LIE ALGEBROIDS
BANACH CENTER PUBLICATIONS, VOLUME 54
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2001

COUPLING TENSORS AND POISSON GEOMETRY NEAR A SINGLE SYMPLECTIC LEAF

YURII VOROBJEV

Department of Applied Mathematics, Moscow State Institute of Electronics and Mathematics
Moscow, Russia, 109028
E-mail: vorob@amath.msk.ru

Departamento de Matemáticas, Universidad de Sonora
Hermosillo, México, 83000
E-mail: yurimv@guaymas.uson.mx

Abstract. In the framework of the connection theory, a contravariant analog of the Sternberg coupling procedure is developed for studying a natural class of Poisson structures on fiber bundles, called coupling tensors. We show that every Poisson structure near a closed symplectic leaf can be realized as a coupling tensor. Our main result is a geometric criterion for the neighborhood equivalence between Poisson structures over the same leaf. This criterion gives a Poisson analog of the relative Darboux theorem due to Weinstein. Within the category of the algebroids, coupling tensors are introduced on the dual of the isotropy of a transitive Lie algebroid over a symplectic base. As a basic application of these results, we show that there is a well defined notion of a "linearized" Poisson structure over a symplectic leaf which gives rise to a natural model for the linearization problem.

1. Introduction. The notion of a coupling form due to Sternberg [St] naturally arises from the study of fiber compatible (pre)symplectic structures on the total space of a symplectic bundle (for various aspects of this problem, see, for example, [Tu, We$_2$, We$_3$, GoLSW, GSt, GLS, KV$_1$]). This notion is based on the concept of connection and curvature and can be introduced for a wide class of bundles [GLS]. A derivation of the coupling procedure for the associated bundle $P \times_G \mathfrak{g}^*$ (called the universal phase space) via reduction was suggested in [We$_2$].

We are interested in a contravariant version of the Sternberg coupling procedure in the Poisson category. Suppose we start with a (locally trivial) Poisson fiber bundle

2000 *Mathematics Subject Classification*: 53D17, 53D35, 70G45.

Key words and phrases: Poisson manifold, fiber bundle, Ehresmann connection, coupling tensors, Lie algebroid, vertical Poisson structure, symplectic leaf.

Research partially supported by CONACYT Grants 28291-E and 35212-E.

The paper is in final form and no version of it will be published elsewhere.

$(E \xrightarrow{\pi} B, \mathcal{V}^{\mathrm{fib}})$ equipped with a smooth field $\mathcal{V}^{\mathrm{fib}} = \{B \ni b \mapsto \mathcal{V}^{\mathrm{fib}}_b \in \chi^2(E_b)\}$ of Poisson structures on the fibers. Unlike the symplectic case [GoLSW], the fiberwise Poisson structure $\mathcal{V}^{\mathrm{fib}}$ admits always a unique extension to a vertical Poisson tensor $\mathcal{V}$ on the total space E. Every Ehresmann connection $\mathrm{Hor} \to TE \xrightarrow{\Gamma} \mathrm{Vert}$ gives rise to the space of horizontal multivector fields on E. A connection is called Poisson if the parallel transport preserves $\mathcal{V}^{\mathrm{fib}}$. Given a Poisson connection Γ, the problem is to describe Poisson bivector fields on E of the form: $\Pi = (\Gamma\text{-horizontal bivector field}) + \mathcal{V}$. Putting this decomposition into the Jacobi identity, we get two quadratic equations for the horizontal part of Π: (I) the horizontal Jacobi identity and (II) the curvature identity. Under the assumption: Π is nondegenerate on the annihilator of the vertical subbundle Vert, equations (I) and (II) are reduced to linear equations for a horizontal 2-form $\mathbb{F}$. If $\mathbb{F}$ is a solution of these equations, then the corresponding Poisson tensor Π is just what we call a coupling tensor associated with data $(\Gamma, \mathcal{V}, \mathbb{F})$. Note that in the case of coadjoint bundles, Poisson structures of such a type arise naturally from the study of Wong's equations [MoMR, Mo].

In this paper we give a systematic treatment of coupling tensors. Our first observation is: in a tubular neighborhood E of a closed symplectic leaf B every Poisson structure Ψ is realized as a coupling tensor (locally, this follows from the splitting theorem [We_4]). As a consequence, Ψ induces an intrinsic Poisson connection Γ on E which gives rise to a geometric characteristic of the leaf. Moreover, the vertical part $\mathcal{V}$ of the coupling tensor at the leaf B is of rank 0 and gives a "global" realization of local transverse Poisson structures [We_4]. If the symplectic leaf B is regular, then $\mathcal{V} = 0$ and the coupling tensor is the horizontal lift of the nondegenerate Poisson structure on B via the flat connection associated with the symplectic foliation. We are interested in the non-flat case when the rank of a Poisson structure Ψ is not locally constant at B.

So, for the study of Poisson structures near a single symplectic leaf we may restrict our attention to the class of coupling tensors. Our main result is a geometric criterion for a neighborhood equivalence of two coupling near a common closed symplectic leaf. The criterion is formulated in terms of the intrinsic Poisson connection and its curvature and implies a Poisson version of the relative Darboux theorem [We_1]. This result continues our previous investigations of the formal Poisson equivalence [IKV] . To state the result, we use a contravariant analog of the homotopy method due to Moser [Mos] and Weinstein [We_1]. The technical part is based on the Schouten calculus [Li, KM, KV_2, Va, K-SM] and the (covariant) connection theory for general fiber bundles [GHV, GLS]. Note also that a geometric approach, based on the notion of a contravariant Poisson connection due to Vaisman [Va], was developed in [Fe].

As a basic application of the Poisson neighborhood theorem, we show that there exists a well defined notion of a linearized Poisson structure over a closed symplectic leaf which is well known in the zero-dimensional case [We_4]. The linearized Poisson structure is completely determined by the transitive Lie algebroid of a symplectic leaf [KV_2]. To derive this fact, we introduce and study a class of coupling tensors associated with transitive Lie algebroids over a symplectic base. This class consists of isomorphic Poisson structures parametrized by connections on the Lie algebroid in the sense of Mackenzie [Mz]. Here we use the technique of adjoint connections on the dual of a Lie algebriod. Adjoint connections naturally arise in the general theory of Lie algebriods [Mz] (also see [Ku]) as well as in the context of infinitesimal Poisson geometry [KV_2, IKV]. We show that the holonomy of adjoint connections is related with the notion of a linear Poisson holonomy introduced in [GiGo] (the definition of this notion in terms of contravariant connections can be found in [Fe]).

In this paper we do not discuss the linearization problem. But we hope that the linearized Poisson model, introduced here, can be used for extension of results on the linearizability at a point [We_4, Cn_1, Cn_2, Du] to the higher-dimentional case.

The body of the paper is organized as follows. In Section 2, a general description of coupling tensors in terms of geometric data is given in Theorem 2.1. In Section 3, we formulate main results on a neighborhood equivalence of two coupling tensors with the same symplectic leaf, Theorem 3.1. and Theorem 3.2. The important technical part of the proof of Lemma 3.1., is given in Appendix A. Section 4 is devoted to coupling tensors associated with transitive Lie algebroids. Here we show that the criterion in Theorem 3.1 leads to the equivalence relation for Lie algebroids. In Section 5, using results of Section 4, we give a definition of the linearized Poisson structure of a symplectic leaf and discuss some applications.

Acknowledgments. The main results of the paper were presented at the Conference "Poisson 2000" held at CIRM, Luminy, France, June 26–30, 2000. I am grateful to a number of people for helpful discussions and comments on this work: especially, to A. Bolsinov, J.-P. Dufour, R. Flores Espinoza, J. Huebschmann, M. V. Karasev, Y. Kosmann-Schwarzbach, P. Libermann, K. Mackenzie, and A. Weinstein.

I would like also to thank G. Dávila Rascón for his help in the last stages of preparation of this paper.

2. Coupling tensors. Let $\pi : E \to B$ be a fiber bundle (that is, a surjective submersion). Let $\mathrm{Vert} = \ker(d\pi) \subset TE$ be the vertical subbundle. Smooth sections of Vert form the Lie algebra of vertical vector fields on the total space E which will be denoted by $\mathcal{X}_V(E)$. Consider the annihilator $\mathrm{Vert}^0 \subset T^*E$ of the vertical subbundle. Sections of Vert^0 are called horizontal 1-forms. Denote by $\chi^k(E) = \Gamma(\Lambda^k TE)$ the space of k-vector fields on E. A k-vector field $T \in \chi^k(E)$ is said to be vertical if $\alpha\rfloor T = 0$ for every horizontal 1-form α. The space of vertical k-vector fields will be denoted by $\chi^k_V(E)$.

We say that a bivector field $\Pi \in \chi^2(E)$ is *horizontally nondegenerate* if for every $e \in E$ the antisymmetric bilinear form $\Pi_e : T^*_eE \times T^*_eE \to \mathbb{R}$ is nondegenerate on the subspace $\mathrm{Vert}^0_e \subset T^*_eE$. In other words,

$$\Pi^{\#}(\mathrm{Vert}^0) \cap \mathrm{Vert} = \{0\}, \tag{2.1}$$

$$\operatorname{rank} \Pi^{\#}(\mathrm{Vert}^0) = \dim B. \tag{2.2}$$

Here $\Pi^{\#} : T^*E \to TE$ is the vector bundle morphism associated with Π, $\Pi^{\#}(\alpha) := \alpha\rfloor\Pi$ ($\alpha \in \Omega^1(E)$).

Recall that a bivector field Π on E is said to be a *Poisson tensor* if Π satisfies the Jacobi identity [Li]

$$[\![\Pi, \Pi]\!]_E = 0. \tag{2.3}$$

Here $[\![\cdot,\cdot]\!]_E$ denotes the *Schouten bracket* for multivector fields on the total space E. The corresponding Poisson bracket is given by

$$\{F,G\}_\Pi = \Pi(dF,dG) = \left\langle dG, \Pi^\#(dF)\right\rangle .$$

The correspondence $C^\infty(E) \ni F \mapsto \Pi^\#(dF) \in \mathcal{X}(E)$ is a homomorphism from the Poisson algebra $(C^\infty(E),\{\ ,\ \}_\Pi)$ onto the Lie algebra $\mathcal{X}^{Ham}(E)$ of Hamiltonian vector fields.

Our goal is to describe horizontally nondegenerate Poisson tensors on E. First, we formulate some preliminary facts.

2.1. *Geometric data.* Suppose we are given a horizontally nondegenerate bivector field $\Pi \in \chi^2(E)$. By conditions (2.1), (2.2), we deduce that Π induces an *intrinsic Ehresmann connection* $\Gamma \in \Omega^1(E)\otimes\mathcal{X}_V(E)$ whose horizontal subbundle $\mathrm{Hor} = \ker\Gamma \subset TE$ is defined as the image of Vert^0 under the bundle map $\Pi^\#$,

$$\mathrm{Hor} := \Pi^\#(\mathrm{Vert}^0). \tag{2.4}$$

So, we have the splitting

$$TE = \mathrm{Hor} \oplus \mathrm{Vert}\,. \tag{2.5}$$

Then $\mathbb{H} = \mathrm{id} - \Gamma$ is the horizontal projection. Let $\mathrm{Hor}^0 \subset T^*E$ be the annihilator of the horizontal subbundle. Sections of Hor^0 are called vertical 1-forms. The set of k-vector fields $T \in \chi^k(E)$ such that $\beta\rfloor T = 0$ for every vertical 1-form β forms the space of horizontal k-vector fields denoted by $\chi^k_H(E)$. In particular, $\mathcal{X}_H(E) = \chi^1_H(E)$ will denote the space of horizontal vector fields on E.

The splitting (2.5) induces a $C^\infty(B)$ homomorphism

$$\mathrm{hor}:\ \mathcal{X}(B) \to \mathcal{X}_H(E)$$

sending a smooth vector field u on B to a smooth section $\mathrm{hor}(u)$ of Hor and satisfying $L_{\mathrm{hor}(u)}(\pi^* f) = \pi^*(L_u f)$ for every $f \in C^\infty(B)$. The vector field $\mathrm{hor}(u)$ is called the *horizontal lift* of a base vector field u, associated with the connection Γ. Notice that the flow Fl_t of $\mathrm{hor}(u)$ is π-related with the flow φ_t of u, $\pi\circ\mathrm{Fl}_t = \varphi_t\circ\pi$.

It follows from (2.4) that the subbundles Hor^0 and Vert^0 are Π-orthogonal. Thus there is a unique decomposition of Π into horizontal and vertical parts:

$$\Pi = \Pi_H + \Pi_V, \qquad \text{where} \quad \Pi_H \in \chi^2_H(E), \quad \Pi_V \in \chi^2_V(E). \tag{2.6}$$

Consider the horizontal part Π_H. The horizontal nondegeneracy of Π implies that the restriction

$$\overset{\circ}{\Pi}{}^\#_H := \Pi^\#_H\big|_{\mathrm{Vert}^0}\ :\ \mathrm{Vert}^0 \to \mathrm{Hor} \tag{2.7}$$

is a vector bundle isomorphism. Note that Π_H, as a horizontal bivector field, is uniquely determined by $\overset{\circ}{\Pi}{}^\#_H$.

Consider the tensor product $\Omega^k(B)\otimes C^\infty(E)$ of $C^\infty(B)$-modules. One can think of the elements of this space as k-forms on the base B with values in the space $C^\infty(E)$, that is, antisymmetric k-linear over $C^\infty(B)$ mappings $\mathcal{X}(B)\times\ldots\times\mathcal{X}(B) \to C^\infty(E)$. Hence if $\mathcal{F} \in \Omega^k(B)\otimes C^\infty(E)$ and $u_1,\ldots,u_k \in \mathcal{X}(B)$, then the restriction of $\mathcal{F}(u_1,\ldots,u_k)$ to the fiber E_b depends only on $u_1(b),\ldots,u_k(b)$ and we have a k-linear (over $\mathbb{R}$) mapping

$$\mathcal{F}_b : T_bB\times\ldots\times T_bB \to C^\infty(E_b).$$

Moreover, there is a natural identification of $\Omega^k(B)\otimes C^\infty(E)$ with the space $\Omega^k_H(E)$ of horizontal k-forms on E: $\Omega^k(B)\otimes C^\infty(E)\ni\mathcal{F}\mapsto\mathcal{F}^h\in\Omega^k_H(E)$, where $\mathcal{F}^h$ is a horizontal k-form defined by

$$\mathcal{F}^h(Y_1,\dots,Y_k):=\mathcal{F}_{\pi(e)}(d_e\pi Y_1,\dots,d_e\pi Y_k)(e)$$

for $Y_1,\dots,Y_k\in T_eE$, $e\in E$. In particular, $\mathcal{F}^h(\mathrm{hor}(u_1),\dots,\mathrm{hor}(u_k))=\mathcal{F}(u_1,\dots,u_k)$ and $(\omega\otimes 1)^h=\pi^*\omega$ for $\omega\in\Omega^k(B)$. We will say that $\mathcal{F}\in\Omega^2(B)\otimes C^\infty(E)$ is *nondegenerate* at a point $e\in E$ if the restriction of $\mathcal{F}^h$ to the horisontal space $\mathrm{Hor}_e\approx T_eE/\operatorname{Vert}_e$ is a nondegenerate bilinear form.

Let us associate to Π the 2-form $\mathbb{F}\in\Omega^2(B)\otimes C^\infty(E)$ defined by

$$\mathbb{F}(u_1,u_2):=-\big\langle\big(\overset{\circ}{\Pi}{}^{\#}_H\big)^{-1}\mathrm{hor}(u_1),\mathrm{hor}(u_2)\big\rangle \tag{2.8}$$

for $u_1,u_2\in\mathcal{X}(B)$. Note that the 2-form $\mathbb{F}$ is *nondegenerate,*

$$u\rfloor\mathbb{F}=0\qquad\text{for } u\in\mathcal{X}(B)\text{ implies } u=0. \tag{2.9}$$

Here the interior product $u\rfloor\mathbb{F}$ is an element of the space $\Omega^1(B)\otimes C^\infty(E)$.

Now we claim that horizontally nondegenerate bivector fields on E can be parametrized by some geometric data. By *geometric data* we mean a triple $(\Gamma,\mathcal{V},\mathbb{F})$ consisting of

- an Ehresmann connection Γ on π;
- a vertical bivector field $\mathcal{V}\in\chi^2_V(E)$;
- a nondegenerate $C^\infty(E)$-valued 2-form $\mathbb{F}\in\Omega^2(B)\otimes C^\infty(E)$ on the base B.

Direct mapping $\Pi\mapsto(\Gamma,\mathcal{V},\mathbb{F})$. We associate to a given horizontally nondegenerate bivector field Π on E the geometric data $(\Gamma,\mathcal{V},\mathbb{F})$, where

- $\Gamma: TE\to\mathrm{Vert}$ is the projection along the subbundle (2.4);
- $\mathcal{V}=\Pi_V$ is the vertical part of Π in (2.6);
- $\mathbb{F}$ is the 2-form in (2.8).

Inverse mapping $(\Gamma,\mathcal{V},\mathbb{F})\mapsto\Pi$. Taking geometric data $(\Gamma,\mathcal{V},\mathbb{F})$, we introduce the horizontally nondegenerate bivector field

$$\Pi=\tau_\Gamma(\mathbb{F})+\mathcal{V}, \tag{2.10}$$

where the Γ-dependent correspondence $\mathbb{F}\mapsto\tau_\Gamma(\mathbb{F})\in\chi^2_H(E)$ is defined in the following way. The nondegenerate 2-form $\mathbb{F}\in\Omega^2(B)\otimes C^\infty(E)$ induces the vector bundle isomorphism

$$\mathbb{F}^\flat:\ \mathrm{Hor}\to\mathrm{Vert}^0\approx\mathrm{Hor}^* \tag{2.11}$$

such that

$$\langle\mathbb{F}^\flat(\mathrm{hor}(u_1)),\mathrm{hor}(u_2)\rangle=\mathbb{F}(u_1,u_2) \tag{2.12}$$

for every $u_1,u_2\in\mathcal{X}(B)$. Then the horizontal bivector field $\tau_\Gamma(\mathbb{F})$ is determined by the condition

$$\tau_\Gamma(\mathbb{F})(\beta_1,\beta_2)=-\langle\beta_2,(\mathbb{F}^\flat)^{-1}\beta_1\rangle \tag{2.13}$$

for all $\beta_1,\beta_2\in\Gamma(\mathrm{Vert}^0)$.

2.2. *Coupling tensors.* Now we can try to rewrite the Jacobi identity (2.3) for a horizontally nondegenerate bivector field in terms of its geometric data. To formulate the result, we recall some definitions.

If Γ is an Ehresmann connection on a fiber bundle $\pi : E \to B$, then the curvature form is a vector valued 2-form $\mathrm{Curv}^\Gamma \in \Omega^2(B, \mathrm{Vert}) \approx \Omega^2(B) \otimes \mathcal{X}_V(E)$ on the base defined as

$$\mathrm{Curv}^\Gamma(u_1, u_2) := -\Gamma\big([\mathrm{hor}(u_1), \mathrm{hor}(u_2)]\big)$$

for $u_1, u_2 \in \mathcal{X}(B)$.

The connection Γ induces the *covariant exterior derivative* [GHV]

$$\partial_\Gamma : \Omega^k(B) \otimes C^\infty(E) \to \Omega^{k+1}(B) \otimes C^\infty(E)$$

taking a k-form $\mathcal{F}$ to a $(k+1)$-form $\partial_\Gamma \mathcal{F}$, which at vector fields $u_0, u_1, \dots, u_k \in \mathcal{X}(B)$ is:

$$\begin{aligned}(\partial_\Gamma \mathcal{F})(u_0, u_1, \dots, u_k) := \sum_{i=0}^{k} (-1)^i L_{\mathrm{hor}(u_i)} \mathcal{F}(u_0, u_1, \dots, \hat{u}_i, \dots, u_k) \\ + \sum_{0 \le i < j \le k} (-1)^{i+j} \mathcal{F}\big([u_i, u_j], u_0, u_1, \dots, \hat{u}_i, \dots, \hat{u}_j, \dots, u_k\big).\end{aligned} \tag{2.14}$$

Notice that ∂_Γ is a coboundary operator, $\partial_\Gamma^2 = 0$ if and only if $\mathrm{Curv}^\Gamma = 0$, that is, Γ is *flat*. Moreover, we have $(\partial_\Gamma \mathcal{F})^h = \mathbb{H}_* \circ d(\mathcal{F}^h)$. Here $\mathbb{H}_* : \Omega^k(E) \to \Omega^k_H(E)$ is the horizontal projection and d is the usual differential of forms. In particular,

$$\partial_\Gamma(\omega \otimes 1) = d\omega \otimes 1 \tag{2.15}$$

for every k-form ω on the base B.

Theorem 2.1. *Let $\pi : E \to B$ be a fiber bundle. A horizontally nondegenerate bivector field $\Pi \in \chi^2(E)$ is a Poisson tensor if and only if its geometric data $(\Gamma, \mathcal{V}, \mathbb{F})$ satisfy the following conditions:*

(i) *the vertical part $\mathcal{V}$ of Π is a Poisson tensor,*

$$[\![\mathcal{V}, \mathcal{V}]\!]_E = 0; \tag{2.16}$$

(ii) *the connection Γ preserves $\mathcal{V}$, that is, for every $u \in \mathcal{X}(B)$ the horizontal lift* $\mathrm{hor}(u)$ *is an infinitesimal automorphism of $\mathcal{V}$,*

$$L_{\mathrm{hor}(u)} \mathcal{V} \equiv [\![\mathrm{hor}(u), \mathcal{V}]\!]_E = 0; \tag{2.17}$$

(iii) *the nondegenerate 2-form $\mathbb{F} \in \Omega^2(B) \otimes C^\infty(E)$ satisfies*

$$\partial_\Gamma \mathbb{F} = 0, \tag{2.18}$$

and the "curvature identity"

$$\mathrm{Curv}^\Gamma(u_1, u_2) = \mathcal{V}^\# d\mathbb{F}(u_1, u_2) \tag{2.19}$$

for $u_1, u_2 \in \mathcal{X}(B)$.

The proof is a direct verification with the use of the Poisson–Ehresmann calculus. For a symplectic version of Theorem 2.1 see [GLS].

So, if geometric data $(\Gamma, \mathcal{V}, \mathbb{F})$ satisfy conditions (2.16)–(2.19), then formula (2.10) determines a Poisson tensor Π on E which will be called a *coupling tensor* associated with $(\Gamma, \mathcal{V}, \mathbb{F})$.

The hypotheses in Theorem 2.1 have the following interpretations. Suppose we are given some geometric data $(\Gamma, \mathcal{V}, \mathbb{F})$ satisfying conditions (2.16)–(2.19).

It follows from (2.16) that $(E, \mathcal{V})$ is a Poisson manifold with vertical Poisson tensor $\mathcal{V}$. Let $\mathrm{Casim}_{\mathcal{V}}(E)$ be the space of *Casimir functions* on $(E, \mathcal{V})$, that is, the center of the Poisson algebra $(C^\infty(E), \{\ ,\ \}_{\mathcal{V}})$. Clearly, $\pi^* C^\infty(B) \subset \mathrm{Casim}_{\mathcal{V}}(E)$. Every fiber $E_b = \pi^{-1}(b)$ $(b \in B)$ is a *Poisson submanifold* of $(E, \mathcal{V})$ which carries a unique Poisson structure $\mathcal{V}_b^{\mathrm{fib}}$ with the property: the inclusion $E_b \hookrightarrow E$ is a Poisson mapping (see [We$_4$]). Thus, $\mathcal{V}$ induces a smooth field of Poisson structures on the fibers: $B \ni b \mapsto \mathcal{V}_b^{\mathrm{fib}} \in \chi^2(E_b)$ called a *fiberwise Poisson structure*. Notice that if we start with a locally trivial Poisson fiber bundle (the typical fiber is a Poissin manifold), then the fiberwise Poisson structure induces a unique compatible vertical Poisson tensor.

Condition (2.17) means that the horizontal lift $\mathrm{hor}(u)$ of every base vector field u is a *Poisson vector field* (an *infinitesimal Poisson automorphism*) of the vertical Poisson structure $\mathcal{V}$. The Ehresmann connection Γ is compatible with the fiberwise Poisson structure in the sense that the (local) flow Fl_t of $\mathrm{hor}(u)$ is a fiber preserving Poisson morphism. In other words, the parallel transport associated with the connection Γ preserves the fiberwise Poisson structure. Such a connection is called a *Poisson connection* on a bundle of Poisson manifolds.

Denote by $\mathcal{X}_V^{\mathrm{Poiss}}(E)$ the Lie algebra of vertical Poisson vector fields and by $\mathcal{X}_V^{\mathrm{Ham}}(E)$ the Lie subalgebra of vertical Hamiltonian vector fields on $(E, \mathcal{V})$. Then $\mathcal{X}_V^{\mathrm{Ham}}(E)$ is an ideal in $\mathcal{X}_V^{\mathrm{Poiss}}(E)$. Consider the quotient space

$$\mathcal{H}_V^1(E; \mathcal{V}) = \mathcal{X}_V^{\mathrm{Poiss}}(E)/\mathcal{X}_V^{\mathrm{Ham}}(E). \tag{2.20}$$

Notice that the symplectic leaf of $\mathcal{V}$ through a point $e \in E$ coincides with the symplectic leaf of the Poisson structure $\mathcal{V}_b^{\mathrm{fib}}$ on the fiber $E_{\pi(e)}$. Hence every Hamiltonian vector field on $(E, \mathcal{V})$ is an element of $\mathcal{X}_V^{\mathrm{Ham}}(E)$. Moreover, every Poisson vector field of $\mathcal{V}$ is represented as a sum of a horozontal lift of some base field and an element of $\mathcal{X}_V^{\mathrm{Poiss}}(E)$. From here we deduce: the first Poisson cohomology space [Li, KM, Va] of $\mathcal{V}$ is isomorphic to the direct sum $\mathcal{X}(B) \oplus \mathcal{H}_V^1(E; \mathcal{V})$.

One can show that condition (2.17) implies the property: for every $u_1, u_2 \in \mathcal{X}(B)$ the curvature vector field $\mathrm{Curv}^\Gamma(u_1, u_2)$ is a vertical Poisson vector field,

$$\mathrm{Curv}^\Gamma(u_1, u_2) \in \mathcal{X}_V^{\mathrm{Poiss}}(E). \tag{2.21}$$

The curvature identity (2.19) leads to the stronger requirement: $\mathrm{Curv}^\Gamma(u_1, u_2)$ is a vertical Hamiltonian vector field with the Hamiltonian function $\mathbb{F}(u_1, u_2)$,

$$\mathrm{Curv}^\Gamma(u_1, u_2) \in \mathcal{X}_V^{\mathrm{Ham}}(E) \tag{2.22}$$

and hence the equivalence class of $\mathrm{Curv}^\Gamma(u_1, u_2)$ in $\mathcal{H}_V^1(E, \mathcal{V})$ is trivial.

Remark also that conditions (2.18) and (2.19) are independent in general. Indeed, the curvature identity (2.19) and the Bianchi identity for the curvature form of Γ imply only that

$$\partial_\Gamma \mathbb{F} \in \Omega^2(B) \otimes \mathrm{Casim}_{\mathcal{V}}(E). \tag{2.23}$$

Consider the following two "extreme" cases.

EXAMPLE 2.1 (Flat Poisson bundles). Suppose we start with a Poisson bundle over symplectic base: $(E \xrightarrow{\pi} B, \mathcal{V}, \omega)$, where $\mathcal{V}$ is a vertical Poisson tensor and ω is a base symplectic structure. Then we can assign to every flat Poisson connection Γ on π the coupling tensor Π^Γ defined by (2.10), where $\mathbb{F} = \omega \otimes 1$. In this case, (2.19) holds because of the flatness, $\mathrm{Curv}^\Gamma = 0$. And (2.18) follows from the closedness of ω. The horizontal part Π_H^Γ is just lifting of the nondegenerate Poisson structure on (B, ω) via Γ. In the nonflat case, to satisfy the curvature identity (2.19) we have to deform the symplectic structure on the base.

EXAMPLE 2.2 (Coupling forms [St, GLS]). Under the same starting point as in Example 2.1, assume also that the Poisson structure $\mathcal{V}_b$ is nondegenerate on each fiber E_b. Then $\mathcal{V}$ induces a fiberwise symplectic structure $B \ni b \mapsto \sigma_b \in \Omega^2(E_b)$ and the bundle E becomes a symplectic fiber bundle (E, σ) over a symplectic base (B, ω). In this case, every Poisson connection Γ on E is also symplectic, that is, the parallel transport preserves the fiberwise symplectic structure σ. Furthermore, $\mathrm{Casim}_{\mathcal{V}}(E) \approx C^\infty(B)$. Let us make the extra assumption: the fiber bundle $\pi : E \to B$ is locally trivial and the typical fiber is compact, connected and simply connected. Then we have (see [GLS])

$$\mathcal{H}^1_V(E; \mathcal{V}) = 0. \tag{2.24}$$

Let Γ be a Poisson connection. It follows from (2.21) and (2.24) that (2.22) holds. The problem now is to find $\mathbb{F}$ in (2.19) satisfying also condition (2.18). Taking into account our assumption, introduce a $C^\infty(B)$ linear mapping

$$\mathcal{X}_V^{\mathrm{Ham}}(E) \ni Z \mapsto \mathbf{m}(Z) \in C^\infty(E)$$

which is determined by the conditions

$$Z \rfloor \sigma_b = -d\mathbf{m}_b(Z) \qquad \text{on} \quad E_b, \tag{2.25}$$

$$\int_{E_b} \mathbf{m}_b(Z) \sigma_b^n = 0, \qquad 2n = \dim(\mathit{fiber}) \tag{2.26}$$

for every $b \in B$. Here $\mathbf{m}_b(Z) = \mathbf{m}(Z)\big|_{E_b}$ and $\sigma_b^n = \frac{1}{n!}\sigma_b \wedge \ldots \wedge \sigma_b$ (n-times) is the volume form. Then we claim that the formula

$$\mathbb{F}^\Gamma(u_1, u_2) = \pi^*\omega(u_1, u_2) + \mathbf{m}(\mathrm{Curv}^\Gamma(u_1, u_2))$$

defines just the desired form $\mathbb{F}^\Gamma \in \Omega^2(B) \otimes C^\infty(E)$ satisfying (2.18) and (2.19). Indeed, (2.18) holds automatically. The symplecticity of Γ and the normalization condition (2.26) imply that the "collective Hamiltonian" $\mathbf{m}$ is Γ-invariant, $\mathbf{m}((\mathrm{Fl}_t^{-1})^* Z) = \mathrm{Fl}_t^*(\mathbf{m}(Z))$, where Fl_t is the flow of $\mathrm{hor}(u)$. This leads to (2.19). Denote by σ^v the vertical 2-form on E which coincides with σ_b on each fiber E_b. Then the closed 2-form

$$\Omega^\Gamma = (\mathbb{F}^\Gamma)^h + \sigma^v \tag{2.27}$$

is called a *coupling form* associated with the symplectic connection Γ [GLS]. The closedness of Ω^Γ is just equivalent to conditions (2.18) and (2.19) for $\mathbb{F} = \mathbb{F}^\Gamma$. In a domain where $\mathbb{F}^\Gamma$ is nondegenerate, Ω^Γ is symplectic and its nondegenerate Poisson structure is the coupling tensor Π^Γ generated by the triple $(\Gamma, \mathcal{V}, \mathbb{F}^\Gamma)$.

3. Neighborhood equivalence. Here we show that coupling tensors naturally appear in the classification problem of Poisson structures near a single symplectic leaf.

3.1. *Geometric splitting.* Let $\nu : \mathcal{N} \to B$ be a fiber bundle over a connected base B. Suppose we have a cross-section $\mathbf{s} : B \to \mathcal{N}$ of ν.

We say that a Poisson tensor Π on $\mathcal{N}$ is compatible with section $\mathbf{s}$, or briefly, $\mathbf{s}$-*compatible* if $\mathbf{s}(B)$ is a *symplectic leaf* of Π.

PROPOSITION 3.1. *Let $\Pi \in \chi^2(\mathcal{N})$ be an $\mathbf{s}$-compatible Poisson tensor. Then there exists a tubular neighborhood E of $\mathbf{s}(B)$ in $\mathcal{N}$ such that Π is a coupling tensor on E. In particular, there is an intrinsic Ehresmann connection Γ on $\pi = \nu\big|_E : E \to B$ which induces a unique decomposition*

$$\Pi = \Pi_H + \Pi_V \quad on \quad E, \tag{3.1}$$

where

- $\Pi_H \in \chi^2_H(E)$ *is a Γ-horizontal bivector field,*
- $\Pi_V \in \chi^2_V(E)$ *is a vertical Poisson tensor such that*

$$\operatorname{rank} \Pi_V = 0 \quad at\ every\ point\ in \quad \mathbf{s}(B). \tag{3.2}$$

The connection Γ is determined by (2.4).

Proof. Since $\mathbf{s}(B)$ is a symplectic leaf of Π, the bivector field Π is nondegenerate on subspaces $\mathrm{Vert}^0_e \subset T^*_e N$ at points $e \in \mathcal{N}$ sufficiently close to $\mathbf{s}(B)$. Hence Π is a horizontally nondegenerate on a tubular neighborhood E of $\mathbf{s}(B)$ in $\mathcal{N}$. By Theorem 2.1, Π is a coupling tensor on E associated with geometric data $(\Gamma, \mathcal{V} = \Pi_V, \mathbb{F})$. Here the intrinsic connection Γ and the bivector fields Π_H, $\mathcal{V}$ are defined by (2.4) and (2.6) respectively. Property (3.2) follows from $\mathbf{s}$-compatibility assumption. ∎

REMARK 3.1. Each fiber E_b over $b \in B$ inherits from Π a Poisson structure in a neighborhood of $\mathbf{s}(b)$ called the transverse Poisson structure at the point b [We$_4$]. By the splitting theorem, transverse Poisson structure is independent of the choice of a point $\mathbf{s}(b)$ up to local isomorphism. The vertical part Π_V in (3.1) gives rise to a fiberwise Poisson structure which fit together local transverse Poisson structures.

Note that the maximal domain, where splitting (3.1) holds, consists of the points $e \in \mathcal{N}$ such that $\operatorname{rank}_e \Pi^{\#}(\mathrm{Vert}^0) = \dim B$. A given $\mathbf{s}$-compatible Poisson structure Π with geometric data $(\Gamma, \Pi_V, \mathbb{F})$ possesses the following properties on E.

(i) The symplectic structure ω on $\mathbf{s}(B)$ is

$$\omega = \mathbb{F}^h\big|_{\mathbf{s}(B)}. \tag{3.3}$$

The horizontal distribution Hor associated with Γ is tangent to $\mathbf{s}(B)$,

$$T_e\mathbf{s}(B) = \mathrm{Hor}_e \qquad \text{for} \quad e \in \mathbf{s}(B). \tag{3.4}$$

and hence

$$\mathrm{Curv}^\Gamma(u_1, u_2) = 0 \quad \text{on} \quad \mathbf{s}(B). \tag{3.5}$$

The projection $\pi : E \to B$ is a Poisson morphism if and only if the curvature of Γ is zero, $\mathrm{Curv}^\Gamma = 0$ on E. In the flat case, the symplectic leaf $\mathbf{s}(B)$ is an integral leaf of the integrable horizontal distribution $\mathrm{Hor} \subset TE$. Hence, there is the holonomy of $\mathbf{s}(B)$ (as a leaf of the corresponding foliation), called the *strict Poisson holonomy* of the leaf [Fe].

(ii) For every $f \in C^\infty(B)$ the Hamiltonian vector field of the pull back $\pi^* f$ is horizontal,

$$\Pi^\#(\pi^* df) = \sum_i \Pi(\pi^* df, \pi^* d\xi^i)\,\mathrm{hor}(\partial/\partial\xi^i), \tag{3.6}$$

where (ξ^i) are (local) coordinates on the base B. Let $\mathcal{D} = \mathrm{Ann}(\ker \Pi^\#)$ be the *characteristic distribution* of Π on E. Let $\mathcal{V}_b^{\mathrm{fib}} \in \chi^2(E_b)$ be the Poisson tensor on the fiber E_b generated by Π_V, and let $\mathcal{D}_b^{\mathrm{fib}}$ be the characteristic distribution of $\mathcal{V}_b^{\mathrm{fib}}$. Then for every $e \in E$ we have $\mathcal{D}_e = \mathrm{Hor}_e \oplus (\mathcal{D}_b^{\mathrm{fib}})_e$. Hence the rank of the Poisson structure Π at e is $\mathrm{rank}_e \Pi = \dim B + \mathrm{rank}_e \mathcal{V}_b^{\mathrm{fib}}$, $b = \pi(e)$. If $(\mathcal{S}, \Omega)$ is a symplectic leaf of (E, Π) with symplectic structure Ω, then at every $e \in E$ we have the decomposition

$$\Omega_e = (\mathbb{F}^h)_e \oplus \sigma_e,$$

where σ is the symplectic form on the leaf of $(E_{\pi(e)}, \mathcal{V}_{\pi(e)}^{\mathrm{fib}})$ passing through the point e. The Poisson tensor Π is of constant rank on E if and only if $\Pi_V \equiv 0$. In this case, Γ is flat.

(iii) Let $f : \widetilde{E} \to E$ be a fiber preserving diffeomorphism from an open neighborhood $\widetilde{E}$ of $\mathbf{s}(B)$ onto E that descends to the identity map on $\mathbf{s}(B)$. Then $f^*\Pi$ is an $\mathbf{s}$-compatible Poisson tensor on $\widetilde{E}$ with the intrinsic connection $f^*\Gamma$ and the splitting $f^*\Pi = (f^*\Pi)_H + (f^*\Pi)_V = f^*(\Pi_H) + f^*(\Pi_V)$.

As a direct consequence of Proposition 3.1, we get the fact: in a tubular neighborhood of a closed symplectic leaf every Poisson structure is realized as a coupling tensor (see Section 5). Thus, the problem on the neighborhood equivalence between Poisson structures near a common symplectic leaf is reduced to the investigation of coupling tensors over a compatible cross-section.

3.2. *Neighborhood equivalence.* Let $\pi : E \to B$ be a fiber bundle over a connected base. Suppose we have two coupling tensors Π and $\widetilde{\Pi}$ on E associated with geometric data $(\Gamma, \mathcal{V}, \mathbb{F})$ and $(\widetilde{\Gamma}, \widetilde{\mathcal{V}}, \widetilde{\mathbb{F}})$, respectively. Assume that Π and $\widetilde{\Pi}$ are compatible with a cross-section $\mathbf{s} : B \to E$ and

$$\mathbb{F}(u_1, u_2)\big|_{\mathbf{s}(B)} = \widetilde{\mathbb{F}}(u_1, u_2)\big|_{\mathbf{s}(B)} \tag{3.7}$$

for $u_1, u_2 \in \mathcal{X}(B)$. Condition (3.7) means that the symplectic structures on $\mathbf{s}(B)$ with respect to Poisson structures Π and $\widetilde{\Pi}$ coincide.

We say that the geometric data $(\Gamma, \mathcal{V}, \Lambda)$ and $(\widetilde{\Gamma}, \widetilde{\mathcal{V}}, \widetilde{\Lambda})$ are *equivalent* over $\mathbf{s}(B)$ if there exist open neighborhoods $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ of $\mathbf{s}(B)$ in E and a pair (g, ϕ) consisting of

- a fiber preserving diffeomorphism $g : \mathcal{E} \to \widetilde{\mathcal{E}}$ ($\pi \circ g = \pi$) such that $g \circ \mathbf{s} = \mathbf{s}$;
- a base 1-form $\phi \in \Omega^1(B) \otimes C^\infty(\mathcal{E})$

which satisfy the relations:

$$g^*\widetilde{\mathcal{V}} = \mathcal{V}, \tag{3.8}$$

$$g^* = \Gamma - (\mathcal{V}^\# \, d\phi)^h, \tag{3.9}$$

$$g^*\widetilde{\mathbb{F}} = -\partial_\Gamma \phi - \frac{1}{2}\{\phi \wedge \phi\}_{\mathcal{V}}. \tag{3.10}$$

Here $\mathcal{V}^{\#}\, d\phi$ is an element of the space $\Omega^1(B)\otimes\mathcal{X}_V^{Ham}(\mathcal{E})$ determined by $(\mathcal{V}^{\#}\, d\phi)(u) = \mathcal{V}^{\#}\, d\phi(u)$ and $\{\phi_1\wedge\phi_2\}_{\mathcal{V}}$ denotes an element of $\Omega^2(B)\otimes C^\infty(\mathcal{E})$ given by

$$\frac{1}{2}\{\phi_1\wedge\phi_2\}_{\mathcal{V}}(u_1,,u_2) := \mathcal{V}\big(d\phi_1(u_1), d\phi_2(u_2)\big) - \mathcal{V}\big(d\phi_1(u_2), d\phi_2(u_1)\big)$$

for $\phi_1,\phi_2\in\Omega^1(B)\otimes C^\infty(\mathcal{E})$ and $u_1,u_2\in\mathcal{X}(B)$.

Theorem 3.1. *Let* Π *and* $\widetilde{\Pi}$ *be two* $\mathbf{s}$*-compatible coupling tensors satisfying condition* (3.7)*. If the corresponding geometric data* $(\Gamma,\mathcal{V},\mathbb{F})$ *and* $(\widetilde{\Gamma},\widetilde{\mathcal{V}},\widetilde{\mathbb{F}})$ *are equivalent over* $\mathbf{s}(B)$*, then there exist neighborhoods* $\mathcal{O}$ *and* $\widetilde{\mathcal{O}}$ *of* $\mathbf{s}(B)$ *in* E *and a diffeomorphism* $\mathbf{f}:\mathcal{O}\to\widetilde{\mathcal{O}}$ *such that*

$$\mathbf{f}\circ\mathbf{s} = \mathbf{s}, \tag{3.11}$$

$$\mathbf{f}^*\widetilde{\Pi} = \Pi. \tag{3.12}$$

Proof. We will use a contravariant analog of the homotopy method due to Moser [Mos] and Weinstein [We$_1$] (see also [GSt, LMr].

Step 1. *Homotopy between coupling tensors.* By the property (iii) in section 3.1, without loss of generality we can assume that the vertical parts of Π and $\widetilde{\Pi}$ coincide, $\widetilde{\mathcal{V}}=\mathcal{V}$ on $\mathcal{E}$ and $g=\mathrm{id}$. By the $\mathbf{s}$-compatibility assumption we deduce that rank $\mathcal{V}=0$ at $\mathbf{s}(B)$. It follows from this property and (3.7) that we can choose ϕ in (3.9), (3.10) so that

$$\phi(u)\big|_{\mathbf{s}(B)} = 0, \qquad \text{for all } u\in\mathcal{X}(B). \tag{3.13}$$

Consider the following t-parameter families of forms:

$$\Gamma_t = \Gamma - t(\mathcal{V}^{\#}d\phi)^h \in\Omega^1(E)\otimes\mathcal{X}_V(\mathcal{E}), \tag{3.14}$$

$$\mathbb{F}_t = \mathbb{F} - t\partial_\Gamma\phi - \frac{t^2}{2}\{\phi\wedge\phi\}_{\mathcal{V}}\in\Omega^2(B)\otimes C^\infty(\mathcal{E}). \tag{3.15}$$

Then Γ_t is a time-dependent connection 1-form on $\mathcal{E}$. By (3.13) and the nondegeneracy of $\mathbb{F}$, there is a neighborhood $\mathcal{E}_0$ of $\mathbf{s}(B)$ in E such that $\mathbb{F}_t$ is nondegenerate on $\mathcal{E}_0$ for all $t\in[0,1]$. This means that for every $t\in[0,1]$ and $e\in\mathcal{E}_0$ the horizontal lift $(\mathbb{F}_t)^h$ induces a nondegenerate bilinear form on the qoutient space $T_eE/Vert_e$. Moreover, we observe that the triple $(\Gamma_t,\mathcal{V},\mathbb{F}_t)$ defines a geometric data on $\mathcal{E}_0$ satisfying conditions (2.16)–(2.19) for every $t\in[0,1]$. Thus the time-dependent coupling tensor Π_t ($t\in[0,1]$) associated to $(\Gamma_t,\mathcal{V},\mathbb{F}_t)$ gives a homotopy from Π to $\widetilde{\Pi}$, $\Pi_t\big|_{t=0}=\Pi$, $\Pi_t\big|_{t=1}=\widetilde{\Pi}$.

Step 2. *Homological equation.* By the nondegeneracy of $\mathbb{F}_t$ on $\mathcal{E}_0$, there exists a unique solution $X_t\in\mathcal{X}(B)\otimes C^\infty(\mathcal{E}_0)$ of the following equation

$$X_t\rfloor\mathbb{F}_t = \phi. \tag{3.16}$$

Clearly

$$\langle df, X_t\rangle\big|_{\mathbf{s}(B)} = 0 \quad \text{for every} \quad f\in C^\infty(B). \tag{3.17}$$

One can associate to X_t the time-dependent horizontal vector field $X_t^h\in\mathcal{X}_H(\mathcal{E}_0)$ defined by

$$X_t^h(\pi^*f) = \langle df, X_t\rangle \tag{3.18}$$

for all $f \in C^\infty(B)$. Property (3.17) implies

$$X_t^h\big|_{\mathbf{s}(B)} = 0. \tag{3.19}$$

LEMMA 3.1. *X_t^h satisfies the equation*

$$L_{X_t^h}\Pi_t + \frac{\partial}{\partial t}\Pi_t = 0 \qquad (t \in [0,1]). \tag{3.20}$$

Here $L_X\Pi$ is the Lie derivative of a bivector field Π along a vector field X, that is, the Schouten bracket $[\![X,\Pi]\!]_E$. The proof of Lemma 3.1 is given in Appendix A.

STEP 3. Let Φ_t be the flow of the time-dependent horizontal vector field X_t^h: $\frac{d}{dt}\Phi_t = X_t^h \circ \Phi_t$, $\Phi_0 = id$. By (3.20) and the usual properties of the Lie derivative (see, for example, [KM, LMr, Va]) we get $\Phi_t^*\Pi_t = \Pi$. Because of (3.19) for every $e \in \mathbf{s}(B)$, we have $\Phi_t(e) = e$ for all $t \in [0,1]$. Hence there exists a neighborhood $\mathcal{O}$ of $\mathbf{s}(B)$ in $\mathcal{E}_0$ that lies in the domain of the flow Φ_t for $t \in [0,1]$. Finally, the time 1 flow Φ_1 of X_t^h generates a diffeomorphism $\mathbf{f} : \mathcal{O} \to \widetilde{\mathcal{O}}$ satisfying (3.11), (3.12). ∎

REMARK 3.2. If $\mathbf{s}(B)$ is a regular symplectic leaf, then $\mathbb{F} = \widetilde{\mathbb{F}} = 0$ and $\mathcal{V} = 0$. Condition (3.9) means that the flat connections $\widetilde{\Gamma}$ and Γ associated with corresponding symplectic foliations over $\mathbf{s}(B)$, are gauge equivalent.

Now suppose we are given a triple $(E \xrightarrow{\pi} B, \mathcal{V}, \mathbf{s})$ consisting of a fiber bundle over a connected base, a vertical Poisson tensor $\mathcal{V}$ and a cross-section $\mathbf{s} : B \to E$. Assume that $\operatorname{rank}\mathcal{V} = 0$ at $\mathbf{s}$(B) and

$$\mathcal{H}_V^1(\mathcal{E};\mathcal{V}) = 0 \tag{3.21}$$

for a certain open neighborhood $\mathcal{E}$ of $\mathbf{s}(B)$ in E. Assume also that there exists a $C^\infty(B)$-linear map $\mathbf{m} : \mathcal{X}_V^{\mathrm{Ham}}(\mathcal{E}) \to C^\infty(\mathcal{E})$ such that

$$Z = \mathcal{V}^\#(d\mathbf{m}(Z)). \tag{3.22}$$

for every $Z \in \mathcal{X}_V^{\mathrm{Ham}}(\mathcal{E})$.

We say that a connection Γ on E is $\mathbf{s}$-*compatible* if condition (3.4) holds. Denote by $C_B^\infty(\mathcal{E})$ the subspace of smooth funcions on $\mathcal{E}$ vanishing at $\mathbf{s}(B)$. Let $\mathrm{Casim}_{\mathcal{V}}^0(\mathcal{E}) \approx \mathrm{Casim}_{\mathcal{V}}(\mathcal{E})/\pi^*C^\infty(B)$ be the subspace of Casimir funcions of $(\mathcal{E},\mathcal{V})$ vanishing at $\mathbf{s}(B)$.

From (3.21), (3.22) we deduce: if Γ and $\widetilde{\Gamma}$ are two $\mathbf{s}$-compatible Poisson connections on $\mathcal{E}$ (condition (2.17) holds), then there exists $\phi_0 \in \Omega^1(B) \otimes C_B^\infty(\mathcal{E})$ such that

$$\Gamma - \widetilde{\Gamma} = (\mathcal{V}^\# d\phi_0)^h. \tag{3.23}$$

Note that ϕ_0 in (3.23) is uniquely determined up to elements from the space $\Omega^1(B) \otimes \mathrm{Casim}_{\mathcal{V}}^0(\mathcal{E})$.

Consider the $C^\infty(B)$-module $\mathcal{M}^k(\mathcal{E}) = \Omega^k(B) \otimes \mathrm{Casim}_{\mathcal{V}}^0(\mathcal{E})$. Notice that the covariant derivative ∂_Γ associated to an $\mathbf{s}$-compatible Poisson connection Γ on $\mathcal{E}$ sends the subspace $\mathcal{M}^k(\mathcal{E}) \subset \Omega^k(B) \otimes C^\infty(\mathcal{E})$ to subspace $\mathcal{M}^{k+1}(\mathcal{E}) \subset \Omega^{k+1}(B) \otimes C^\infty(\mathcal{E})$. It follows from (3.21) that there exists a unique operator $\partial_0 : \mathcal{M}^k(\mathcal{E}) \to \mathcal{M}^{k+1}(\mathcal{E})$ with the property: for every $\mathbf{s}$-compatible Poisson connection Γ on $\mathcal{E}$ the restriction of ∂_Γ to $\mathcal{M}^k(\mathcal{E})$ coincides with ∂_0, $\partial_\Gamma\,|_{\mathcal{M}^k(\mathcal{E})} = \partial_0$. Moreover, ∂_0 is coboundary operator, $\partial_0 \circ \partial_0 = 0$.

We say that the germ of $\mathcal{V}$ at $\mathbf{s}(B)$ is *trivial* if there exists an open neighborhood $\mathcal{E}$ of $\mathbf{s}(B)$ in E such that apart from conditions (3.21), (3.22) the second cohomology space of ∂_0 is trivial,

$$\frac{\ker(\partial_0 : \mathcal{M}^2(\mathcal{E}) \to \mathcal{M}^3(\mathcal{E}))}{\operatorname{im}(\partial_0 : \mathcal{M}^1(\mathcal{E}) \to \mathcal{M}^2(\mathcal{E}))} = 0. \tag{3.24}$$

From Theorem 3.1 we derive the following Poisson analog of the relative Darboux theorem due to [We$_1$] .

THEOREM 3.2. *Assume that the germ of $\mathcal{V}$ at $\mathbf{s}(B)$ is trivial. Then every two $\mathbf{s}$-compatible Poisson tensors Π and $\widetilde{\Pi}$ on E with the same symplectic structure on $\mathbf{s}(B)$ (condition (3.7)) and the same vertical part*

$$\Pi_V = \widetilde{\Pi}_V = \mathcal{V} \qquad \text{on } \mathcal{E} \tag{3.25}$$

are isomorphic in the sense of (3.11), (3.12).

Proof. Let Π and $\widetilde{\Pi}$ be two $\mathbf{s}$-compatible Poisson tensors on E satisfying the above hypotheses. Let $(\Gamma, \mathcal{V}, \mathbb{F})$ and $(\widetilde{\Gamma}, \mathcal{V}, \widetilde{\mathbb{F}})$ be the geometric data associated with Π and $\widetilde{\Pi}$, respectavely. Thus, Poisson connections Γ and $\widetilde{\Gamma}$ are $\mathbf{s}$-compatible and hence (3.23) holds. Pick a ϕ_0 in (3.23) and define

$$\mathcal{C} := \widetilde{\mathbb{F}} - \mathbb{F} + \partial_\Gamma \phi_0 + \frac{1}{2}\{\phi_0 \wedge \phi_0\}_{\mathcal{V}}. \tag{3.26}$$

It follows from (3.23) and the curvature identity (2.19) for $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ that $\mathcal{C} \in \mathcal{M}^2(\mathcal{E})$. Using (2.18), we deduce: $\mathcal{C}$ is a 2-cocycle, $\partial_0 \mathcal{C} = 0$ whose cohomology class does not depend on the choice of ϕ_0 in (3.26). If this class vanishes, then $\mathcal{C} = \partial_0 \beta$ for a $\beta \in \Omega^1(B) \otimes \mathrm{Casim}^0_{\mathcal{V}}(\mathcal{E})$ and $\widetilde{\mathbb{F}}$ and $\mathbb{F}$ satisfy (3.10) for $\phi = \phi_0 - \beta$ and $g = \mathrm{id}$. ∎

It remains to note: for the equivalence of two individual $\mathbf{s}$-compatible Poisson tensors Π and $\widetilde{\Pi}$ instead of (3.24) we can assume that the cohomology class of the relative 2-cocycle (3.26) is trivial.

4. Poisson structures from Lie algebroids. Our goal is to describe a class of connection-dependent coupling tensors on the dual of the isotropy of a transitive Lie algebroid over a symplectic base.

To begin, we recall some definitions and facts in the theory of Lie algebroids (for more detail see [Mz, Ku, Va, IKV, CWe] and references given there).

A *Lie algebroid* over a manifold B is a vector bundle $A \to B$ together with a bundle map $\rho : A \to TB$, called the *anchor*, and a Lie algebra structure $\{\ ,\ \}_A$ on the space $\Gamma(A)$ of smooth sections of A such that

1. For any $a_1, a_2 \in \Gamma(A)$,
$$\rho(\{a_1, a_2\}_A) = [\rho(a_1), \rho(a_2)]. \tag{4.1}$$
2. For any $a_1, a_2 \in \Gamma(A)$ and $f \in C^\infty(B)$,
$$\{a_1, f a_2\}_A = f\{a_1, a_2\}_A + (L_{\rho(a_1)} f) a_2. \tag{4.2}$$

The kernel of the anchor ρ is called *isotropy*.

If A and $\widetilde{A}$ are two Lie algebroids over the same base manifold B, then a morphism of Lie algebroids over B is a vector bundle morphism $\imath : A \to \widetilde{A}$ over B such that $\widetilde{\rho} \circ \imath = \rho$ and such that $\imath(\{a_1, a_2\}_A) = \{\imath(a_1), \imath(a_2)\}_{\widetilde{A}}$ for all $a_1, a_2 \in \Gamma(A)$. If $\imath$ is a vector bundle isomorphism we say that A and $\widetilde{A}$ are isomorphic.

A Lie algebroid is called *transitive* if the anchor is a fiberwise surjection.

Let $(A, \rho, [\ ,\ ]_A)$ be a transitive Lie algebroid over a connected base B. Then there is an exact sequence of vector bundles

$$\ker\rho \to A \overset{\rho}{\to} TB. \tag{4.3}$$

It follows from the Lie algebroid axioms that the restriction of the bracket $\{\ ,\ \}_A$ to $\Gamma(\ker\rho)$ defines a fiberwise Lie algebra structure on $\ker\rho$ which will be denoted by $[\ ,\ ]$. Further, $(\ker\rho, [\ ,\ ])$ is a locally trivial Lie algebra bundle with a typical fiber $\mathfrak{g}$, that is, the structure group of $\ker\rho$ reduces from $\mathrm{GL}(\mathfrak{g})$ to the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ (see [Mz]).

A *connection* on the transitive Lie algebroid A due to Mackenzie [Mz], is defined as a right splitting of the exact sequence of vector bundles (4.3), that is, a vector bundle morphism $\gamma : TB \to A$ such that $\rho\circ\gamma = \mathrm{id}$. Thus γ induces $A = \gamma(TB)\oplus\ker\rho$. The *curvature* of γ is the vector valued 2-form $\mathcal{R}^\gamma \in \Omega^2(B)\otimes\Gamma(\ker\rho)$ defined by

$$\mathcal{R}^\gamma(u_1, u_2) := \{\gamma(u_1), \gamma(u_2)\}_A - \gamma([u_1, u_2]) \tag{4.4}$$

for $u_1, u_2 \in \mathcal{X}(B)$.

Given a connection γ on A, there is a linear Koszul connection $\nabla^\gamma : \Gamma(\ker\rho) \to \Gamma(T^*B\otimes\ker\rho)$ on the vector bundle $\ker\rho$, called an *adjoint connection* [Mz], and defined by

$$\nabla^\gamma_u\eta = \{\gamma(u), \eta\}_A \qquad (u\in\mathcal{X}(B), \eta\in\Gamma(\ker\rho)). \tag{4.5}$$

This connection preserves the fiberwise Lie structure on $\ker\rho$,

$$\nabla^\gamma([\eta_1, \eta_2]) = [\nabla^\gamma\eta_1, \eta_2] + [\eta_1, \nabla^\gamma\eta_2] \tag{4.6}$$

for $\eta_1, \eta_2 \in \Gamma(\ker\rho)$. The curvature form $\mathrm{Curv}^{\nabla^\gamma} : TB\oplus TB \to \mathrm{End}(\ker\rho)$ is given by

$$\mathrm{Curv}^{\nabla^\gamma}(u_1, u_2) := [\nabla_{u_1}, \nabla_{u_2}] - \nabla_{[u_1,u_2]}$$

and related to the curvature of γ by the adjoint representation

$$\mathrm{Curv}^{\nabla^\gamma} = \mathrm{ad}\circ\mathcal{R}^\gamma. \tag{4.7}$$

Here we use the notation $\mathrm{ad}\circ\eta = [\eta, \cdot]$ for $\eta\in\Gamma(\ker\rho)$. Furthermore, one can show that $\mathcal{R}^\gamma$ satisfies the Bianchi identity:

$$\underset{(u_0,u_1,u_2)}{\mathfrak{S}} \left(\nabla^\gamma_{u_0}\mathcal{R}^\gamma(u_1, u_2) + \mathcal{R}^\gamma(u_0, [u_1, u_2])\right) = 0 \tag{4.8}$$

for any $u, u_1, u_2 \in \mathcal{X}(B)$. Identity (4.8) means that the ∇^γ-covariant derivative of $\mathcal{R}^\gamma$ vanishes, $\partial^{\nabla^\gamma}\mathcal{R}^\gamma = 0$.

Example 4.1. An important class of transitive Lie algebriods comes from principle bundles. If we have a G-principle bundle $P \xrightarrow{\tau} B$, then there is an exact sequence of of vector bundles

$$\mathrm{ad}(P) = P\times_G\mathfrak{g} \to TP/G \to TB$$

called the Atiyah sequence. Here $\mathfrak{g}$ is the Lie algebra of G, TP/G is the quotient manifold with respect to the (right) lifted action to the cotangent bundle and $\mathrm{ad}(P)$ is the bundle over B associated with P via the adjoint action of G on $\mathfrak{g}$. The natural isomorphism between smooth sections of TP/G and the space of right invariant vector fields on P induces the Lie bracket on $\Gamma(TP/G)$. Thus, $A = TP/G$ becomes a transitive Lie algebriod over B whose isotropy is the adjoint bundle $\mathrm{ad}(P)$ (see [Mz, Ku]). A given principle connection $\vartheta : TB \to TP$ with G-invariant horizontal subbundle $\vartheta(TB)$ induces the connection $\gamma : TB \to TP/G$ on the Lie algebroid A. The $\mathfrak{g}$-valued curvature form $\mathcal{K}^\vartheta \in \Omega^2(B;\mathfrak{g})$ of ϑ is related with the curvature $\mathcal{R}^\gamma : TB \oplus TB \to P \times_G \mathfrak{g}$ by the formula $\mathcal{R}^\gamma = \mathrm{pr} \circ \tau^*\mathcal{K}^\vartheta$. Here $\tau^*\mathcal{K}^\vartheta : TP \oplus TP \to P \times \mathfrak{g}$ is the pull back via the projection τ and $\mathrm{pr} : P \times \mathfrak{g} \to P \times_G \mathfrak{g}$ is the natural projection. As is known [AM] there are "nonintegrable" Lie algebriods, which are transitive and can not be realized as the Lie algebroids of principle bundles (also see [Mz, Ku]).

4.1. *Connection-dependent coupling tensors.* Let $\nu : \mathcal{N} \to B$ be a vector bundle over a connected symplectic base (B, ω). Suppose we are given

- a transitive Lie algebroid $(A, \rho, \{\ ,\ \}_A)$ over B such that the isotropy of A coincides with the dual of $\mathcal{N}$
$$\mathcal{N}^* \to A \to TB, \qquad \mathcal{N}^* = \ker\rho; \tag{4.9}$$
- a connection $\gamma : TB \to A$.

Recall that $\mathcal{N}^*$ is a Lie algebra bundle with fiberwise Lie algebra structure [,] and typical fiber $\mathfrak{g}$. Hence $\mathcal{N}$ can be viewed as a *bundle* of *Lie–Poisson manifolds* with typical fiber $\mathfrak{g}^*$.

Denote by $C^\infty_{\mathrm{lin}}(\mathcal{N})$ the space of *fiberwise linear functions* on $\mathcal{N}$. Then we have the natural identification
$$\ell : \Gamma(\mathcal{N}^*) \to C^\infty_{\mathrm{lin}}(\mathcal{N}) \tag{4.10}$$
given by $\ell(\eta)(x) = \langle \eta(\nu(x)), x\rangle$ for $x \in \mathcal{N}$ and $\eta \in \Gamma(\mathcal{N}^*)$.

We say that an *Ehresmann connection* on the vector bundle $\mathcal{N}$ is *homogeneous* if the horizontal lift of every base vector field (as a differential operator) preserves the space $C^\infty_{\mathrm{lin}}(\mathcal{N})$. Equivalently, the horizontal subbundle is invariant with respect to dilations $\lambda_t : \mathcal{N} \to \mathcal{N}$ ($\lambda_t(x) = t \cdot x$, $x \in \mathcal{N}$, $t \in \mathbb{R}$). Notice that there is a bijective correspondence between homogeneous Ehresmann connections on $\mathcal{N}$ and linear connections (covariant derivatives) in the sense of Koszul [GHV].

Now let us assign to the pair (A, γ) a triple $(\Gamma^{A,\gamma}, \Lambda, \mathbb{F}^{A,\gamma})$ consisting of

- the homogeneous Ehresmann connection $\Gamma^{A,\gamma}$ on $\mathcal{N}$ whose horizontal lift is defined by
$$L_{\mathrm{hor}(u)}\varphi = \ell(\{\gamma(u), \ell^{-1}(\varphi)\}_A) \tag{4.11}$$
for $u \in \mathcal{X}(B)$, $\varphi \in C^\infty_{\mathrm{lin}}(\mathcal{N})$;
- the fiberwise linear vertical Poisson tensor $\Lambda \in \chi^2_V(\mathcal{N})$ given by
$$\Lambda(d\varphi_1, d\varphi_2) = \ell([\ell^{-1}(\varphi_1), \ell^{-1}(\varphi_2)]) \tag{4.12}$$
for $\varphi_1, \varphi_2 \in C^\infty_{\mathrm{lin}}(\mathcal{N})$;
- the base 2-form $\mathbb{F}^{A,\gamma} \in \Omega^2(B) \otimes C^\infty_{\mathrm{aff}}(\mathcal{N})$:
$$\mathbb{F}^{A,\gamma} = \omega \otimes 1 - \ell \circ \mathcal{R}^\gamma. \tag{4.13}$$

For the second term in (4.13) we have $\ell \circ \mathcal{R}^\gamma(u_1, u_2)(e) = \langle \mathcal{R}^\gamma(u_1(b), u_2(b)), e\rangle$ for $u_1, u_2 \in \mathcal{X}(B)$ and $e \in \mathcal{N}$, here $b = \nu(e)$. Thus, the homogeneous Ehresmann connection $\Gamma^{A,\gamma}$ is generated by the linear connection on $\mathcal{N}$ which is conjugate to the adjoint connection ∇^γ in (4.5). The bivector field Λ defines the fiberwise Lie–Poisson structure on the bundle $\mathfrak{g}^* \to \mathcal{N} \xrightarrow{\nu} B$. The 2-form $\mathbb{F}^{A,\gamma}$ takes values in the space of fiberwise affine functions $C^\infty_{\mathrm{aff}}(\mathcal{N}) \approx C^\infty(B) \oplus C^\infty_{\mathrm{lin}}(\mathcal{N})$ and includes the base symplectic 2-form ω and the curvature form $\mathcal{R}^\gamma : TB \oplus TB \to \mathcal{N}^*$ in (4.4).

Now we observe that properties (4.6), (4.8) and (4.7) imply relations (2.17)–(2.19) for $(\Gamma^{A,\gamma}, \Lambda, \mathbb{F}^{A,\gamma})$. Moreover, since $\ell \circ \mathcal{R}^\gamma \in \Omega^2(B) \otimes C^\infty_{\mathrm{lin}}(\mathcal{N})$, there is a neighborhood E of the zero section $B \hookrightarrow \mathcal{N}$, where the 2-form $\mathbb{F}^{A,\gamma}$ is nondegenerate. So applying Theorem 2.1, we arrive at the following assertion.

Theorem 4.1. *In a neighborhood E of the zero section $B \hookrightarrow \mathcal{N}$ the transitive Lie algebroid A with a connection γ induces a coupling tensor $\Pi^{A,\gamma}$ associated with the geometric data $(\Gamma^{A,\gamma}, \Lambda, \mathbb{F}^{A,\gamma})$ in* (4.11)–(4.13). *If the kernel $\ker \mathcal{R}^\gamma \subset TB$ of the curvature 2-form $\mathcal{R}^\gamma$ is a coisotropic distribution with respect to the base symplectic form ω, then the coupling tensor $\Pi^{A,\gamma}$ is well-defined on the entire total space $\mathcal{N}$.*

To justify the second part of Theorem 4.1, let us consider the coordinate representation for $\Pi^{A,\gamma}$.

Let $(\xi, x) = (\xi^1, \ldots, \xi^{2k}; x^1, \ldots, x^r)$ be a (local) coordinate system on $\mathcal{N}$, where (ξ^i) are coordinates on the base B and (x^σ) are coordinates on the fibers of $\mathcal{N}$ associated with a basis of local sections (X_σ). Then we have

- the *symplectic form on the base*: $\omega = \frac{1}{2}\sum_{i,j} \omega_{ij}(\xi)\, d\xi^i \wedge d\xi^j$, $\omega^{is}\omega_{sj} = \delta^i_j$;
- the *curvature form*: $\mathcal{R}^\gamma = \frac{1}{2}\sum_{i,j,\sigma} \mathcal{R}_{ij\sigma}(\xi) d\xi^i \wedge d\xi^j \otimes dx^\sigma$;
- the *connection form*: $\Gamma^{A,\gamma} = \sum_{i,\sigma} \Gamma^\sigma_i d\xi^i \otimes \frac{\partial}{\partial x^\sigma}$, $\qquad \Gamma^\sigma_i = \Gamma^\sigma_{i\sigma'}(\xi) x^{\sigma'}$;
- the *base 2-form* (4.13): $\mathbb{F}^{A,\gamma} = \frac{1}{2}\sum_{i,j} d\xi^i \wedge d\xi^j \otimes F_{ij}$, where

$$F_{ij} = \omega_{ij} - \sum_\sigma \mathcal{R}_{ij\sigma} x^\sigma. \tag{4.14}$$

Let (η^σ) be the dual basis of local sectons of $\mathcal{N}^*$, $\langle \eta^\sigma, X_{\sigma'}\rangle = \delta^\sigma_{\sigma'}$. Then with respect to the induced basis of local sections $(\Xi_i = \gamma(\frac{\partial}{\partial \xi^i}), \eta^\sigma)$ of A the Lie algebroid structure takes the form:

$$\{\Xi_i, \Xi_j\}_A = \sum_\nu \mathcal{R}_{ij\sigma}\eta^\nu, \quad \{\Xi_i, \eta^\sigma\}_A = -\sum_\nu \Gamma^\sigma_{i\nu}\eta^\nu, \quad \{\eta^\sigma, \eta^{\sigma'}\}_A = \sum_\nu \lambda^{\sigma\sigma'}_\nu \eta^\nu.$$

Consider the open domain containing the zero section $B = \{x^1 = 0, \ldots, x^r = 0\}$:

$$E = \{(\xi, x) \in \mathcal{N} \mid \det(\!(\omega_{ij} - \sum_\sigma \mathcal{R}_{ij\sigma} x^\sigma)\!) \neq 0\}. \tag{4.15}$$

Then the coupling tensor $\Pi^{A,\gamma}$ is well-defined on E and has the representation

$$\Pi^{A,\gamma} = \frac{1}{2}\sum_{i,j} H^{ij}(\xi, x)\, \mathrm{hor}(\partial_i) \wedge \mathrm{hor}(\partial_j) + \frac{1}{2}\sum_{\sigma\sigma'} \Lambda^{\sigma\sigma'}(\xi, x) \frac{\partial}{\partial x^\sigma} \wedge \frac{\partial}{\partial x^{\sigma'}}. \tag{4.16}$$

Here $\mathrm{hor}(\partial_i) = \partial/\partial\xi^i - \sum_\sigma \Gamma^\sigma_i \partial/\partial x^\sigma$ and the matrix functions $(\!(H^{ij})\!)$ and $(\!(\Lambda^{\sigma\sigma'})\!)$ are defined by

$$\sum_s H^{is} F_{sj} = -\delta^i_j, \qquad \Lambda^{\sigma\sigma'} = \sum_\nu \lambda^{\sigma\sigma'}_\nu(\xi) x^\nu, \tag{4.17}$$

where $\lambda^{\sigma\sigma'}_\nu(\xi)$ are the structure constants of the Lie algebra $\mathcal{N}_\xi \approx \mathfrak{g}$.

The Poisson brackets of the coupling tensor $\Pi^{A,\gamma}$ on the domain (4.15) take the form:

$$\begin{aligned}
\{\xi^i,\xi^j\} &= H^{ij} = (-\omega^{ij} + \omega^{ii'}\mathcal{R}_{i'j'\sigma}\omega^{j'j}x^\sigma) + O_2,\\
\{\xi^i,x^\sigma\} &= -H^{is}\Gamma^\sigma_s = \omega^{is}\Gamma^\sigma_{s\sigma'}x^{\sigma'} + O_2,\\
\{x^\sigma,x^{\sigma'}\} &= \Lambda^{\sigma\sigma'} + H^{ij}\Gamma^\sigma_i\Gamma^{\sigma'}_j = (\lambda^{\sigma\sigma'}_\nu x^\nu - \omega^{ij}\Gamma^\sigma_{i\nu}\Gamma^{\sigma'}_{j\nu'}x^\nu x^{\nu'}) + O_3.
\end{aligned} \tag{4.18}$$

Here the summation is taken with respect to repeated indices and O_k denotes a term having zero of order k at every point in B.

Finally, using standard facts from linear symplectic geometry, it is easy to show that under the coisotropic hypothesis for $\ker\mathcal{R}^\gamma$, the matrix $((F^{ij}))$ in (4.14) is totally nondegenerate and hence domain (4.15) coincides with the total space $\mathcal{N}$.

REMARK 4.1. In the case when A is the coadjoint bundle of a principle bundle, connection dependent Poisson structures of type $\Pi^{A,\gamma}$ were studied in [MoMR, Mo].

EXAMPLE 4.2. Suppose we are given a vector bundle $\nu:\mathcal{L}\to Q$ equipped with

- a fiberwise Lie algebra structure $[\eta^\sigma,\eta^{\sigma'}]_{\mathcal{L}} = \sum_\nu \lambda^{\sigma\sigma'}_\nu(q)\eta^\nu$,
- a linear connection $\nabla_{\partial/\partial q_i}\eta^\sigma = -\sum_{\sigma'}\theta^\sigma_{i\sigma'}(q)\eta^{\sigma'}$.

Here (η^σ) is a basis of local sections of $\mathcal{L}$ and $q=(q^i)$ are local coordinates on the base Q. Assume that

(i) ∇ preserves $[\ ,\ ]_{\mathcal{L}}$ (condition (4.6));

(ii) there exists a vector bundle morphism $\mathcal{R}: TQ\times TQ\to\mathcal{L}$

$$\mathcal{R}\left(\frac{\partial}{\partial q^i},\frac{\partial}{\partial q^j}\right) = \sum_\nu \mathcal{R}_{ij\nu}(q)\eta^\nu$$

which is related to the curvature 2-form Curv^∇ on Q by formula (4.7);

(iii) $\mathcal{R}$ satisfies the modified Bianchi identity (4.8).

Then the triple $(\nabla,\mathcal{R},[,]_{\mathcal{L}})$ defines the transitive Lie algebroid on $A=TQ\oplus\mathcal{L}$ ([Mz]) such that $\mathrm{pr}_1: TQ\oplus\mathcal{L}\to TQ$ is the anchor, $\mathcal{L}$ is the isotropy, ∇ and $\mathcal{R}$ is the adjoint connection and the curvature of the connection $\gamma_0: TQ\to TQ\oplus\mathcal{L}$ (canonical injection). Consider the pull back $\widetilde{A}\to T^*Q$ of A via the natural projection $T^*Q\to Q$. Denote also by $(\widetilde{\nabla},\widetilde{\mathcal{R}}[,]_{\widetilde{\mathcal{L}}})$ the cotangent pull back of the original triple $(\nabla,\mathcal{R},[,]_{\mathcal{L}})$ and by $\widetilde{\mathcal{L}}\to TQ$ the pull back of the bundle $\mathcal{L}\to Q$. Consider the canonical symplectic structure $\omega=\sum_i dp^i\wedge dq^i = dp\wedge dq$ on T^*Q . Then the triple $(\widetilde{\nabla},\widetilde{\mathcal{R}},[,]_{\widetilde{\mathcal{L}}})$ induces a transitive Lie algebroid on $\widetilde{A}$ over the symplectic base $(B=T^*Q,\omega=dp\wedge dq)$ (this is an inverse-image algebroid [Mz, Ku]). Moreover, $\widetilde{\mathcal{L}}$ is the isotropy of $\widetilde{A}$ and the pull back $\widetilde{\gamma_0}: T(T^*Q)\to\widetilde{A}$ is the connection on $\widetilde{A}$ whose curvature is just $\widetilde{\mathcal{R}}$. Thus, the kernel of $\widetilde{\mathcal{R}}$ is a Lagrangian distribution on T^*Q with respect to the form $dp\wedge dq$. Hence the coupling tensor associated with the pair $(\widetilde{A},\widetilde{\gamma_0})$ is well defined on the entire total space of the dual $\widetilde{\mathcal{L}}^*$ and the Poisson bracket in (4.18) takes the following coordinate form

$$\begin{aligned}
&\{p^i,p^j\} = \mathcal{R}_{ij\nu}(q)x^\nu, \quad \{p^i,q^j\} = \delta^{ij}, \qquad \{q^i,q^j\} = 0,\\
&\{p^i,x^\sigma\} = -\theta^\sigma_{i\sigma'}(q)x^{\sigma'}, \quad \{p^i,x^\sigma\} = \{q^i,x^\sigma\} = 0,\\
&\{x^\sigma,x^{\sigma'}\} = \lambda^{\sigma\sigma'}_\nu(q)x^\nu.
\end{aligned}$$

On the other hand, it is of interest to note: this Poisson structure coincides with the Courant structure [Co] on the dual $A^* = TQ^* \oplus \mathcal{L}^*$ of the Lie algebroid A. Notice also that such a type of Poisson structures arises from the study of Hamiltonian structures for Wong's equations [MoMR, Mo, La].

4.2. *Varying the connection and the Lie algebroid structure.* Let us address the following question: how does the coupling tensor $\Pi^{A,\gamma}$ defined in Theorem 4.1 depend on the choice of the connection γ and the Lie algebroid structure on A? We will investigate this issue in two steps. Let A be a transitive Lie algebroid over a connected symplectic base (B,ω). Let $\mathcal{L}$ be the isotropy of A and let $\mathcal{N} = \mathcal{L}^*$ be the dual. The fiberwise Lie structure on $\mathcal{L}$ will be denoted by $[\ ,\ ]_{\mathcal{L}}$.

I. Suppose that we have two connections on A:

$$\gamma : TB \to A \quad \text{and} \quad \widetilde{\gamma} : TB \to A.$$

Consider adjoint connections and curvature forms $\nabla^{\gamma}, \mathcal{R}^{\gamma}$ and $\nabla^{\tilde{\gamma}}, \mathcal{R}^{\tilde{\gamma}}$ associated to γ and $\tilde{\gamma}$ respectively. There is a vector bundle map $\mu : TB \to \mathcal{L}$ such that

$$\tilde{\gamma}(u) = \gamma(u) + \mu(u) \qquad \text{for} \quad u \in \mathcal{X}(B).$$

We can think of μ as an $\mathcal{L}$-valued 1-form on B, $\mu \in \Omega^1(B) \otimes \Gamma(\mathcal{L})$. Then we have [Mz]:

$$\nabla_u^{\tilde{\gamma}} = \nabla_u^{\gamma} + \operatorname{ad} \circ \mu(u), \qquad u \in \mathcal{X}(B), \tag{4.19}$$

$$\mathcal{R}^{\tilde{\gamma}} = \mathcal{R}^{\gamma} + \partial_{\nabla^{\gamma}} \mu + \frac{1}{2}[\mu \wedge \mu]_{\mathcal{L}}. \tag{4.20}$$

Here $\partial_{\nabla^{\gamma}} : \Omega^k(B) \otimes \Gamma(\mathcal{L}) \to \Omega^{k+1}(B) \otimes \Gamma(\mathcal{L})$ is the covariant exterior derivative associated with the linear connection ∇^{γ}, and in the last term in (4.20) we use the standard bracket on the graded algebra of L-valued forms on B generated by the fiberwise Lie algebra structure $[\ ,\ ]_{\mathcal{L}}$. Now let us consider the geometric data $(\Gamma^{A,\gamma}, \Lambda, \mathbb{F}^{A,\gamma})$ and $(\Gamma^{A,\tilde{\gamma}}, \Lambda, \mathbb{F}^{A,\tilde{\gamma}})$ defined in (4.11)–(4.13).

It follows from (4.19), (4.20) that $\Gamma^{A,\tilde{\gamma}}, \Gamma^{A,\gamma}$ and $\mathbb{F}^{A,\tilde{\gamma}}, \mathbb{F}^{A,\gamma}$ satisfy relations (3.9), (3.10) for $g = \mathrm{id}$, $\mathcal{V} = \Lambda$, and

$$\phi = \ell \circ \mu \in \Omega^1(B) \otimes C^{\infty}_{\mathrm{lin}}(\mathcal{N}). \tag{4.21}$$

Thus the geometric data $(\Gamma^{A,\gamma}, \Lambda, \mathbb{F}^{A,\gamma})$ and $(\Gamma^{A,\tilde{\gamma}}, \Lambda, \mathbb{F}^{A,\tilde{\gamma}})$ are equivalent. Consider the corresponding coupling tensors $\Pi^{A,\gamma}$, $\Pi^{A,\tilde{\gamma}}$ on $\mathcal{N}$. Then the zero section $B \hookrightarrow \mathcal{N}$ with a given symplectic form ω is a common symplectic leaf of $\Pi^{A,\gamma}$ and $\Pi^{A,\tilde{\gamma}}$. So, we can apply to $\Pi^{A,\gamma}$ and $\Pi^{A,\tilde{\gamma}}$ the neighborhood equivalence Theorem 3.1.

Proposition 4.1. *Coupling tensors $\Pi^{A,\gamma}$ and $\Pi^{A,\tilde{\gamma}}$ associated with arbitrary connections γ and $\tilde{\gamma}$ on A are isomorphic over B, that is, there are open neighborhoods $\mathcal{O}$, $\widetilde{\mathcal{O}}$ of the zero section $B \hookrightarrow \mathcal{N}$ and a diffeomorphism $\mathbf{f} : \mathcal{O} \to \widetilde{\mathcal{O}}$ identical on B such that $\mathbf{f}^*\Pi^{A,\tilde{\gamma}} = \Pi^{A,\gamma}$.*

The equivalence class of isomorphic Poisson structures $\Pi^{A,\gamma}$ will be called an *ω-coupling structure* of a transitive Lie algebroid A.

II. Let A and $\widetilde{A}$ be two transitive Lie algebroids over the same connected base (B,ω). Assume that A and $\widetilde{A}$ are isomorphic and $\imath : \widetilde{A} \to A$ is a Lie algebroid isomorphism.

Without loss of generality, we can also assume that

$$A = TB \oplus \mathcal{L}, \tag{4.22}$$

$$\widetilde{A} = TB \oplus \widetilde{\mathcal{L}}, \tag{4.23}$$

and the corresponding anchors $\rho : A \to TB$, $\widetilde{\rho} : \widetilde{A} \to TB$ coincide with the canonical projections $\rho = \mathrm{pr}_1$, $\widetilde{\rho} = \widetilde{\mathrm{pr}}_1$. It is clear that the restriction

$$g = \imath\big|_{\widetilde{\mathcal{L}}} : \widetilde{\mathcal{L}} \to \mathcal{L} \tag{4.24}$$

is a vector bundle isomorphism preserving the fiberwise Lie algebra structure on $\mathcal{L}$ and $\widetilde{\mathcal{L}}$. We observe that $\imath$ takes an element $u \oplus \eta$ in $\widetilde{A}$ into the element $\imath(u \oplus \eta)$ in A of the form

$$\imath(u \oplus \eta) = u \oplus (g(\eta) + \mu(u)), \tag{4.25}$$

where $\mu : TB \to \mathcal{L}$ is a vector bundle morphism. Thus, $\imath$ is characterized by the pair (g, μ). Define connections γ_0 on A and $\tilde{\gamma}_0$ on $\widetilde{A}$ as the canonical injections:

$$u \mapsto \gamma_0(u) = u \oplus 0 \in TB \oplus \mathcal{L}, \tag{4.26}$$

$$u \mapsto \tilde{\gamma}_0(u) = u \oplus 0 \in TB \oplus \widetilde{\mathcal{L}}. \tag{4.27}$$

Then we get

$$g([a_1, a_2]_{\mathcal{L}}) = [g(a_1), g(a_2)]_{\widetilde{\mathcal{L}}} \qquad (a_1, a_2 \in \Gamma(\mathcal{L})), \tag{4.28}$$

$$g \circ \nabla_u^{\tilde{\gamma}_0} \circ g^{-1} = \nabla_u + \mathrm{ad} \circ \mu(u) \qquad (u \in \mathcal{X}(B)), \tag{4.29}$$

$$g \circ \mathcal{R}^{\tilde{\gamma}_0} = \mathcal{R}^{\gamma_0} + \partial_{\nabla_{\gamma_0}} \mu + \frac{1}{2}[\mu \wedge \mu]_{\mathcal{L}}. \tag{4.30}$$

Relations (4.28)–(4.30) lead to the equivalence relations (3.8)–(3.10) for geometric data $(\Gamma^{\gamma_0}, \Lambda, \mathbb{F}^{A,\gamma_0})$ and $(\Gamma^{\tilde{\gamma}_0}, \widetilde{\Lambda}, \mathbb{F}^{\widetilde{A},\tilde{\gamma}_0})$ associated to pairs (A, γ_0) and $(\widetilde{A}, \tilde{\gamma}_0)$, respectively. As a consequence of Theorem 3.1, we get the proposition.

Proposition 4.2. *There is the neighborhood equivalence between coupling tensors* Π^{A,γ_0} *and* $\Pi^{\widetilde{A},\tilde{\gamma}_0}$.

Finally, combining Proposition 4.1 with Proposition 4.2, we obtain the main result.

Theorem 4.2. *Let A and $\widetilde{A}$ be two transitive Lie algebroids over the same connected symplectic base (B, ω), and let $\gamma : TB \to A$, $\tilde{\gamma} : TB \to \widetilde{A}$ be two connections. Consider coupling tensors $\Pi^{A,\gamma}$ and $\Pi^{\widetilde{A},\tilde{\gamma}}$ associated to (A, γ) and $(\widetilde{A}, \tilde{\gamma})$, respectively.*

(i) Assume that A is isomorphic to $\widetilde{A}$. Then under the arbitrary choice of connections γ, $\tilde{\gamma}$, there exists a diffeomorphism $\mathbf{f}$; $\mathcal{O} \to \widetilde{\mathcal{O}}$ from a neighborhood $\mathcal{O}$ of the zero section $B \hookrightarrow \mathcal{N} = \mathcal{L}^*$($\mathcal{L}$ is the isotropy of A) onto a neighborhood $\widetilde{\mathcal{O}}$ of the zero section $B \hookrightarrow \widetilde{\mathcal{N}} = \widetilde{\mathcal{L}}^*$($\widetilde{\mathcal{L}}$ is the isotropy of $\widetilde{A}$) such that $\mathbf{f}\big|_B = \mathrm{id}_B$ and

$$\mathbf{f}^* \Pi^{\widetilde{A},\tilde{\gamma}} = \Pi^{A,\gamma} \quad \textit{and} \quad \mathbf{f}\big|_B = \mathrm{id}_B\,. \tag{4.31}$$

(ii) Conversely, the equivalence between coupling tensors $\Pi^{A,\gamma}$ and $\Pi^{\widetilde{A},\tilde{\gamma}}$ (in the sense of (4.31)) implies the isomorphism between the corresponding Lie algebroids A and $\widetilde{A}$.

Now suppose we start with some data $(\mathcal{L}, [\ ,\ ]_{\mathcal{L}}, \mathfrak{g})$, where $(\mathcal{L}, [\ ,\ ]_{\mathcal{L}})$ is locally trivial bundle of Lie algebras over a connected symplectic base (B, ω), $\mathfrak{g}$ is the typical fiber.

Let ∇ be a linear connection in $\mathcal{L}$ preserving the fiberwise Lie algebra structure $[\ ,\ ]_{\mathcal{L}}$ (condition (4.6)) and $\mathcal{R} \in \Omega^2(B) \otimes \Gamma(\mathcal{L})$ be a vector valued 2-form which is compatible with $(\nabla, [\ ,\ ]_{\mathcal{L}})$ by means of (4.7) and (4.8). In this case, we say that the pair $(\nabla, \mathcal{R})$ is admissible for $[\ ,\ ]_{\mathcal{L}}$. Accoding to [Mz] the pair $(\nabla, \mathcal{R})$ induces a unique transitive Lie algebroid structure $\{\ ,\ \}_{\nabla,\mathcal{R}}$ on $A = TB \oplus \mathcal{L}$ such that the anchor is the natural projection, $(\mathcal{L}, [\ ,\ ]_{\mathcal{L}})$ is the isotropy,∇ is the adjoint connection associated with connection γ^0 in (4.26) and $\mathcal{R}$ is the curvature of γ^0. The coupling tensor on $\mathcal{L}$ associated to $\{\ ,\ \}_{\nabla,\mathcal{R}}$ and γ^0 will be denoted by $\Pi^{\nabla,\mathcal{R}}$. Consider the subbundle $\mathrm{Cent}(\mathcal{L}) \subset \mathcal{L}$ whose typical fiber is the center of the Lie algebra $\mathfrak{g}$. Then $\mathrm{Cent}(\mathcal{L})$ is invariant with respect to the connection ∇ and the restriction $\nabla_0 = \nabla \mid_{\mathrm{Cent}(\mathcal{L})}$is a flat connection which does not depend on the choice of ∇ in the class of adjoint connections of the Lie algebroid (see [IKV]). Thus the covariant derivative $\partial_0 : \Omega^k(B; \mathrm{Cent}(\mathcal{L})) \to \Omega^{k+1}(B; \mathrm{Cent}(\mathcal{L}))$ associated with ∇_0 is a coboundary operator, $\partial_0 \circ \partial_0 = 0$. Notice that the comology of ∂_0 coincides with the cohomology of the *abelian* Lie subalgebroid $A_0 = TB \oplus \mathrm{Cent}(\mathcal{L})$ in $(A, \{,\}_{\nabla,\mathcal{R}})$.

Let $(\widetilde{\nabla}, \widetilde{\mathcal{R}})$ be a second admissible pair for $[\ ,\ ]_{\mathcal{L}}$ and $(A, \{\ ,\ \}_{\widetilde{\nabla},\widetilde{\mathcal{R}}})$ be the corresponding Lie algebroid. Assume that connection $\widetilde{\nabla}$ and ∇ on $\mathcal{L}$ are related by (4.19) for a certain $\mu \in \Omega^1(B) \otimes \Gamma(\mathcal{L})$. This condition means that the structures of abelian Lie algebroids on A_0 coming from the brackets $\{\ ,\ \}_{\nabla,\mathcal{R}}$ and $\{\ ,\ \}_{\widetilde{\nabla},\widetilde{\mathcal{R}}}$ coincide. It follows from (4.8) and (4.19) that

$$\mathcal{C} := \widetilde{\mathcal{R}} - \mathcal{R} - \partial_\nabla \mu - \frac{1}{2}[\mu \wedge \mu]_{\mathcal{L}}. \tag{4.32}$$

is a 2-cocyle $\mathcal{C} \in \Omega^2(B) \otimes \mathrm{Cent}(\mathcal{L})$, $\partial_0 \mathcal{C} = 0$ whose cohomology class does not depend on the choice of μ in (4.19). Moreover we observe: the Lie algebroid structures $\{\ ,\ \}_{\nabla,\mathcal{R}}$ and $\{\ ,\ \}_{\widetilde{\nabla},\widetilde{\mathcal{R}}}$ are isomorphic if and only if $[\mathcal{C}] = 0$. Then as a consequence of Theorem 4.2, we get the following "linear" analog of Theorem 3.2.

Proposition 4.3. *Under assuption (4.19) the coupling tensors $\Pi^{\nabla,\mathcal{R}}$ and $\Pi^{\widetilde{\nabla},\widetilde{\mathcal{R}}}$ are isomorphic over B if and only if the cohomology class of the relative 2-cocycle $\mathcal{C}$ in (4.32) is zero. In particular, this is true in the case when the second cohomology space of the abelian Lie algebroid A_0 is trivial.*

Remark 4.2. Assume that the typical fiber $\mathfrak{g}$ is reductive, that is, $\mathfrak{g} = \mathrm{Cent}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, where $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra. Then vanishing of the second cohomology of A_0 leads to the same property for the second cohomology of the transitive Lie algebroid A, $\mathcal{H}^2(A) = 0$[IKV]. This condition appears also under the study of the formal Poisson equivalence [IKV].

5. Linearized Poisson models over a single symplectic leaf. In this section we will show that for every Poisson manifolds with a given closed symplectic leaf B there is a well defined notion of a *linearized Poisson structure* at B. This linearized structure is defined as an equivalence class of isomorphic Poisson structures which live naturally on the normal bundle to the symplectic leaf B. In the zero-dimensional case ($\dim B = 0$), our definition coincides with the notion of a linear approximation of a Poisson structure at a point of rank 0 arising in the context of the linearization problem [We_4].

5.1. *First approximations.* Let (M, Ψ) be a Poisson manifolds equipped with a Poisson bracket

$$\{F, G\} = \Psi(dF, dG). \tag{5.1}$$

Suppose that we are given a closed (embedded) symplectic leaf (B, ω) of M with symplectic structure ω. Consider the *normal bundle* to the symplectic leaf B:

$$\mathcal{N} = T_B M / TB. \tag{5.2}$$

The well known fact is that the original Poisson structure on M induces a fiberwise Lie–Poisson structure on the normal bundle $\mathcal{N}$ which is given by the vertical Poisson bivector field $\Lambda \in \chi^2(\mathcal{N})$ called a *linearized transverse Poisson structure* of the leaf B [We$_4$]. At each fiber N_b over $b \in B$ the Lie–Poisson structure $\Lambda_b \in \chi^2(\mathcal{N}_b))$ can be defined as the linearization of the transverse Poisson structure at b due to the splitting theorem. To compare the original Poisson tensor Ψ with Λ, it is natural to consider a pull back of Ψ onto $\mathcal{N}$ via an exponential map.

By an *exponential map*, we mean a diffeomorphism $\mathbf{f} : \mathcal{N} \to M$ from the normal bundle $\mathcal{N}$ onto a tubular neighborhood of the leaf B in M such that

(i) $\mathbf{f}$ is compatible with the zero section $s_0 : B \hookrightarrow \mathcal{N}$, that is, $\mathbf{f} \circ s_0 = s_0$; and
(ii) the composite map

$$\mathcal{N}_b \hookrightarrow T_b(\mathcal{N}) \xrightarrow{d_b\mathbf{f}} T_b M \xrightarrow{\nu_b} \mathcal{N}_b$$

is the identity. Here the last mapping is the canonical projection $\nu : T_B M \to T_B M/TB$.

It follows from the tubular neighborhood theorem that an exponential map always exists [LMr]. By Proposition 3.1 we deduce the following statement.

Proposition 5.1. *Let $\mathbf{f}^*\Psi \in \chi^2(\mathcal{N})$ be the pull back of the Poisson tensor Ψ via an exponential map $\mathbf{f}$. Then the zero section $B \hookrightarrow \mathcal{N}$ is a closed symplectic leaf of $\mathbf{f}^*\Psi$ with symplectic structure ω. Moreover, there exists an open neighborhood E of B in $\mathcal{N}$ such that $\mathbf{f}^*\Psi$ is a coupling tensor on E. For the vertical tensor $(\mathbf{f}^*\Psi)_V$ defined in* (3.1), *we have*

$$(\mathbf{f}^*\Psi)_V = \Lambda + O_2 \qquad \text{on} \quad E, \tag{5.3}$$

that is, the linearized transverse Poisson structure Λ gives a linear approximation to the vertical part of $\mathbf{f}^*\Psi$.

Definition 5.1. A 0-section compatible Poisson tensor Π defined (as a coupling tensor) on an open (tubular) neighborhood E of B in $\mathcal{N}$ is said to be a *first approximation* to Ψ at the leaf B if

(i) the intrinsic Ehresmann connection Γ (2.4) of Π is *homogeneous* on E;
(ii) the vertical part Π_V in (3.1) coincides with the linearized transverse Poisson structure Λ of B,

$$\Pi_V = \Lambda \qquad \text{on} \quad E; \tag{5.4}$$

(iii) there exists an exponential map $\mathbf{f} : \mathcal{N} \to M$ such that

$$\mathbf{f}^*\Psi = \Pi + O_2 \qquad \text{on} \quad E. \tag{5.5}$$

Theorem 5.1. *Let (M, Ψ, B, ω) be a Poisson manifold with a closed symplectic leaf (B, ω). Then for a given exponential map $\mathbf{f}$ there exists a unique first approximation $\Pi^{\mathbf{f}}$ to Ψ at B satisfying* (5.5). *The Poisson bivector field $\Pi^{\mathbf{f}}$ does not depend on the choice of $\mathbf{f}$ up to 0-section neighborhood isomorphism.*

Definition 5.2. The equivalence class of isomorphic Poisson tensors $\Pi^{\mathbf{f}}$ is said to be the *linearized Poisson structure* of the leaf B.

Remark 5.1. If the symplectic leaf B is not closed, then in the definition of the exponential map we can require $\mathbf{f}$ to be a smooth immersion. In this case, the notion of the linearized Poisson structure is still well defined. But the pull back $\mathbf{f}^*\Psi$ does not isomorphic to the original Poisson structure Ψ in general.

To prove Theorem 5.1, we will use results obtained in Section 4.

5.2. *The transitive Lie algebroid of a symplectic leaf.* As is well known, the Poisson bracket (5.1) on M admits the natural extension to the bracket for 1-forms on M:

$$\{\alpha, \beta\}_{T^*M} = \Psi^{\#}(\alpha)\rfloor d\beta - \Psi^{\#}(\beta)\rfloor d\alpha - d\langle\alpha, \Psi^{\#}(\beta)\rangle. \tag{5.6}$$

This structure makes the cotangent bundle T^*M a Lie algebroid:

$$\big(T^*M, \{\ ,\ \}_{T^*M}, \rho = \Psi^{\#}\big) \tag{5.7}$$

which is called the Lie algebroid of the Poisson manifold (M, Ψ) [We$_5$]. Notice that if M is not regular, then the Lie algebroid (5.7) is not transitive.

Given a symplectic leaf (B, ω) of M, one can restrict the bracket $\{\ ,\ \}_{T^*M}$ to a bracket $\{\ ,\ \}_{T^*_BM}$ on smooth sections of the restricted cotangent bundle T^*_BM. The result is the transitive Lie algebroid [IKV] (also see [Ku] for general criteria of Lie subalgebroids):

$$\big(T^*_BM, \{\ ,\ \}_{T^*_BM}, \rho = \rho_B\big) \tag{5.8}$$

with anchor

$$\rho_B : T^*_BM \to T^*B \xrightarrow{-(\omega^\flat)^{-1}} TB, \tag{5.9}$$

where the first morphism is induced by the inclusion $TB \hookrightarrow T_BM$ and $\omega^\flat : TB \to T^*B$ is the bundle map associated with the symplectic structure ω ($\omega^\flat(u) = u \lrcorner \omega$) The isotropy of this Lie algebroid coincides with the annihilator $TB^0 = \ker_B \Psi^{\#}$ of TB in T_BM. We will call (5.8) the *transitive Lie algebroid* of the *symplectic leaf* B.

Let $\mathcal{N}$ be the normal bundle to the leaf B and $\mathbf{f} : \mathcal{N} \to M$ be an exponential map. Then the differential

$$d_B\mathbf{f} : T_B\mathcal{N} = TB \oplus \mathcal{N} \to T_BM$$

is identical on TB and takes the subbundle $\mathcal{N}$ to the complementary subbundle $S = d_B\mathbf{f}(\mathcal{N})$ to TB. Let S^0 be the annihilator of S in T_BM. The natural splitting

$$T^*_BM = S^0 \oplus TB^0 \tag{5.10}$$

defines the connection $\gamma_{\mathbf{f}} : TB \to T^*_BM$ in the Lie algebroid (5.8). On the other hand, the exact sequence of vector bundles $TB \to T_BM \xrightarrow{\nu} \mathcal{N}$ induces the dual exact sequence

$$\mathcal{N}^* \xrightarrow{\nu^*} T^*_BM \to T^*B. \tag{5.11}$$

Using (5.10) and (5.11), we define the vector bundle isomorphism

$$\iota_{\mathbf{f}} = \gamma_{\mathbf{f}} \oplus \nu^* : TB \oplus \mathcal{N}^* \to T^*_B M \tag{5.12}$$

which induces the Lie algebroid structure on $A = TB \oplus \mathcal{N}^*$:

$$\{ a_1, a_2\}_A = \iota_{\mathbf{f}}^{-1}(\{\iota_{\mathbf{f}}(a_1), \iota_{\mathbf{f}}(a_2)\}_{T^*_B M}).$$

Thus, we get the transitive Lie algebroid over B with distinguished connection:

$$(A = TB \oplus \mathcal{N}^*, \{\ ,\ \}_A, \rho = \mathrm{pr}_1, \gamma_0). \tag{5.13}$$

Here the anchor is the projection onto the first factor, the conormal bundle $\mathcal{N}^*$ is the isotropy and the connection γ_0 is the canonical injection (4.26) whose pull back via $\iota_{\mathbf{f}}$ coincides with the $\mathbf{f}$-dependent connection, $\gamma_{\mathbf{f}} = \iota_{\mathbf{f}} \circ \gamma_0$.

Now we can proceed to the proof of Theorem 5.1. Given an exponential map $\mathbf{f}$, we define the coupling tensor Π^{A,γ_0} on the normal bundle $\mathcal{N}$ associated with the transitive Lie algebroid A in (5.13) and connection γ_0. Clearly Π^{A,γ_0} is equivalent to the coupling tensor associated with the transitive Lie algebroid of B (5.8) and the connection $\gamma_{\mathbf{f}}$. Finally, we observe that Π^{A,γ_0} is just the first approximation to Ψ at B generated by the exponential map $\mathbf{f}$,

$$\Pi^{\mathbf{f}} = \Pi^{A,\gamma_0}. \tag{5.14}$$

Here we use the following equivalent reformulation of Definition 5.1: a coupling tensor Π with an exponential map $\mathbf{f}$ defines a first approximation to Ψ at B if the geometric data of Π are obtained from the geometric data of $\mathbf{f}^*\Psi$ by means of the linearization at B. The independence of $\Pi^{\mathbf{f}}$ of the choice of $\mathbf{f}$ (up to a neighborhood equivalence) follows from Theorem 4.2.

We can conclude: *the linearized Poisson structure of Ψ at a closed symplectic leaf (B, ω) coincides with the ω-coupling structure of the transitive Lie algebroid of the leaf.*

Now it is natural to say that a Poisson stucture Ψ is *linearizable* at a closed symplectic leaf (B, ω) if there exists an exponential map $\mathbf{f}$ such that the pull back $\mathbf{f}^*\Psi$ and the first approximation $\Pi^{\mathbf{f}}$ are isomorphic over the zero section $B \hookrightarrow \mathcal{N}$. This definition does not depend on the choice of $\mathbf{f}$.

REMARK 5.2. If $\Lambda = 0$, then one can try to introduce second approximations to Ψ at B, using, for example, results [Du].

To end this section, as a consequence of the above results, we give an affirmative answer to the question on the Poisson realization of transitive Lie algebroids.

THEOREM 5.2. *Every transitive Lie algebroid A over a connected symplectic base (B, ω) can be realized as the transitive Lie algebroid of the symplectic leaf (B, ω) of a certain Poisson manifold.*

5.3. *Homotopy invariants.* The notion of the *reduced linear Poisson holonomy* of a symplectic leaf B, introduced in [GiGo] (also see [Fe]), can be defined as a homotopy invariant of the transitive Lie algebroid of B (5.8). To see that, pick two coonections γ and $\widetilde{\gamma}$ in the Lie algebroid $T^*_B M$ and consider the corresponding adjoint connections ∇^{γ} and $\nabla^{\widetilde{\gamma}}$ on the isotropy $\mathcal{L} = TB^0$. Fix a point $b_0 \in B$ and consider a smooth path

$[0,1] \ni t \mapsto \sigma(t) \in B$ starting at b_0, $\sigma(0) = b_0$. Denote by $\mathcal{P}_t : \mathcal{L}_{b_0} \to \mathcal{L}_{\sigma(t)}$ and $\widetilde{\mathcal{P}_t} : \mathcal{L}_{b_0} \to \mathcal{L}_{\sigma(t)}$ parallel transport operators associated with linear connections ∇^γ and $\widetilde{\nabla^\gamma}$, respectively. Define a time dependent field of linear operators on the fiber $\mathcal{L}_{b_0}$ as follows:

$$\Xi_t := \mathcal{P}_t^{-1} \circ \left(\operatorname{ad} \circ \mu\left(\frac{d\sigma(t)}{dt} \right) \right) \circ \mathcal{P}_t, \tag{5.15}$$

where μ is an $\mathcal{L}$-valued 1-form on B, defined in (4.19). It follows from (4.6) that $\Xi_t \in \operatorname{ad}(\mathcal{L}_{b_0}) \approx \operatorname{ad}(\mathfrak{g})$(the adjoint algebra of the typical fiber $\mathfrak{g}$) for all $t \in [0,1]$. Consider the evolution operator $\mathbb{T}_t \in \operatorname{Ad}(\mathcal{L}_{b_0}) \approx \operatorname{Ad}(\mathfrak{g})$:

$$\frac{d\mathbb{T}_t}{dt} = -\Xi_t \circ \mathbb{T}_t, \qquad \mathbb{T}_0 = \operatorname{id}. \tag{5.16}$$

Then we get the following relationship between parallel transports of two adjoint connections [KV$_1$]: $\widetilde{\mathcal{P}_t} = \mathcal{P}_t \circ \mathbb{T}_t$. This implies that for every loop $\sigma \in \Omega(B; b_0)$ based at b_0, the corresponding elements of holonomy groups $\widetilde{\mathcal{P}_\sigma} \in \operatorname{Hol}_{b_0}^{\widetilde{\nabla^\gamma}} \subset \operatorname{Aut}(\mathfrak{g})$ and $\mathcal{P}_\sigma \in \operatorname{Hol}_{b_0}^{\nabla^\gamma} \subset \operatorname{Aut}(\mathfrak{g})$ are related by $\widetilde{\mathcal{P}_\sigma} = \mathcal{P}_\sigma \circ \mathbb{T}_\sigma$, where $\mathbb{T}_\sigma \in \operatorname{Inn}(\mathcal{L}_{b_0}) \approx \operatorname{Inn}(\mathfrak{g})$ (the normal subgroup of inner automorphisms of $\mathfrak{g}$, see [GiGo]). Thus, there is a well defined homomorphism $\Omega(B; b_0) \to \operatorname{Aut}(\mathfrak{g}) / \operatorname{Inn}(\mathfrak{g})$, which does not depend on the choice of an adjoint connection. If we consider the conjugate homomorphism $\Omega(B; b_0) \to \operatorname{Aut}(\mathfrak{g}^*) / \operatorname{Inn}(\mathfrak{g}^*)$, then its cotangent lift coincides with the definition of the reduced linear Poisson holonomy of B given in [GiGo, Fe].

A. Appendix: the proof of Lemma 3.1. First, remark that if Γ is an Ehresmann connection on a fiber bundle $\pi : E \to B$, then the horizontal lift and the covariant exterior derivative (2.14) satisfy the modified Cartan formula

$$L_{\operatorname{hor}(u)} = \imath_u \circ \partial_\Gamma + \partial_\Gamma \circ \imath_u, \qquad u \in \mathcal{X}(B). \tag{A.1}$$

Here $\imath_u$ is the interior product. Moreover, the commutator of the horizontal lift $\operatorname{hor}(u)$ with an arbitrary vertical vector field is again a vertical vector field,

$$[\operatorname{hor}(u), \mathcal{X}_V(E)] \in \mathcal{X}_V(E). \tag{A.2}$$

Let Π_t be the time-dependent coupling tensor associated with geometric data $(\Gamma_t, \mathcal{V}, \mathbb{F}_t)$ in (3.14), (3.15), and let $X_t^h \in \mathcal{X}_H(\mathcal{E}_0)$ be an arbitrary time-dependent horizontal vector field. Using properties (A.1), (A.2) and the standard properties of the Schouten bracket, from relations (2.16)–(2.19) for $(\Gamma_t, \mathcal{V}, \mathbb{F}_t)$ we deduce the key formula

$$\begin{aligned} L_{X_t^h} \Pi_t = &-\frac{1}{2} H^{ii'} H^{jj'} (\partial_{\Gamma_t}(X_t \rfloor \mathbb{F}_t))_{i'j'} \operatorname{hor}_t(\partial_i) \wedge \operatorname{hor}_t(\partial_j) \\ &+ H^{is} (\mathcal{V}^{\#} d\mathbb{F}_t(X_t, \partial_s))^\sigma \partial_\sigma \wedge \operatorname{hor}_t(\partial_i). \end{aligned} \tag{A.3}$$

Here hor_t is the horizontal lift associated with Γ_t and we use the local representations

$$\Pi_t = \frac{1}{2} H^{ij} \operatorname{hor}_t(\partial_i) \wedge \operatorname{hor}_t(\partial_j) + \frac{1}{2} \mathcal{V}^{\sigma\sigma'} \partial_\sigma \wedge \partial_{\sigma'},$$

where $\partial_i = \partial / \partial \xi^i$, $\partial_\sigma = \partial / \partial x^\sigma$, (ξ^i) and (x^σ) are local coordinates on the base and the fiber of π, respectively. Let $\mathbb{F}_t = \frac{1}{2} F_{ij} d\xi^i \wedge d\xi^j$. Taking into account $H^{is} F_{sj} = -\delta_j^{i'}$ and relations (3.14), (3.15), we get also

$$\frac{\partial}{\partial t}\Pi_t = -\frac{1}{2}H^{ii'}H^{jj'}\frac{\partial}{\partial t}F_{i'j'}\operatorname{hor}_t(\partial_i)\wedge\operatorname{hor}_t(\partial_j) - H^{is}(\mathcal{V}^{\#}d\phi(\partial_s))^{\sigma}\partial_\sigma\wedge\operatorname{hor}_t(\partial_i). \tag{A.4}$$

Now a direct consequence of (A.3) and (A.4) is that a time-dependent horizontal vector field X_t^h is a solution of the homological equation (3.20) if and only if the associated element $X_t \in \mathcal{X}(B)\otimes C^\infty(\mathcal{E}_0)$ satisfies the following two equations

$$\partial_{\Gamma_t}(X_t\rfloor\mathbb{F}_t) + \frac{\partial}{\partial t}\mathbb{F}_t = 0, \tag{A.5}$$

$$X_t\rfloor\mathbb{F}_t = \phi + c, \tag{A.6}$$

where $c \in \Omega^1(B)\otimes\mathrm{Casim}_{\mathcal{V}}(\mathcal{E}_0)$ is arbitrary. Taking $c = 0$ and X_t as the solution of (3.16), we reduce (A.5) to the identity

$$\partial_{\Gamma_t}\phi = \partial_\Gamma\phi + t\{\phi\wedge\phi\}_{\mathcal{V}}, \qquad t\in[0,1],$$

which holds because of the assumption (3.9). This completes the proof.

References

[AM] R. Almeida and P. Molino, *Suites d'Atiyah et feuilletages transversalement complets*, C. R. Acad. Sci. Paris Sér. I Math. 300 (1985), 13–15.

[CWe] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes, AMS, Providence, 1999.

[Cn$_1$] J. Conn, *Normal forms for analytic Poisson structures*, Ann. of Math. 119 (1984), 576–601.

[Cn$_2$] J. Conn, *Normal forms for smooth Poisson structures*, Ann. of Math. 121 (1985), 565–593.

[Co] T. J. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. 319 (1990), 631–661.

[Du] J.-P. Dufour, *Quadratisation de structures de Poisson à partie quadratique diagonale*, Séminaire Gaston Darboux de Géométrie et Topologie Différentielle, 1992–1993 (Montpellier), iii, 10–13, Univ. Montpellier II, Montpellier, 1994.

[Fe] R. L. Fernandes, *Connections in Poisson geometry: Holonomy and invariants*, Preprint 2000,math. DG/001129.

[GiGo] V. Ginzburg and A. Golubev, *Holonomy on Poisson manifolds and the modular class*, Preprint 1998, math. DG/9812153.

[GoLSW] M. Gotay, R. Lashof, J. Sniatycki, and A. Weinstein, *Closed forms on symplectic fiber bundles*, Comment. Math. Helv. 58 (1983), 617–621.

[GHV] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature, and Cohomology*, Vol. II, Academic Press, New York–London, 1973.

[GLS] V. Guillemin, E. Lerman, and S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge Univ. Press, Cambridge, 1996.

[GSt] V. Guillemin and S. Sternberg, *Symplectic Technique in Physics*, Cambridge Univ. Press, Cambridge, 1984.

[IKV] V. M. Itskov, M. Karasev, and Yu. M. Vorobjev, *Infinitesimal Poisson geometry*, in: Amer. Math. Soc. Transl. (2) 187, 1998, 327–360.

[KM] M. V. Karasev and V. P. Maslov, *Nonlinear Poisson Brackets. Geometry and Quantization*, Transl. of Math. Monographs 119, AMS, Providence, 1993.

[KV_1] M. V. Karasev and Yu. M. Vorobjev, *Adapted connections, Hamilton dynamics, geometric phases, and quantization over isotropic submanifolds*, in: Amer. Math. Soc. Transl. (2) 187, 1998, 203–326.

[KV_2] M. V. Karasev and Yu. M. Vorobjev, *Deformations and cohomology of Poisson manifolds*, in: Lecture Notes in Math. 1453, Springer, 1990, 271–289.

[K-SM] Y. Kosmann–Schwarzbach and F. Magri, *Poisson–Nijenhuis structures*, Ann. Inst. H. Poincaré Phys. Théor. 53 (1990), 35–81.

[Ku] J. Kubarski, *The Chern-Weil homomorphism of regular Lie algebroids*, Publ. Dep. Math. Nouvelle Sér., Univ. Claude-Bernard Lyon 1, 1991, 1–69.

[La] N. P. Landsman, *Mathematical Topics between Classical and Quantum Mechanics*, Springer, 1998.

[LMr] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.

[Li] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Differential Geom. 12 (1977), 253–300.

[Mz] K. C. H. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, LMS Lecture Note Ser. 124, Cambridge Univ. Press, Cambridge, 1987.

[Mos] J. Moser, *On the volume element on a manifold*, Trans. Amer. Math. Soc. 120 (1965), 286–294.

[Mo] R. Montgomery, *Canonical formalism of a classical particle in a Yang–Mills field and Wong's equations*, Lett. Math. Phys. 8 (1984), 59–67.

[MoMR] R. Motgomery, J. E. Marsden, and T. Ratiu, *Gauged Lie–Poisson structures*, in Contemp. Math. 28, AMS, 1984, 101–114.

[St] S. Sternberg, *Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Young–Mills field*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 5253–5254.

[Tu] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (1976), 467–468.

[Va] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progr. in Math. 118, Birkhäuser, Boston, 1994.

[We_1] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. 6 (1971), 329–346.

[We_2] A. Weinstein, *A universal phase space for particles in Yang–Mills fields*, Lett. Math. Phys. 2 (1978), 417–420.

[We_3] A. Weinstein, *Fat bundles and symplectic manifolds*, Adv. Math. 37 (1980), 239–250.

[We_4] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom. 18 (1983), 523–557.

[We_5] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc. 16 (1987), 101-104.