0^{1-\theta} \frac{t}{(1-t)^3}\, \nu(dt) + \frac{C}{\theta^4} \int_{1-\theta}^{1} t(1-t)\, \nu(dt)$$

for every $\theta \in (0, 1/2]$.

Notice that there exists $\alpha > 0$ such that for all $t \in [0,1]$ and $\theta \in (0,1/2]$,
$|2t^2 - t(1+t^2)\cos\theta)| = t|(1+t^2)(1-\cos\theta) - (1-t)^2| \le \alpha \max(\theta^2, (1-t)^2)$.
Combining this estimate with (4.16), we get

CLAIM 8. *There exists $C > 0$ such that*

$$|\mathrm{Im}\, \hat{\mu}''(\theta)| \le C\theta \int_0^{1-\theta} \frac{t}{(1-t)^4}\, \nu(dt) + \frac{C}{\theta^3} \int_{1-\theta}^{1} t\, \nu(dt),$$

for every $\theta \in (0, 1/2]$.

We already saw that (2.13) holds. Let us prove that (**H**)(i)–(iv) hold.
(**H**)(i) follows from (4.5) and (4.3) (see the proof of Proposition 3.10).
(**H**)(ii) follows from Claims 5 and 6 combined with (4.3) and (4.12).
(**H**)(iii) follows from (**H**)(ii) combined with (2.13).
(**H**)(iv) follows from Claims 7 and 8 combined with (4.3) and (4.12). ■

PROPOSITION 4.5. *Let τ be a centred probability measure on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} |n|\tau(n) < \infty$. Assume that there exists a CCM probability measure μ satisfying* (4.9) *and there exists $a > 0$ such that $\sum_{n\in\mathbb{Z}} n^2|\tau(n) - a\mu(n)| < \infty$. Then the conclusion of Theorem* 4.4 *holds for τ.*

Proof. We shall assume that $\sum_{n\in\mathbb{Z}} n^2\tau(n) = \infty$, otherwise, the result holds by Theorem 1.6. In particular we must have $\sum_{n\in\mathbb{Z}} n^2\mu(n) = \infty$ and by (4.10) and (4.11),

$$\liminf_{\theta\to 0,\, \theta\in(0,\pi]} \psi(\theta)/\theta^2 = \infty \quad \text{and} \quad \liminf_{\theta\to 0,\, \theta\in(0,\pi]} \psi'(\theta)/\theta = \infty. \tag{4.17}$$

It follows from the proof of Theorem 4.4 that there exists an even function ψ continuous on $[-\pi,\pi]$ and continuously differentiable on $(0,\pi]$, with $\psi(0) = 0$, such that μ and ψ satisfy (**H**)(i)–(iv) for some $C, c > 0$.

Since $\hat{\tau} = (\hat{\tau} - a\hat{\nu}) + a\hat{\mu}$ is clearly twice differentiable on $(0,\pi]$, the proposition will be proved if we can show that (**H**)(i)–(iv) hold with τ in place of μ with the same ψ, but for possibly different $C, c > 0$.

We already saw that τ must be strictly aperiodic. Hence $|\hat{\mu}| < 1$ on $(0,\pi]$. In particular, to prove (**H**)(i) it suffices to consider $\theta \in (0,\eta]$ for some small enough $\eta > 0$.

For every $\theta \in (0,\pi]$, we have

$$\begin{aligned}\hat{\tau}(\theta) &= \sum_{n\in\mathbb{Z}}(\tau(n)-a\mu(n))(e^{in\theta}-1) + \Big[1-a+a\sum_{n\in\mathbb{Z}}\mu(n)e^{in\theta}\Big]\\ &:= \chi(\theta)+\phi(\theta).\end{aligned}$$

As $\sum_{n\in\mathbb{Z}} n^2|\tau(n)-a\mu(n)|<\infty$ and $\sum_{n\in\mathbb{Z}} n(\tau(n)-a\mu(n))=0$, we see that $\lim_{\theta\to 0,\,\theta\neq 0}\chi(\theta)/\theta^2=\chi''(0)$ exists. In particular, $\lim_{\theta\to 0,\,\theta\neq 0}\chi(\theta)/\psi(\theta)=0$.

Now, since μ has BAR, there exists $C>0$ such that

$$\begin{aligned}|\phi(\theta)|^2 &= (1-a+a\operatorname{Re}\hat{\mu}(\theta))^2+a^2(\operatorname{Im}\hat{\mu}(\theta))^2\\ &= 1-2a(1-\operatorname{Re}\hat{\mu}(\theta))+a^2(1-\operatorname{Re}\hat{\mu}(\theta))^2+a^2(\operatorname{Im}\hat{\mu}(\theta))^2\\ &\le 1+(1-\operatorname{Re}\hat{\mu}(\theta))\big(-2a+a^2C(1-\operatorname{Re}\hat{\mu}(\theta))\big).\end{aligned}$$

Hence, using (4.5), (4.3) and (4.12), we infer that there exists $\eta>0$ such that for every $\theta\in(0,\eta]$, $|\hat{\tau}(\theta)|\le 1-\delta\psi(\theta)$ for some $\delta>0$. Since μ is strictly aperiodic and $\hat{\mu}$ is continuous, on taking c smaller if necessary, the latter holds for every $\theta\in[0,\pi]$.

The proofs of (**H**)(ii)–(iv) are similar (but simpler), hence we leave them to the reader. ■

4.2. Proof of Theorem 1.8 and example

Proof of Theorem 1.8. Let μ be a CCM probability measure on $\mathbb{Z}$ with representing measure ν. Item (i) follows from Proposition 4.2 combined with Proposition 4.3. Item (ii) is just Proposition 4.5.

Example 3. The next example is based on Theorem 1.8. Let $\alpha\in(2,\infty)$ and $\alpha_1,\dots,\alpha_k\ge 0$. Let μ be a probability on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}}|n|\mu(n)<\infty$, $\sum_{n\in\mathbb{Z}}n\mu(n)=0$ and $\sum_{n\in\mathbb{Z}}n^2|\mu(n)-af_{\alpha,\alpha_1,\dots,\alpha_k}(n+1)| <\infty$ for some $a>0$, where $f_{\alpha,\alpha_1,\dots,\alpha_k}$ is extended to $\mathbb{Z}^-$ by setting $f_{\alpha,\alpha_1,\dots,\alpha_k}(-n)=0$ for every $n\in\mathbb{N}$.

5. Symmetric probability measures.

If μ is symmetric (i.e. $\check{\mu}=\mu$), then $\hat{\mu}$ is real-valued, hence has BAR. It is known that if moreover $(\mu(n))_{n\in\mathbb{N}}$ is non-increasing then (2.6) holds. However, we are not aware of any result concerning the Ritt property or (2.7) with $m\ge 1$.

We shall again deal with completely monotone coefficients. To be more precise we consider the following situation.

Definition 5.1. We say that a probability measure μ on $\mathbb{Z}$ is *SCM* if it is symmetric and there exists a finite positive measure on $[0,1]$ such that

$$\int_0^1 \frac{1}{1-t}\,\nu(dt)=1/2,\qquad \mu(0)=2\int_0^1\nu(dt),\qquad \mu(n)=\int_0^1 t^n\,\nu(dt)\quad \forall n\ge 1.$$

Let μ be an SCM probability measure on $\mathbb{Z}$ with representing measure ν. Define another measure on $\mathbb{Z}$, supported on $\mathbb{N}$, by setting

$$\mu_1(0) = 2\int_0^1 \nu(dt), \qquad \mu_1(n) = 2\int_0^1 t^n\,\nu(dt) \quad \forall n \geq 1.$$

Then μ_1 is a probability measure, $(\mu_1(n))_{n\in\mathbb{N}}$ is completely monotone and $\mu = \frac{1}{2}(\check{\mu}_1 + \mu_1)$. In particular, it follows from Propositions 2.4 and 3.10 that μ satisfies hypothesis $(\mathbf{H})$ as soon as μ satisfies (3.14). The fact that μ is Ritt when it satisfies (3.14) may be proved similarly (but more easily).

We could use a similar argument based on Theorem 4.4. However, doing so, we would miss some symmetric probability measures satisfying hypothesis $(\mathbf{H})$.

Let us be more precise. Let μ be an SCM probability measure. It follows from previous computations that, for every $\theta \in \mathbb{R}$,

$$1 - \hat{\mu}(\theta) = 1 - \operatorname{Re}\hat{\mu}(\theta) = \int_0^1 \frac{t(1-\cos\theta)}{(1-t)((1-t)^2 + 2t(1-\cos\theta))}\,\nu(dt).$$

Consider the following condition on ν: there exists $L > 0$ such that for every $x \in [0,1)$,

$$\int_x^1 \frac{t}{1-t}\,\nu(dt) \leq L(1-x)^2 \int_0^x \frac{t}{(1-t)^3}\,\nu(dt). \tag{5.1}$$

This condition can be proved to be equivalent to: there exists $D > 0$ such that for every $n \geq 1$,

$$n^2 \sum_{k\geq n} \mu(k) \leq L \sum_{k=1}^{n} k^2 \mu(k). \tag{5.2}$$

One can prove that if (5.1) holds, then μ satisfies hypothesis $(\tilde{\mathbf{H}})$ with ψ given by

$$\psi(\theta) = \theta^2 \int_0^1 \frac{t}{(1-t)(1-t+|\theta|)^2}\,\nu(dt) \qquad \forall \theta \in [-\pi,\pi] - \{0\}.$$

Notice that $\psi'(\theta) = 2\theta \int_0^1 \frac{t}{(1-t+\theta)^3}\,\nu(dt)$ for every $\theta \in (0,\pi]$.

Then one can prove that an SCM probability measure satisfying (5.2) is Ritt and satisfies (2.7) for every $m \in \mathbb{N}$ and some $C_m > 0$.

In particular, we have the following.

THEOREM 5.2. *Let μ be an SCM probability measure such that $(\mu(n))_{n\in\mathbb{N}}$ satisfies either* (3.14) *or* (5.2). *Then μ is Ritt and satisfies* (2.7) *for every $m \in \mathbb{N}$ and some $C_m > 0$.*

EXAMPLE 4. Let $\alpha > 1$ and $\alpha_1, \ldots, \alpha_k \geq 0$. Let μ be a symmetric probability measure defined by $\mu(0) = 2cf_{\alpha,\alpha_1,\ldots,\alpha_k}(1)$ and $\mu(n) = cf_{\alpha,\alpha_1,\ldots,\alpha_k}(n+1)$ for every $n \geq 1$, where c is a normalizing constant. Then μ is an SCM probability measure for which the above theorem applies.

6. Discussion and open questions. Most of the examples of (strictly aperiodic) probability measures on $\mathbb{Z}$ that have BAR are known to be Ritt. We believe that the BAR property and the Ritt property are not equivalent, but one has to find a counterexample. This problem was also formulated by Dungey [10, remarks on p. 1729].

One may wonder whether, in the symmetric case, the condition "$(\mu(n))_{n\in\mathbb{N}}$ is non-increasing" is sufficient for the Ritt property or for weak type maximal inequalities (1.5) to hold, since it is sufficient for the weak type maximal inequality (1.4). At least, for an SCM probability measure on $\mathbb{Z}$, can one remove the conditions (3.14) and (5.2) from Theorem 5.2?

Let μ be a probability measure on $\mathbb{Z}$. Let $f \in \ell^p(\mathbb{Z})$, $p \geq 1$. Consider the square function defined by

$$s_\mu(f)(k) := \Big(\sum_{n\geq 1} n\big((\mu^{*n} - \mu^{*(n+1)}) * f(k)\big)^2\Big)^{1/2}.$$

Assume that μ has BAR. When $p > 1$, it follows from the work of Le Merdy and Xu [17] that there exists $C_p > 0$ such that for every $f \in \ell^p(\mathbb{Z})$, $\|s_\mu(f)\|_p \leq C_p\|f\|_p$, i.e. $s(f)$ satisfies a strong p-p inequality. A natural question is whether $s(f)$ satisfies a weak 1-1 inequality.

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