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TOEPLITZ OPERATORS ON HARDY SPACES OVER $\mathrm{SL}(2,\mathbf{R})$: IRREDUCIBILITY AND REPRESENTATIONS

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Abstract. On the non-abelian, non-compact simple rank 1 Lie group $G = \mathrm{SL}(2,\mathbf{R})$, we consider Hardy spaces $\mathbf{H}^2(G_\pm^{\mathbf{C}})$ defined by $\mathbf{L}^2$-boundary values of holomorphic functions on the complex subsemigroups $G_\pm^{\mathbf{C}}$ of $G^{\mathbf{C}} = \mathrm{SL}(2,\mathbf{C})$. These Hardy spaces are associated to the two parts of the discrete series of G, and give rise to equivariant projections $E_\pm$ and corresponding Toeplitz operators $\mathrm{T}_\pm(f)$, $f \in \mathcal{C}^0(G)$. We show that a stratification of boundary faces for $G_\pm^{\mathbf{C}}$ can be given, and, by a geometric construction, associate to these faces representations of the C^*-algebra generated by the Toeplitz operators for the respective domain, thus achieving a step 2 composition series for this C^*-algebra.

1. Introduction. For a semi-simple Lie group $\mathcal{G}$ of Hermitian type, a major part of harmonic analysis on $\mathcal{G}$ involves Hilbert spaces of holomorphic functions on the associated Hermitian symmetric space $\mathcal{G}/K$.

The basic example is the so-called holomorphic discrete series with can be realized by (vector-valued) holomorphic functions on $\mathcal{G}/K$ admitting a reproducing kernel of Bergman type.

This is the starting point of the Berezin quantization method, in which $\mathcal{G}/K$ is considered as a symplectic manifold (classical phase space), and to every $\mathcal{C}^\infty$ function on $\mathcal{G}/K$ one associates a Bergman type Toeplitz operator T_f on the corresponding Hilbert space. The C^*-algebraic properties of these Toeplitz operators have been extensively studied, cf. [Upm96].

In order to develop the Berezin–Toeplitz quantization procedure in a wider setting, it is important to study more general phase spaces which admit a $\mathcal{G}$-action but without

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the quite restrictive requirement of transitivity. The framework of general symplectic (or Poisson) $\mathcal{G}$-spaces maintains the close relation to harmonic analysis on $\mathcal{G}$, but adds more flexibility in the geometric setting, allowing foliations by symplectic $\mathcal{G}$-orbits.

An important class of Poisson $\mathcal{G}$-spaces can be realized as domains of holomorphy in the complexification of $\mathcal{G}/G$, for a suitable closed subgroup G. More precisely, $\mathcal{G}/G$ is required to be a semi-simple (non-Riemannian) symmetric space, and we consider suitable non-homogeneous domains in $\mathcal{G}^{\mathbf{C}}/G^{\mathbf{C}}$. The symmetric space $\mathcal{G}/G$ plays the role of 'Shilov boundary' since, as in the classical setting, we will study holomorphic functions and their boundary values on $\mathcal{G}/G$. The associated Hilbert spaces are known in the literature as 'non-commutative' Hardy and Bergman spaces, cf. [Nee00], [HN93], [BH00], [HÓØ91], [KØ96], [KØ97], and play a crucial role in the well-known 'Gelfand–Gindikin program', cf. [GG77].

Every semi-simple Lie group G can be realized as a (non-Riemannian) symmetric space for $\mathcal{G} = G \times G$, endowed with the flip involution. In this case one studies domains of holomorphy in $G^{\mathbf{C}}$ and the structure of the corresponding non-commutative Hardy spaces is well-established, cf. [Nee00], [HN93], [HÓØ91].

On the other hand, virtually nothing is known about the corresponding Toeplitz operators and their C*-algebraic properties. The main difficulty lies in the fact that the underlying group G is neither commutative nor compact. In addition, new features such as the existence of non-conjugate Cartan subgroups lead to profound new properties of the Toeplitz C*-algebras.

In this paper, the program outlined above is carried out in detail for the basic case $G = \mathrm{SL}(2, \mathbf{R})$ which already exhibits the main features of non-Riemannian symmetric spaces.

For $\mathrm{SL}(2, \mathbf{R})$, the (holomorphic) discrete series consists of the well-known Bergman spaces on the upper half plane, which are the building blocks of the 'non-commutative' Hardy space over $\mathrm{SL}(2, \mathbf{R})$. Its reproducing kernel, leading to the definition of Toeplitz operators, was already determined in [GG77], but it is still quite difficult to construct (irreducible) representations of the associated C*-algebra, since standard techniques, such as groupoid realizations, are not available.

The main idea to overcome these problems is to realize the Toeplitz C*-algebra via a C*-algebra 'cocrossed product' for a natural coaction of $G = \mathrm{SL}(2, \mathbf{R})$, an approach first chosen by [Was84]. The fundamental facts concerning coactions and cocrossed products are presented here in a general setting to allow for the generalizations mentioned above.

In the second part of the paper, we use the C*-algebraic framework and a detailed geometric study of the underlying domain in $\mathrm{SL}(2, \mathbf{C})$ and its symplectic foliation to construct the irreducible representations of the Toeplitz C*-algebra. The main new feature, related to the existence of two non-conjugate Cartan subgroups, is a step 2 composition series for the Toeplitz C*-algebra over the rank 1 group $\mathrm{SL}(2, \mathbf{R})$.

2. Preliminaries

2.1. *Conventions.* To denote function spaces, we use the Bourbaki notations. So $\mathcal{K}(X, Y)$ denotes the space of continuous compactly supported functions $X \to Y$, endowed with the final topology w.r.t. the compact-open topology on compact subsets of X. In fact, generically, we use $\mathcal{K}E$ to denote compactly supported elements of E whenever this makes sense. $\mathcal{C}^0(X)$ shall be the space of continuous functions $X \to \mathbf{C}$ vanishing at ∞, with $\|\cdot\|_\infty$. $\mathcal{E}(X)$ denotes the space of smooth functions $X \to \mathbf{C}$, endowed with the topology of uniform convergence of all derivatives on compact subsets of X. $\mathcal{D}(X)$ is the space of compactly supported smooth functions $X \to \mathbf{C}$, endowed with the final locally convex topology w.r.t. uniform convergence of all derivatives on compact subsets of X. $\mathcal{D}'(X)$, its dual, is the space of distributions on X, usually endowed with the $\sigma(\mathcal{D}'(X), \mathcal{D}(X))$-topology.

The space of linear maps $E \to F$ will be denoted $\mathrm{L}(E, F)$. The algebra of bounded operators on a Hilbert space $\mathcal{H}$ will be denoted $\mathcal{L}(\mathcal{H})$. Here, we usually use the norm topology. However, we also use the weak, ultraweak (i.e. $\sigma(\mathcal{L}(\mathcal{H}), \mathcal{L}^1(\mathcal{H}))$) and strong topologies. We avoid using abbreviations for these topologies and always state their use explicitly. The compact operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{LC}(\mathcal{H})$.

We use $\mathrm{C}^* \prec \cdots \succ$ to denote the C^*-algebra generated by a specified sets of operators. E.g.,

$$\mathrm{C}^* \prec af \mid a \in A, f \in \mathcal{C}^0(G) \succ \subset \mathcal{L}(\mathbf{L}^2(G))$$

will denote the C^*-subalgebra of $\mathcal{L}(\mathbf{L}^2(G))$ generated by the set of products af where $a \in A$ and $f \in \mathcal{C}^0(G)$. If μ is a linear functional on the vector space E, we shall write

$$\langle \alpha : \mu \rangle = \mu(\alpha) \quad \text{for all} \quad \alpha \in E.$$

X^Y will denote the set of maps $Y \to X$. Finally, identity maps will always be written id, and we use the symbol $\diamond$ to denote placeholders (i.e. anonymous variables).

2.2. *Group (co-) algebras.* Let G be a locally compact group, and consider a left Haar measure $dm_G(g) = dg$ on G. The left regular representation of G on $\mathbf{L}^2(G)$ will be denoted $g \mapsto g^\#$, so that we have $\mu * \xi = \mu^\# \xi$ for any bounded measure μ on G and any $\xi \in \mathbf{L}^2(G)$. Similarly, we denote right convolution by $g_\#$ resp. $\mu_\#$. Consequently, the *reduced group* C^**-algebra*, which is the C^*-algebra generated by $\mathbf{L}^1(G)$ in the left regular representation, will be denoted by $\mathrm{C}^*_\#(G)$. We have $\mathrm{C}^*_\#(G \times G) = \mathrm{C}^*_\#(G) \otimes \mathrm{C}^*_\#(G)$ where the tensor product is spatial. The *reduced group von Neumann algebra* of G, which is the ultraweak closure of $\mathrm{C}^*_\#(G)$, will be denoted by $\mathrm{W}^*(G)$. If G is commutative, $\mathrm{W}^*(G) \cong \mathbf{L}^\infty(\widehat{G})$.

If $\mathrm{C}^*(G)$ is the universal group C^*-algebra, that is the C^*-algebra generated by $\mathbf{L}^1(G)$ in the universal representation, it is known that the Banach dual $\mathrm{B}(G) = \mathrm{C}^*(G)'$ may be considered as a subalgebra of $\mathcal{C}^b(G)$, and is a commutative unital Banach $*$-algebra in the dual norm, the so-called *Fourier–Stieltjes algebra* of G, cf. [Val85], [Eym64]. The dual $\mathrm{B}_\#(G) = \mathrm{C}^*_\#(G)'$ is a closed $*$-subalgebra of $\mathrm{B}(G)$, the so-called *reduced Fourier–Stieltjes algebra* (a.k.a. Eymard algebra), and carries the induced norm. It is unital if and only if G is amenable. The predual $\mathrm{A}(G) = \mathrm{W}^*(G)_*$ is a closed $*$-ideal of $\mathrm{B}(G)$, the *Fourier algebra* of G. Hence $\mathrm{W}^*(G)$ is a $\mathrm{B}(G)$-module in the natural way:

$$\langle \alpha : \beta \cdot x \rangle := \langle \alpha \cdot \beta : x \rangle \quad \text{for all} \quad \alpha \in \mathrm{A}(G), \beta \in \mathrm{B}(G), x \in \mathrm{W}^*(G).$$

$\mathrm{A}(G)$ is precisely the closure of the elements of compact support in $\mathrm{B}_\#(G)$ (or, equivalently, $\mathrm{B}(G)$). Furthermore, it coincides with the set

$$\{\bar{\eta} * \xi^{\vee} \mid \xi, \eta \in \mathbf{L}^2(G)\}.$$

If G is commutative, $\mathrm{A}(G) \cong \mathbf{L}^1(\widehat{G})$. We shall denote the set of elements of $\mathrm{A}(G)$ with compact support by $\mathcal{K}\mathrm{A}(G)$. Note that by a theorem of Leptin, $\mathrm{A}(G)$ has a bounded approximate unit if and only if G is amenable. This is the main source of technical difficulties in dealing with $\mathrm{A}(G)$, since factorization results such as Cohen's theorem are in general not applicable to $\mathrm{A}(G)$-modules. However, since $\mathrm{A}(G)$ is Shilov-regular, this may often be circumvented by using compactly supported functions in $\mathrm{A}(G)$.

Besides the norm and weak topologies, we need to consider the *strict topology* on $\mathrm{B}(G)$ which is the weakest locally convex topology that makes multiplication by elements of $\mathrm{A}(G)$ norm-continuous. This topology is also called the multiplier topology.

If A is a C^*-algebra, let $\mathrm{M}(A)$ denote its multiplier algebra.

Since the left regular representation $g \mapsto g^{\#} : G \to \mathrm{M}(\mathrm{C}^*_{\#}(G))$ is bounded and strictly continuous, it may be considered as an element

$$W_G \in \mathrm{M}(\mathcal{C}^0(G) \otimes \mathrm{C}^*_{\#}(G)) = \mathcal{C}^b(G, \mathrm{M}(\mathrm{C}^*_{\#}(G))).$$

This is the so-called *Kac–Takesaki fundamental unitary* given by

$$W_G\xi(s,t) = \xi(s, s^{-1}t) \quad \text{for all} \;\; s,t \in G, \xi \in \mathbf{L}^2(G \times G).$$

It gives rise to an injective normal $*$-morphism

$$\delta_G : \mathrm{W}^*(G) \to \mathrm{W}^*(G)\bar{\otimes}\mathrm{W}^*(G) = \mathrm{W}^*(G \times G) : x \mapsto \mathrm{Ad}(W_G)(x \otimes 1)$$

satisfying the identity $(\delta_G \otimes \mathrm{id}) \circ \delta_G = (\mathrm{id} \otimes \delta_G) \circ \delta_G$. Here, $\bar{\otimes}$ denotes the W^*-tensor product. Thus, $(\mathrm{W}^*(G), \delta_G)$ is a *Hopf–von Neumann algebra* with *coproduct* δ_G, cf. [NT79], [ES80]. δ_G is also the (normal extension of the) integrated version of the representation

$$g \mapsto (g,g)^{\#} = g^{\#} \otimes g^{\#} : G \to \mathrm{M}(\mathrm{C}^*_{\#}(G) \otimes \mathrm{C}^*_{\#}(G)).$$

Furthermore, the multiplication of $\mathrm{A}(G)$ is dual to δ_G, i.e.

$$\langle \alpha \cdot \alpha' \,:\, x \rangle = \langle \alpha \otimes \alpha' \,:\, \delta_G(x) \rangle \quad \text{for all} \;\; \alpha, \alpha' \in \mathrm{A}(G), x \in \mathrm{W}^*(G).$$

In particular, the action of $\mathrm{A}(G)$ on $\mathrm{W}^*(G)$ is given by slice maps:

$$\alpha \cdot x = (\mathrm{id} \otimes \alpha)(\delta_G(x)) \quad \text{for all} \;\; \alpha \in \mathrm{A}(G), x \in \mathrm{W}^*(G).$$

When there is no danger of confusion, we often omit the subscript and simply write δ.

3. A convenient setting for Toeplitz C^*-algebras.

In this section, G will denote a locally compact group.

3.1. *Coactions and modules over various subalgebras of* $\mathrm{B}(G)$.

In this subsection, we introduce a notion of support for elements of $\mathrm{A}(G)$-modules which we will use as a tool of local analysis in the C^*-category.

NOTATION 3.1.1. If $\cdot : E \times E' \to E''$ is bilinear, we denote by $E \cdot E'$ the linear span of the set of the $e \cdot e'$, $e \in E$, $e' \in E'$, in E''.

DEFINITION 3.1.2. If A and B are C^*-algebras and $\varphi : A \to \mathrm{M}(B)$ is a $*$-morphism, φ is called *strict* if it has a unital strictly continuous extension $\overline{\varphi} : \mathrm{M}(A) \to \mathrm{M}(B)$ (the extension is unique since A is strictly dense in $\mathrm{M}(A)$). Here, *unital* means that $\overline{\varphi}(1) = 1$. If $\varphi : A \to \mathrm{M}(B)$ is a $*$-morphism such that $\overline{\varphi(A) \cdot B} = B$, then φ is said to be *non-degenerate*. A non-degenerate φ is strict by [LPRS87, lemma 1.1]. Clearly, if φ is injective, then so is $\overline{\varphi}$.

We introduce the following C*-subalgebra of $\mathrm{M}(A \otimes B)$:

$$\overleftarrow{\mathrm{M}}(A, B) := \{m \in \mathrm{M}(A \otimes B) \,\big|\, m(\mathbf{C} \otimes B) \cup (\mathbf{C} \otimes B)m \subset A \otimes B\}.$$

(We find this notation more suggestive than the usual $\widetilde{\mathrm{M}}(A \otimes B)$.) If δ is an injective, non-degenerate $*$-morphism

$$\delta : A \to \mathrm{M}(A \otimes \mathrm{C}^*_{\#}(G)) \quad \text{such that} \quad \delta(A) \subset \overleftarrow{\mathrm{M}}(A, \mathrm{C}^*_{\#}(G))$$

and $\overline{\delta \otimes \mathrm{id}} \circ \delta = \overline{\mathrm{id} \otimes \delta_G} \circ \delta$, then δ or (A, δ, G) shall be called a (*reduced*) C*-*coaction* of G on A. We usually omit the terms 'reduced' and 'C*'. A coaction is said to be *non-degenerate* (*in the sense of Landstad*) if for all $\omega \in A' \setminus \{0\}$ there exists $\beta \in \mathrm{B}_{\#}(G)$ such that $\overline{\omega \otimes \beta} \circ \delta \neq 0$. Here, the overline denotes strict extension, cf. [Tay70, corollary 2.3].

Similarly, we define W*-coactions, cf. [Qui92], [NT79].

Closely related to the notions of coaction are normed modules over certain subalgebras of $\mathrm{B}(G)$.

DEFINITION 3.1.3. Let E be a normed **C**-vector space which is a module over a normed **C**-algebra $\mathcal{J}$. Assume further that the map

$$\mathcal{J} \times E \to E : (\alpha, \mu) \mapsto \alpha \cdot \mu$$

has norm ≤ 1. Then we shall call E a *normed* $\mathcal{J}$-*module*. We call E *non-degenerate* if for any $\mu \neq 0$ there exists $\alpha \in \mathcal{J}$ such that $\alpha \cdot \mu \neq 0$.

The following proposition is folklore. The (more important) C*-case follows from the coaction identity, [LPRS87, 1.5 lemma] and the injectivity of $\overline{\delta}$.

PROPOSITION 3.1.4.

(i) *If* (M, δ, G) *is a* W*-*coaction, then*

$$\alpha \cdot \mu := (\mathrm{id} \otimes \alpha) \circ \delta(\mu) \quad \textit{for all} \quad \alpha \in \mathrm{A}(G), \mu \in M$$

defines on M *the structure of a non-degenerate normed* $\mathrm{A}(G)$-*module.*

(ii) *If* (A, δ, G) *is a* C*-*coaction, then*

$$\beta \cdot a := \overline{(\mathrm{id} \otimes \beta) \circ \delta}(a) \quad \textit{for all} \quad \beta \in \mathrm{B}_{\#}(G), a \in \mathrm{M}(A)$$

defines on $\mathrm{M}(A)$ *the structure of a non-degenerate normed* $\mathrm{B}_{\#}(G)$-*module for which* A *is a submodule.*

REMARK 3.1.5.

(i) The proposition has an obvious generalization to coactions of Hopf–von Neumann resp. Hopf-C*-algebras.

(ii) There are topological obstructions preventing the existence of a general converse to the proposition.

DEFINITION 3.1.6. Let E be a normed $\mathrm{A}(G)$-module. For any $\mu \in E$, let

$$\operatorname{supp}_E \mu := \{g \in G \,\big|\, \alpha \cdot \mu = 0 \;\Rightarrow\; \alpha(g) = 0 \text{ for all } \alpha \in \mathrm{A}(G)\}.$$

We omit the subscript whenever there is no danger of confusion. $\operatorname{supp} \mu$ is obviously a closed subset of G. We denote by $\mathcal{K}E$ resp. $\bar{\mathcal{K}}E$ the set of all compactly supported $\mu \in E$ resp. the norm-closure of this set. We have $\bar{\mathcal{K}}E = \overline{\mathrm{A}(G) \cdot E}$.

REMARK 3.1.7.

(i) Clearly,

$$\operatorname{supp} \mu = \operatorname{hull}(\mu^\perp) = \operatorname{Sp} \mathrm{A}(G)/\mu^\perp,$$

where $\mu^\perp = \{\alpha \in \mathrm{A}(G) \,|\, \alpha \cdot \mu = 0\}$.

(ii) From [Eym64, (4.4) proposition], it is clear that for the usual $\mathrm{A}(G)$-module structure on $\mathrm{W}^*(G)$, the notion of support set forth in that article is the same as the one introduced above. However, unlike Eymard, we need to consider degenerate module structures.

(iii) In [Nak77a] and [Nak77b], cf. also [NT79], one finds a notion of support called the local essential spectrum $\operatorname{sp}_\delta(x)$ of δ near x for a coaction δ of G on a W^*-algebra M. $\operatorname{sp}_\delta(x)$ depends only on the $\mathrm{A}(G)$-module structure induced by δ, and hence coincides precisely with our $\operatorname{supp}_M x$.
This concept was extended to a coaction δ on a C^*-algebra A by Katayama [Kat81]. His definition of $\operatorname{sp}_\delta(x)$ uses the $\mathrm{B}_\#(G)$-module structure induced by δ instead of its restriction to $\mathrm{A}(G)$. However, this is irrelevant: indeed, let $g \in \operatorname{supp}_A x$. Further, let $\beta \in \mathrm{B}_\#(G)$ such that $\beta \cdot x = 0$. Choose $\chi \in \mathrm{A}(G)$ such that $\chi(g) = 1$. Clearly, $\chi \cdot \beta \cdot x = 0$, so $\beta(g) = \chi(g) \cdot \beta(g) = 0$. In particular, Katayama's definition of $\operatorname{sp}_\delta(x)$ coincides with the one given in [Fan94].

The following proposition is mostly an adaptation of [Eym64, (4.8) proposition] to our situation.

PROPOSITION 3.1.8. *Let* $\mathrm{A}(G) \subset \mathcal{J} \subset \mathrm{B}(G)$ *be a closed* $*$*-ideal and* E *be a normed* $\mathcal{J}$*-module. Let* $\mu \in E$.

(i) $\operatorname{supp} \mu = \emptyset$ *implies that* $\alpha \cdot \mu = 0$ *for all* $\alpha \in \mathrm{A}(G)$*, in particular, if* E *is non-degenerate as an* $\mathrm{A}(G)$*-module, that* $\mu = 0$.
(ii) *For any* $\alpha \in \mathcal{J}$*, we have* $\operatorname{supp} \alpha \cdot \mu \subset \operatorname{supp} \alpha \cap \operatorname{supp} \mu$.
(iii) *For any* $\alpha \in \mathcal{K}\mathrm{A}(G)$ *vanishing on a neighbourhood of* $\operatorname{supp} \mu$*, we have* $\alpha \cdot \mu = 0$*. In fact,* $\operatorname{supp} \mu$ *is the smallest closed subset of* G *with this property.*
(iv) *If there exists a closed* $\mathcal{J}$*-invariant subspace* $F \subset \mathrm{W}^*(G)$ *such that* $E = \mathrm{W}^*(G)/F$ *and* $\operatorname{supp}_{\mathrm{W}^*(G)} \widetilde{\mu}$ *is compact for some representative* $\widetilde{\mu} \in \mathrm{W}^*(G)$ *of* μ*, then for any* $\alpha \in \mathcal{J}$ *vanishing in a neighbourhood of* $\operatorname{supp}_E \mu$*, we have* $\alpha \cdot \mu = 0$.

Proof. (i) It is easy to see that

$$\mu^\perp = \{\alpha \in \mathrm{A}(G) \,\big|\, \alpha \cdot \mu = 0\}$$

is a closed ideal in $\mathrm{A}(G)$. We have $\emptyset = \operatorname{supp} \mu = \operatorname{hull}(\mu^\perp)$; hence for all $g \in G$, there is $\alpha \in \mathrm{A}(G)$, so that $\alpha(g) \neq 0$ and $\alpha \cdot \mu = 0$. By the Tauberian theorem [Eym64, (3.38) corollaire], this implies $\mu^\perp = \mathrm{A}(G)$, i.e. $\alpha \cdot \mu = 0$ for all $\alpha \in \mathrm{A}(G)$.

(ii) If $s \notin \operatorname{supp} \alpha$, there exists a neighbourhood $s \in U \subset G$ such that $\alpha|_U = 0$. Let $\beta \in \mathrm{A}(G)$, $\operatorname{supp} \beta \subset U$, such that $\beta(s) \neq 0$. Then $\alpha \cdot \beta = 0$, so

$$\beta \cdot (\alpha \cdot \mu) = (\alpha \cdot \beta) \cdot \mu = 0 \ ;$$

but $\beta(s) \neq 0$, so $s \notin \operatorname{supp} \alpha \cdot \mu$. Now let $s \in \operatorname{supp} \alpha \cdot \mu$. Further, let $\beta \in \mathrm{A}(G)$, so that $\beta \cdot \mu = 0$. In particular,

$$\beta \cdot (\alpha \cdot \mu) = \alpha \cdot (\beta \cdot \mu) = 0,$$

so we deduce $\beta(s) = 0$. Thus $s \in \operatorname{supp} \mu$, and the assertion follows.

(iii) Let $\chi \in \mathcal{K}\mathrm{A}(G)$ such that $\chi|_{\operatorname{supp} \alpha} = 1$ and $\operatorname{supp} \chi \subset G \setminus \operatorname{supp}_E \mu$. Obviously, $\alpha \cdot \mu = (\alpha \cdot \chi) \cdot \mu$. Since

$$\emptyset = \operatorname{supp} \chi \cap \operatorname{supp} \mu \supset \operatorname{supp} \chi \cdot \mu,$$

we deduce $\varphi \cdot \chi \cdot \mu = 0$ for all $\varphi \in \mathrm{A}(G)$ by (i), in particular $\alpha \cdot \mu = 0$.

Now let $C \subset G$ be a closed set satisfying the hypotheses. Let $g \in G \setminus C$, and choose $\alpha \in \mathcal{K}\mathrm{A}(G)$, $\operatorname{supp} \alpha \subset G \setminus C$, such that $\alpha(g) \neq 0$. Now $\alpha \cdot \mu = 0$, and hence $g \notin \operatorname{supp} \mu$.

(iv) Let $\chi \in \mathcal{K}\mathrm{A}(G)$ such that $\chi|_C = 1$ for some compact neighbourhood $C \subset G$ of $\operatorname{supp} \widetilde{\mu}$. We have $\chi \cdot \widetilde{\mu} = \widetilde{\mu}$, and since F is $\mathcal{J}$-invariant, we deduce $\chi \cdot \mu = \mu$. Because $\operatorname{supp} \alpha \cap \operatorname{supp}_E \mu = \emptyset$, we deduce by (i) and (ii): $0 = \chi \cdot \alpha \cdot \mu = \alpha \cdot \mu$. ∎

Proposition 3.1.9. *Let δ be a coaction of G on the C*-algebra A. Then δ is non-degenerate if and only if $A = \overline{\mathrm{A}(G) \cdot A}$, i.e., if and only if A is generated as a Banach module by its compactly supported elements.*

Proof. This is merely another way of stating [Kat85, theorem 5 (i),(iv)]. ∎

3.2. *Generated submodule* C**-algebras.* In this subsection, we collect some basic properties of submodule C*-algebras. These C*-algebras give rise to Toeplitz C*-algebras if they are generated by a single projection, as we shall see in the next subsection.

Definition 3.2.1. Let (A, δ, G) be a non-degenerate coaction which is a submodule of a normed $\mathrm{A}(G)$-module E. If $\mathcal{E} \subset E$ is such that $\mathrm{A}(G) \cdot \mathcal{E} \subset A$, define

$$\mathrm{C}^*_{\mathcal{E}}(\delta) := \mathrm{C}^* \prec \alpha \cdot a \,|\, \alpha \in \mathrm{A}(G), a \in \mathcal{E} \succ \subset A,$$

the *submodule* C**-algebra generated by* $\mathcal{E}$. For the special case $\delta = \delta_G|_A$, we write $\mathrm{C}^*_{\mathcal{E}}(G) := \mathrm{C}^*_{\mathcal{E}}(\delta)$. In particular, $\mathrm{C}^*_{\mathbf{L}^1(G)}(G) = \mathrm{C}^*_{\#}(G)$ and $\mathrm{C}^*_{\mathrm{W}^*(G)}(G) = \bar{\mathcal{K}}\mathrm{W}^*(G)$.

Proposition 3.2.2. *Let (A, δ, G) be a non-degenerate coaction and $\mathcal{E}$ such that $\mathrm{A}(G) \cdot \mathcal{E} \subset A$. Then $(\mathrm{C}^*_{\mathcal{E}}(\delta), \delta|_{\mathrm{C}^*_{\mathcal{E}}(\delta)}, G)$ is a non-degenerate coaction in each of the following cases:*

(i) *$\mathrm{C}^*_{\mathcal{E}}(\delta)$ acts non-degenerately on the same Hilbert space $\mathcal{H}$ as A does.*
(ii) *δ is the restriction of a W*-coaction.*

Proof. Let $\varphi, \psi \in \mathcal{K}\mathrm{A}(G)$, $a \in \mathcal{E}$, $b \in \mathrm{C}^*_{\#}(G)$. By [Kat85, lemma 3], we have

$$\delta(\varphi \cdot a)(1 \otimes \psi^{\vee \#} b) = \int_G (g * \psi) \cdot \varphi \cdot a \otimes g^{\#} b \, dg \in \mathrm{C}^*_{\mathcal{E}}(\delta) \otimes \mathrm{C}^*_{\#}(G),$$

since the integrand is contained in $\mathcal{K}(G, \mathrm{C}^*_{\mathcal{E}}(G) \otimes \mathrm{C}^*_{\#}(G))$. (We may assume $a \in A$ since φ has compact support.) In particular, $\delta(\mathrm{C}^*_{\mathcal{E}}(\delta))$ is contained in the closure of $\mathrm{C}^*_{\mathcal{E}}(\delta) \otimes \mathrm{C}^*_{\#}(G)$

in the ultraweak topology. If condition (i) is satisfied, $\mathrm{M}(\mathrm{C}^*_{\mathcal{E}}(\delta)\otimes\mathrm{C}^*_{\#}(G))$ is the idealizer of $\mathrm{C}^*_{\mathcal{E}}(\delta)\otimes\mathrm{C}^*_{\#}(G)$ in its ultraweak closure as operators on $\mathcal{H}\otimes\mathbf{L}^2(G)$ by [Bus68, 3.9 theorem] and the von Neumann density theorem, so in this case

$$\delta(\mathrm{C}^*_{\mathcal{E}}(\delta))\subset\overleftarrow{\mathrm{M}}(\mathrm{C}^*_{\mathcal{E}}(\delta),\mathrm{C}^*_{\#}(G)).$$

If condition (ii) is satisfied, this statement follows as in [Qui92, proof of lemma 2.2, (1)].

So $\mathrm{C}^*_{\mathcal{E}}(\delta)$ is a closed $\mathrm{B}_{\#}(G)$-submodule of A by proposition 3.1.4 (ii). Moreover, $\mathrm{C}^*_{\mathcal{E}}(\delta)=\overline{\mathrm{A}(G)\cdot\mathrm{C}^*_{\mathcal{E}}(\delta)}$ since $a\in\mathrm{C}^*_{\mathcal{E}}(G)$ can be approximated by elements of $\mathcal{K}\mathrm{C}^*_{\mathcal{E}}(G)$. So δ is non-degenerate if it is a coaction.

If $\varphi,\psi\in\mathcal{K}\mathrm{A}(G)$, $a\in\mathrm{C}^*_{\mathcal{E}}(\delta)$ and $b\in\mathrm{C}^*_{\#}(G)$, we see (cf. [Qui92, lemma 2.3] or [Qui94, lemma 1.3]) that

$$(\varphi\cdot a)\otimes\psi^{\#}b=\int_G\delta((\varphi*g)\cdot a)(1\otimes g^{\#}b)\,dg\in\overline{\delta(\mathrm{C}^*_{\mathcal{E}}(\delta))(\mathbf{C}\otimes\mathrm{C}^*_{\#}(G))},$$

so that

$$\overline{\delta(\mathrm{C}^*_{\mathcal{E}}(\delta))(\mathbf{C}\otimes\mathrm{C}^*_{\#}(G))}=\mathrm{C}^*_{\mathcal{E}}(\delta)\otimes\mathrm{C}^*_{\#}(G).$$

Hence δ defines an injective non-degenerate $*$-morphism

$$\delta:\mathrm{C}^*_{\mathcal{E}}(\delta)\to\mathrm{M}(\mathrm{C}^*_{\mathcal{E}}(\delta)\otimes\mathrm{C}^*_{\#}(G)).$$

Finally, δ satisfies the coaction identity by [LPRS87, lemma 4.4], so $(\mathrm{C}^*_{\mathcal{E}}(\delta),\delta,G)$ is indeed a non-degenerate coaction. ∎

Corollary 3.2.3. *Let $\mathcal{E}\subset\mathrm{W}^*(G)$. Then $(\mathrm{C}^*_{\mathcal{E}}(\delta_G),\delta_G|_{\mathrm{C}^*_{\mathcal{E}}(\delta_G)},G)$ is a non-degenerate coaction.*

Proof. By [Kat85, lemma 3 and remark], we may choose $A=\bar{\mathcal{K}}\mathrm{W}^*(G)$ in proposition 3.2.2. Obviously, condition (ii) is satisfied, so the assertion follows. ∎

The following proposition gives rise to a useful criterion for the irreducibility of Toeplitz C*-algebras, as we shall see in the next subsection.

Proposition 3.2.4. *Let $\mathcal{E}\subset\mathrm{W}^*(G)$.*

- (i) *The C*-algebra $\mathrm{C}^*_{\#}(G)$ is an ideal of $\bar{\mathcal{K}}\mathrm{W}^*(G)$, i.e. $\mathrm{C}^*_{\#}(G)\cdot\bar{\mathcal{K}}\mathrm{W}^*(G)\subset\mathrm{C}^*_{\#}(G)$. That is, we have the inclusion $\bar{\mathcal{K}}\mathrm{W}^*(G)\subset\mathrm{M}(\mathrm{C}^*_{\#}(G))$.*
- (ii) *If there is $\mu\in\mathcal{E}$ such that $\operatorname{supp}\mu=G$ and the set M of points where μ is not locally contained in $\mathrm{A}(G)$ is a zero set, then $\mathrm{C}^*_{\#}(G)\subset\mathrm{C}^*_{\mathcal{E}}(G)$ so that $\mathrm{C}^*_{\#}(G)$ is a closed $*$-ideal of $\mathrm{C}^*_{\mathcal{E}}(G)$.*

Here, by 'μ is locally contained in $\mathrm{A}(G)$ at g' we mean that for some neighbourhood U of g, and any $\alpha\in\mathcal{K}\mathrm{A}(G)$ such that $\operatorname{supp}\alpha\subset U$, there exists $\varphi\in\mathcal{K}\mathrm{A}(G)$ such that

$$\langle\alpha' : \alpha\cdot\mu\rangle=\int_G\alpha'(g)\varphi(g)\,dg\quad\textit{for all}\quad\alpha'\in\mathrm{A}(G).$$

Proof. (i) Clearly $\mathcal{K}\mathrm{W}^*(G)$ is dense in $\bar{\mathcal{K}}\mathrm{W}^*(G)$ and $\mathcal{K}\mathrm{A}(G)$ is dense in $\mathrm{A}(G)$. So, consider $\mu\in\mathrm{W}^*(G)$ with $\operatorname{supp}\mu$ compact and $\alpha\in\mathcal{K}\mathrm{A}(G)$. By [Eym64, (3.17) proposition 3°],

$$\mu^{\#}\alpha^{\#}=(\mu*\alpha)^{\#}\quad\text{ is convolution by }\mu*\alpha\in\mathrm{A}(G).$$

Since $\mu * \alpha$ has compact support, it is contained in $\mathcal{K}\mathrm{A}(G) \subset \mathbf{L}^1(G)$, so we have $\mu^{\#}\alpha^{\#} \in \mathrm{C}^*_{\#}(G)$.

As to the second statement, $\mathrm{C}^*_{\#}(G)$ acts faithfully and non-degenerately on $\mathbf{L}^2(G)$, so $\mathrm{M}(\mathrm{C}^*_{\#}(G))$ is faithfully represented as the idealizer of $\mathrm{C}^*_{\#}(G)$ in its ultraweak closure $\mathrm{W}^*(G)$ (von Neumann density theorem and [Bus68, 3.9 theorem]). What is more, we clearly have $\bar{\mathcal{K}}\mathrm{W}^*(G) \subset \mathrm{W}^*(G)$. Hence the assertion.

(ii) Let $\alpha \in \mathcal{K}\mathrm{A}(G)$, $\operatorname{supp}\alpha \subset G \setminus M$. Let $K \subset G \setminus M$ be a compact neighbourhood of $\operatorname{supp}\alpha$, and $\chi \in \mathcal{K}\mathrm{A}(G)$, $\chi|_K = 1$, $\operatorname{supp}\chi \subset G \setminus M$. Then

$$f := \chi \cdot \mu \in \mathcal{K}\mathrm{A}(G), \quad \text{and} \quad \inf |f(K)| > 0,$$

so, since $\mathrm{A}(G)$ is a Shilov-regular Banach algebra, there exists $\varphi \in \mathcal{K}\mathrm{A}(G)$ such that $(\varphi \cdot f)|_K = 1$ (cf. [Eym64, proof of (4.4) proposition]). Then

$$\alpha \cdot \varphi \cdot \mu = \alpha \cdot \varphi \cdot \chi \cdot \mu = \alpha \cdot \varphi \cdot f = \alpha,$$

i.e. $\alpha \in \mathrm{C}^*_{\mathcal{E}}(G)$. Since M is a zero set, $\{\alpha \in \mathcal{K}\mathrm{A}(G) \mid \operatorname{supp}\alpha \subset G \setminus M\}$ is dense in $\mathbf{L}^1(G)$ and hence in $\mathrm{C}^*_{\#}(G)$. The assertion follows. ∎

The above considerations allow us to define the following refinement of the notion of *singular support* of a distribution. This is a local object better adapted to the C*-context than the former one, defined in the smooth category.

Definition 3.2.5. For $\mu \in \mathrm{W}^*(G)$, let the *singular set* of μ be

$$\operatorname{sing}\mu := \operatorname{supp}_{\mathrm{W}^*(G)/\mathrm{C}^*_{\#}(G)}[\mu] = \{g \in G \,\big|\, \alpha \cdot \mu \in \mathrm{C}^*_{\#}(G) \Rightarrow \alpha(g) = 0 \text{ for all } \alpha \in \mathrm{A}(G)\}.$$

Its applications will become clear in the following sections.

3.3. *Cocrossed products and Toeplitz* C*-*algebras*

Definition 3.3.1. Let $p \in \mathrm{W}^*(G)$ be an orthogonal projection. Define the *Toeplitz operator of symbol f*,

$$\mathrm{T}_p(f) := pfp : \mathbf{L}^2(G) \to \mathbf{L}^2(G) \quad \text{for all} \quad f \in \mathcal{C}^0(G).$$

We denote by

$$\mathcal{T}_p(G) := \mathrm{C}^* \prec \mathrm{T}_p(f) \mid f \in \mathcal{C}^0(G) \succ$$

the *Toeplitz* C*-*algebra* defined by p. We fix p for this subsection.

Typically, Toeplitz operators are hard to describe since they mix convolution and multiplication. However, convolution and multiplication can be separated via the description of Toeplitz C*-algebras as corners of cocrossed product C*-algebras which we achieve in the sequel.

Definition 3.3.2. Given a coaction (A, δ, G), a *covariant pair of representations* (π, μ) in B is given by non-degenerate $*$-morphisms

$$\pi : A \to \mathrm{M}(B) \quad \text{and} \quad \mu : \mathcal{C}^0(G) \to \mathrm{M}(B)$$

such that $\overline{\pi \otimes \mathrm{id}} \circ \delta(a) = \mathrm{Ad}(u)(\pi(a) \otimes 1)$ for all $a \in A$. Here $u = \overline{\mu \otimes \mathrm{id}}(W_G)$ denotes the *corepresentation* u of G associated with μ, cf. [LPRS87, section 3], [Dei00, section 1]. The closed linear subspace

$$\mathrm{C}^*(\pi, \mu) := \overline{\pi(A)\mu(\mathcal{C}^0(G))} \subset \mathrm{M}(B)$$

is a C*-algebra. A covariant pair of representations in B clearly induces a covariant pair of representations in $\mathrm{C}^*(\pi,\mu)$, since due to the non-degeneracy of π and μ, the inclusion of $\mathrm{C}^*(\pi,\mu)\subset B\subset \mathrm{M}(B)$ is non-degenerate.

In particular, since $(\delta, 1\otimes M)$ is covariant by the coaction identity, the *cocrossed product*

$$A\otimes_\delta \mathrm{C}^*_\#(G) := \overline{\delta(A)(\mathbf{C}\otimes\mathcal{C}^0(G))}\subset \mathrm{M}(A\otimes\mathcal{LC}(\mathbf{L}^2(G)))$$

is a C*-subalgebra. Here M is the action of $\mathcal{C}^0(G)$ on $\mathbf{L}^2(G)$ by multiplication operators. The cocrossed product satisfies the following universal property (op. cit.): whenever (π,μ) is a covariant pair of representations in B, there exists a non-degenerate $*$-morphism $\varphi : A\otimes_\delta \mathrm{C}^*_\#(G)\to \mathrm{M}(B)$ (unique by construction of $A\otimes_\delta \mathrm{C}^*_\#(G)$) such that

$$(\overline{\varphi}\circ\delta, \overline{\varphi}\circ(1\otimes M)) = (\pi,\mu).$$

We write $\pi\otimes_\delta\mu = \varphi$.

Similarly, we define W*-cocrossed products, cf. [NT79].

Remark 3.3.3. Let $\mathcal{E}\subset \mathrm{W}^*(G)$. The non-degenerate $*$-morphism

$$\mathrm{id}\otimes_\delta M : \mathrm{C}^*_{\mathcal{E}}(G)\otimes_\delta \mathcal{C}^0(G)\to\mathcal{L}(\mathbf{L}^2(G)) : \delta_G(a)(1\otimes f)\mapsto af$$

given by the universal property of the cocrossed product is *injective* since it is the restriction of the $*$-isomorphism

$$\mathrm{W}^*(G)\bar{\otimes}_\delta\mathbf{L}^\infty(G)\cong\mathcal{L}(\mathbf{L}^2(G))$$

given by the Takesaki duality theorem (cf. [NT79]) applied to the trivial action of G on the von Neumann algebra $\mathbf{C}$. (It easy to check that the dual coaction of this action is δ_G.) In other words, $\mathrm{C}^*(\mathrm{id}, M)$ is a cocrossed product for the coaction $(\mathrm{C}^*_{\mathcal{E}}(G),\delta_G,G)$.

We can now state our main result for this subsection.

Theorem 3.3.4. *Considering* $\mathrm{C}^*_p(G)\otimes_\delta\mathcal{C}^0(G)\subset\mathcal{L}(\mathbf{L}^2(G))$, *we have*

$$\mathcal{T}_p(G) = p(\mathrm{C}^*_p(G)\otimes_\delta\mathcal{C}^0(G))p.$$

The proof of this theorem requires several lemmata.

Lemma 3.3.5. *The coproduct* Δ *defined by* $\Delta\beta(s,t) := \beta(st)$ *is a norm* 1 *linear map*

$$\Delta : \mathrm{B}_\#(G)\to\mathrm{B}_\#(G\times G)\quad \textit{satisfying}\quad (\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta.$$

Proof. Consider multiplication $m : \mathrm{C}^*_\#(G\times G) = \mathrm{C}^*_\#(G)\otimes\mathrm{C}^*_\#(G)\to\mathrm{C}^*_\#(G)$ which has norm 1. (Recall that $\otimes$ is the spatial tensor product!) Define $\Delta = m'$. Then, obviously,

$$\Delta : \mathrm{B}_\#(G)\to\mathrm{B}_\#(G\times G)\quad\text{has norm 1}$$

and satisfies coassociativity. Clearly, for any $\beta\in\mathrm{B}_\#(G)$, $f,g\in\mathbf{L}^1(G)$,

$$\langle\Delta\beta\, :\, f^\#\otimes g^\#\rangle = \int_{G\times G}\beta(st)f(s)g(t)\,ds\,dt,$$

so $\Delta\beta(s,t) = \beta(st)$ for a.e. (s,t), and, by continuity, everywhere. ∎

Remark 3.3.6. Note that the same procedure applied to the *universal* group C*-algebra $\mathrm{C}^*(G)$ gives an extension of Δ to $\mathrm{B}(G) = \mathrm{C}^*(G)'$ whose image, however, will in general not be contained in $\mathrm{B}(G\times G)$ for non-amenable G.

COROLLARY 3.3.7. *For any* $\eta \in \mathbf{L}^2(G)$, $\xi \in \mathcal{K}\mathrm{A}(G)$, $\varphi \in \mathrm{B}(G)$ *and* $a, b \in \mathrm{W}^*(G)$,

$$(\eta \mid a\varphi b\xi) = \int_G \overline{\eta(t_0)} \langle (t_1, t_2) \mapsto \varphi(t_1^{-1}t_0)\xi(t_2^{-1}t_1^{-1}t_0) \,:\, a \otimes b \rangle \, dt_0$$

if either φ *or* a *has compact support.*

Proof. Note that the action of $\mathrm{W}^*(G)$ leaves $\mathrm{A}(G) \cap \mathbf{L}^2(G)$ invariant by [Eym64, (3.17) proposition 3°]. We may assume hat φ has compact support (otherwise chose $\chi \in \mathcal{K}\mathrm{A}(G)$ with $\chi \cdot a = a$). By lemma 3.3.5, the function

$$(t_1, t_2) \mapsto \varphi(t_1^{-1}t_0)\xi(t_2^{-1}t_1^{-1}t_0)$$

lies in $\mathrm{B}_\#(G \times G)$ for any t_0. Since it has compact support, it lies in $\mathcal{K}\mathrm{A}(G \times G)$ (cf. [Eym64]). So we may apply [Eym64, (3.17) proposition 2°] to prove the equation along the lines of the usual Fubini theorem for distributions. ∎

The following lemma is a sharpened version of [HU98, lemma 5.1], valid for any locally compact group.

LEMMA 3.3.8. *Let* $n \in \mathbf{N}$ *and* $a_1, \ldots, a_n \in \mathrm{W}^*(G)$. *Further, choose sequences* $\mathcal{F}, \mathcal{F}'$ *of finite subsets of* $\mathrm{B}(G)^{n+1}$ *resp.* $\mathrm{B}(G)^{2n}$. *Assume further that*

(i) *The* a_j *have compact supports* or
(ii) *for all* $k \in \mathbf{N}$, $(\varphi_0, \ldots, \varphi_n) \in \mathcal{F}(k)$ *and* $(\alpha_1, \ldots, \alpha_n, \psi_1, \ldots, \psi_n) \in \mathcal{F}'(k)$, $\varphi_1, \ldots, \varphi_n$ *have compact support, and for all* $1 \le j \le n-1$, *either* α_j *or* ψ_j *have compact support.*

If f, f' *are such that*

$$(t_0, \ldots, t_n) \mapsto f(t_0, t_1^{-1}, \ldots, t_n^{-1} \ldots t_1^{-1}t_0)$$

lies in $\mathrm{A}(G^{n+1})$, *further*

$$f = \lim_k \sum_{(\varphi_0, \ldots, \varphi_n) \in \mathcal{F}(k)} \varphi_0 \otimes \cdots \otimes \varphi_n$$

in the norm topology on $\mathrm{B}(G^{n+1})$ *and*

$$f' = \lim_k \sum_{(\alpha_1, \ldots, \alpha_n, \psi_1, \ldots, \psi_n) \in \mathcal{F}'(k)} \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \psi_1 \otimes \cdots \otimes \psi_n$$

in the norm topology on $\mathrm{B}(G^{2n})$, *such that*

$$f(t_0, \ldots, t_n^{-1} \cdots t_1^{-1}t_0) = f'(t_1, \ldots, t_n, t_1^{-1}t_0, \ldots, t_n^{-1} \cdots t_1^{-1}t_0)$$

for all $(t_0, \ldots, t_n) \in G^{n+1}$, *then we have the existence in norm and equality of the limits*

$$\lim_k \sum_{(\varphi_0, \ldots, \varphi_n) \in \mathcal{F}(k)} \varphi_0 a_1 \cdots \varphi_{n-1} a_n \varphi_n = \lim_k \sum_{(\alpha_1, \ldots, \psi_n) \in \mathcal{F}'(k)} (\alpha_1 \cdot a_1)\psi_1 \cdots (\alpha_n \cdot a_n)\psi_n.$$

Proof. Because of the compactness assumptions, we may assume w.l.o.g. that the functions $\varphi_j, \alpha_j, \psi_j$ all have compact support, in particular, that the sequences above converge in the Fourier algebra (which carries the induced norm). Applying corollary 3.3.7, it is easy to see that

$$(\eta \mid \varphi_0 a_1 \varphi_1 \cdots a_n \varphi_n \xi) = \int_G \overline{\eta(t_0)} \langle (t_1, \ldots, t_n) \mapsto \varphi_0(t_0) \cdots (\varphi_n \xi)(t_n^{-1} \cdots t_1^{-1}t_0) \,:\, \otimes a \rangle dt_0$$

for all $\eta \in \mathbf{L}^2(G)$, $\xi \in \mathcal{K}\mathrm{A}(G)$, $\varphi_0, \dots, \varphi_n \in \mathcal{K}\mathrm{A}(G)$, where we use the shorthand

$$\otimes a = a_1 \otimes \cdots \otimes a_n.$$

From the two convergence assumptions and the continuity of the $\mathrm{B}(G)$-module structure on $\mathrm{W}^*(G)$, the action of $\mathrm{B}(G)$ by multiplication operators and multiplication in $\mathcal{L}(\mathbf{L}^2(G))$, the limits (LHS) and (RHS) exist.

The indicated transformations on the variables $t_0, \dots, t_n$ define continuous maps on the level of the reduced Fourier–Stieltjes algebras. Again applying compactness and the assumption on f, the sequences and their limits live in the Fourier algebra.

So we have

$$\begin{aligned}(\mathrm{LHS}) &= \int_G \overline{\eta(t_0)} \langle (t_1, \dots, t_n) \mapsto f(t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0) \xi(t_n^{-1} \cdots t_1^{-1} t_0) \,:\, \otimes a \rangle \, dt_0 \\ &= \int_G \overline{\eta(t_0)} \langle (t_1, \dots, t_n) \mapsto f'(t_1, \dots, t_n^{-1} \cdots t_1^{-1} t_0) \xi(t_n^{-1} \cdots t_1^{-1} t_0) \,:\, \otimes a \rangle \, dt_0 \\ &= (\mathrm{RHS}),\end{aligned}$$

proving the assertion. ∎

Lemma 3.3.9.

(i) *Let $n \geq 1$ and $(\alpha_j)_{1 \leq j \leq n}, (\psi_j)_{1 \leq j \leq n} \subset \mathcal{K}\mathrm{A}(G)$. Set $f' = \alpha_1 \otimes \cdots \otimes \psi_n$. There exists a sequence $\mathcal{F}$ of finite subsets of $\mathcal{K}\mathrm{A}(G)^{n+1}$ such that*

$$f = \lim_k \sum_{(\varphi_0, \dots, \varphi_n) \in \mathcal{F}(k)} \varphi_0 \otimes \cdots \otimes \varphi_n \quad \text{exists in } \mathrm{A}(G^{n+1})$$

and

$$f(t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0) = f'(t_1, \dots, t_n, t_1^{-1} t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0)$$

for all $(t_0, \dots, t_n) \in G^{n+1}$. Furthermore,

$$f'' : (t_0, \dots, t_n) \mapsto f(t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0)$$

lies in $\mathrm{A}(G^{n+1})$.

(ii) *Let $n \geq 1$ and $\varphi_0, \varphi_n, (\varphi_j^i)_{1 \leq j \leq n-1, i=1,2} \subset \mathcal{K}\mathrm{A}(G)$. Set*

$$f = \varphi_0 \otimes \varphi_1^1 \cdot \varphi_1^2 \otimes \cdots \otimes \varphi_{n-1}^1 \cdot \varphi_{n-1}^2 \otimes \varphi_n.$$

There exists a sequence $\mathcal{F}'$ of finite subsets of $\mathcal{K}\mathrm{A}(G)^{2n}$ such that

$$f' = \lim_k \sum_{(\alpha_1, \dots, \alpha_n, \psi_1, \dots, \psi_n) \in \mathcal{F}'(k)} \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \psi_1 \otimes \cdots \otimes \psi_n \text{ exists in } \mathrm{A}(G^{2n})$$

and

$$f(t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0) = f'(t_1, \dots, t_n, t_1^{-1} t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0)$$

for all $(t_0, \dots, t_n) \in G^{n+1}$. Furthermore,

$$f'' : (t_0, \dots, t_n) \mapsto f(t_0, \dots, t_n^{-1} \cdots t_1^{-1} t_0)$$

lies in $\mathrm{A}(G^{n+1})$.

Proof. (i) By lemma 3.3.5, the function f defined by

$$f(t_0,\ldots,t_n):=\prod_{j=1}^{n}\alpha_j(t_{j-1}t_j^{-1})\psi_j(t_j)$$

lies in $\mathrm{B}_{\#}(G^{n+1})$. Since it has compact support, it lies in $\mathcal{K}\mathrm{A}(G^{n+1})$. So there exists a sequence $\mathcal{F}$ of finite subsets of $\mathcal{K}\mathrm{A}(G)^{n+1}$ such that

$$f=\lim\nolimits_k\sum_{(\varphi_0,\ldots,\varphi_n)\in\mathcal{F}(k)}\varphi_0\otimes\cdots\otimes\varphi_n\quad\text{ exists in }\mathrm{A}(G^{n+1}).$$

Clearly, f satisfies the second equation. Since f' has compact support, so does f''. Hence $f''\in\mathrm{A}(G^{n+1})$ by lemma 3.3.5.

(ii) Again applying lemma 3.3.5, the function f' defined by

$$f'(s_1,\ldots,t_n):=\varphi_0(s_1t_1)\Bigl[\prod_{j=1}^{n-1}\varphi(t_j)\varphi_j^2(s_{j+1}t_{j+1})\Bigr]\varphi_n(t_n)$$

lies in $\mathcal{K}\mathrm{A}(G^{2n})$, so there exists $\mathcal{F}'$ as required. For the remaining assertion, we proceed as in (i). ■

Proof of theorem 3.3.4. Since $\mathcal{K}\mathrm{A}(G)\subset\mathcal{C}^0(G)$ is dense and $A{\cdot}A=A$ for any C*-algebra A, the set of products

$$p\varphi_0p\varphi_1^1\varphi_1^2\cdots\varphi_{n-1}^1\varphi_{n-1}^2p\varphi_np\quad\text{with }n\geq 2,\varphi_0,\varphi_n,(\varphi_j^i)\subset\mathcal{K}\mathrm{A}(G)$$

is total in $\mathcal{T}_p(G)$. By lemma 3.3.9 and lemma 3.3.8, $\varphi_0p\varphi_1^1\varphi_1^2\cdots\varphi_np\in\mathrm{C}^*_p(G)\otimes_\delta\mathcal{C}^0(G)$. Hence, we have the first inclusion. Also, the set of products

$$(\alpha_1\cdot p)\varphi_1\cdots(\alpha_n\cdot p)\varphi_n\quad\text{with }n\geq 1,(\alpha_j),(\varphi_j)\subset\mathcal{K}\mathrm{A}(G)$$

is total in $\mathrm{C}^*_p(G)\otimes_\delta\mathcal{C}^0(G)$, since products

$$(\alpha_1\cdot p)\cdots(\alpha_n\cdot p)\quad\text{with }(\alpha_j)\subset\mathcal{K}\mathrm{A}(G)$$

can be approximated in this way by lemma 3.3.9 and lemma 3.3.8. These two lemmata also show that $p(\alpha_1\cdot p)\varphi_1\cdots(\alpha_n\cdot p)\varphi_np\in\mathcal{T}_p(G)$, so we are done. ■

Remark 3.3.10. In the proof of theorem 3.3.4, we have also shown that

$$\mathrm{C}^*_p(G)\otimes_\delta\mathcal{C}^0(G)=\mathrm{C}^*\prec\delta(\alpha\cdot p)(1\otimes f)\,|\,\alpha\in\mathrm{A}(G),f\in\mathcal{C}^0(G)\succ\ .$$

From the theorem, we deduce the following criterion for the irreducibility of $\mathcal{T}_p(G)$.

Proposition 3.3.11. *Let p be such that $\mathrm{C}^*_{\#}(G)\subset\mathrm{C}^*_p(G)$. Then*

(i) *$\mathcal{LC}(\mathbf{L}^2(G))\subset\mathrm{C}^*_p(G)\otimes_\delta\mathcal{C}^0(G)$, in particular, the latter acts irreducibly on $\mathbf{L}^2(G)$.*
(ii) *$\mathcal{LC}(p\,\mathbf{L}^2(G))\subset\mathcal{T}_p(G)$, so the latter acts irreducibly on $p\,\mathbf{L}^2(G)$. In particular, the multiplier algebra $\mathrm{M}(\mathcal{T}_p(G))$ is the idealizer in $p\mathcal{L}(\mathbf{L}^2(G))p$ of $\mathcal{T}_p(G)$ acting on $\mathbf{L}^2(G)$, that is, the largest C*-subalgebra containing $\mathcal{T}_p(G)$ as a closed ideal.*

The condition is satisfied if p has full support and is a.e. locally contained in $\mathrm{A}(G)$, in particular if p has full support, G is a Lie group and sing supp p *has zero measure.*

Proof. (i) $\mathcal{LC}(\mathbf{L}^2(G))=\mathrm{C}^*_{\#}(G)\otimes_\delta\mathcal{C}^0(G)$ by Takesaki's duality theorem. So the assertion follows from Schur's lemma.

(ii) We have $\mathcal{LC}(p\,\mathbf{L}^2(G)) = p\mathcal{LC}(\mathbf{L}^2(G))p$, so we may apply theorem 3.3.4. The statement about the multiplier algebra follows from [Bus68, 3.9 theorem].

That the condition is satisfied under the given assumption follows from proposition 3.2.4, (ii). For the special case of a Lie group G, $\mathcal{D}(G) \subset \mathrm{A}(G)$ is dense and the induced topology is weaker than the Schwartz topology by [Eym64, (3.26) proposition], so $\mathrm{W}^*(G) \subset \mathcal{D}'(G)$. That $\operatorname{sing\,supp} p$ has zero measure means that p is a.e. locally contained in $\mathcal{D}(G) \subset \mathrm{A}(G)$, whence the assertion. ∎

We also note the following proposition which will prove useful later on.

PROPOSITION 3.3.12. *If $a, b \in \mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G)$, then $apb \in \mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G)$.*

Proof. It suffices to prove that

$$(\alpha_1 \cdot p)\psi_1 p(\alpha_2 \cdot p)\psi_2 \in \mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G) \text{ for } \alpha_1, \alpha_2, \psi_1, \psi_2 \in \mathcal{K}\mathrm{A}(G).$$

So, by lemmata 3.3.9 and 3.3.8, it suffices to prove that $\varphi_0 p \varphi_1 p \varphi_2 p \varphi_3$ lies in the cocrossed product for any $\varphi_0, \ldots, \varphi_3 \in \mathcal{K}\mathrm{A}(G)$. But this follows from the same lemmata. ∎

4. Behaviour of $\mathrm{W}^*(G)$ at ∞ and representations of Toeplitz C^*-algebras.

In this section, let G be any locally compact group and let $\bar{G} \sqsubset G$ be a closed subgroup.

4.1. *Restriction to subgroups.* Although the *universal* group von Neumann algebra of a locally compact group manifestly behaves cofunctorially under continuous group homomorphisms, the behaviour of the reduced group von Neumann algebra $\mathrm{W}^*(G)$ is far from clear, since the corresponding $\mathbf{L}^2$ spaces are not easily related. Also, it is known that positive definite functions living on subgroups $\bar{G}$ usually do not extend as positive definite functions, cf. [Eym64].

At the other extreme, for G a Lie group, smooth functions of compact support on $\bar{G}$ are easily seen to extend smoothly, with compact support contained in any tubular neighbourhood of the subgroup. Hence distributions whose (locally finite) order in directions transversal to $\bar{G}$ vanishes have preimages on $\bar{G}$ (in fact, these properties are equivalent). Since it is known that the finitely supported elements of $\mathrm{W}^*(G)$ have order 0 ([Eym64, (4.9) théorème]), it is therefore reasonable to suspect that a similar statement holds for $\mathrm{W}^*(G)$ in place of the space of distributions of transversal order zero.

Indeed, this fact is recorded in the literature. We state it as the following theorem, which is a combination of [Her73, theorem A and theorem 1] and [TT72, theorem 3].

THEOREM 4.1.1.

(i) *The restriction map $\mathrm{res}_{\bar{G}} : \mathrm{A}(G) \to \mathrm{A}(\bar{G}) : \alpha \mapsto \alpha|_{\bar{G}}$ is an extremal epimorphism of Banach spaces, i.e.*

$$\mathrm{A}(\bar{G}) \cong \mathrm{A}(G)/\ker \mathrm{res}_{\bar{G}}$$

as Banach spaces.

(ii) *The dual map of $\mathrm{res}_{\bar{G}}$ coincides with the extension by zero on the set of bounded measures on $\bar{G}$. It is an isometry $\mathrm{W}^*(\bar{G}) \to \mathrm{W}^*(G)$, whose image is precisely the set*

$$\mathrm{W}^*_{\bar{G}}(G) := \{\mu \in \mathrm{W}^*(G) \,\big|\, \operatorname{supp}\mu \subset \bar{G}\}\ ;$$

in particular, this set is an ultraweakly closed unital $*$-subalgebra of $\mathrm{W}^*(G)$ *isometrically isomorphic to* $\mathrm{W}^*(\bar{G})$.

In particular, for all $\mu \in \mathrm{W}^*_{\bar{G}}(G)$, *there exists a unique* $\mu_{\bar{G}} \in \mathrm{W}^*(\bar{G})$ *such that*

$$\langle \alpha|_{\bar{G}} : \mu_{\bar{G}} \rangle = \langle \alpha : \mu \rangle \quad \text{for all} \quad \alpha \in \mathrm{A}(\bar{G}), \tag{4.1.1}$$

and the $*$-algebra isomorphism $\mathrm{W}^*(\bar{G}) \cong \mathrm{W}^*_{\bar{G}}(G)$ *is a homeomorphism for the respective ultraweak topologies.*

Remark 4.1.2. The maps in the above theorem are both module maps and algebra morphisms. Indeed, $\mathrm{res}_{\bar{G}}$ is obviously multiplicative. From this we deduce $(\alpha \cdot \mu)_{\bar{G}} = \alpha|_{\bar{G}} \cdot \mu_{\bar{G}}$. On the other hand, for $h \in \bar{G}$, we have by [Eym64, (3.17) proposition 2°]

$$(\mu\alpha^\vee)(h) = \langle \alpha(h^{-1}\diamond) : \mu \rangle = \langle \alpha(h^{-1}\diamond)|_{\bar{G}} : \mu_{\bar{G}} \rangle = \mu_{\bar{G}}\, \alpha^\vee|_{\bar{G}}(h),$$

i.e. $(\mu\alpha)|_{\bar{G}} = \mu_{\bar{G}}\, \alpha|_{\bar{G}}$ since $\diamond^\vee$ is an automorphism of the Fourier algebras.

By [Eym64, (3.16)], $\diamond^\vee$ on $\mathrm{W}^*(G)$ is the dual map of $\diamond^\vee$ on $\mathrm{A}(G)$, so, since the latter commutes with $\mathrm{res}_{\bar{G}}$, the former commutes with $\mu \mapsto \mu_{\bar{G}}$. Hence, by [Eym64, (3.16) définition], we see that

$$\begin{aligned}\langle \alpha|_{\bar{G}} : (\mu\mu')_{\bar{G}} \rangle &= \langle \alpha : \mu\mu' \rangle = \langle \mu^\vee\alpha : \mu' \rangle \\ &= \langle (\mu^\vee\alpha)|_{\bar{G}} : \mu'_{\bar{G}} \rangle = \langle \mu_{\bar{G}}^\vee\, \alpha|_{\bar{G}} : \mu'_{\bar{G}} \rangle = \langle \alpha|_{\bar{G}} : \mu_{\bar{G}}\, \mu'_{\bar{G}} \rangle,\end{aligned}$$

thus $(\mu\mu')_{\bar{G}} = \mu_{\bar{G}}\, \mu'_{\bar{G}}$ since $\mathrm{res}_{\bar{G}}$ is surjective.

4.2. *Passage to subsequential limits.* It is a well-known fact that a sequence in a compact metric space converges if and only if all its convergent subsequences possess the same limit. In probability theory, this fact is exploited to prove convergence assertions, cf. Prohorov's lemma on the convergence of tight sequences of random variables. We shall proceed in the same fashion. For the convenience of the reader, we include the necessary propositions.

Notation 4.2.1. For a normed vector space E, let $\mathbf{B}(E)$ denote the closed unit ball. If Λ is a directed set, we write $\gamma \prec \Lambda$ if γ is a cofinal subset.

The following proposition is contained in [Mia99, proof of theorem 3.7].

Proposition 4.2.2. *If G is countable at infinity, then* $\mathbf{B}(\mathrm{W}^*(G))$ *is a compact metrizable space in the* $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$*-topology. In particular, it is sequentially compact.*

Proof. The unit ball $\mathbf{B}(\mathrm{W}^*(G))$ is a $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$-compact subset of $\mathrm{W}^*(G)$ by the Alaoğlu theorem. Since G is countable at infinity, $\mathbf{L}^2(G)$ contains a dense countably-dimensional subspace and is hence $\|\cdot\|_2$-separable. By [Eym64, (3.25) théorème] $\mathrm{A}(G)$ consists of the elements $\bar{\xi} * \eta^\vee$ with $\xi, \eta \in \mathbf{L}^2(G)$, and we have

$$\|\bar{\xi} * \eta^\vee\|_{\mathrm{A}(G)} \leq \|\xi\|_2 \cdot \|\eta\|_2$$

by [Eym64, (3.1) lemme]. We conclude that $\mathrm{A}(G)$ is $\|\cdot\|_{\mathrm{A}(G)}$-separable. Now, on norm bounded subsets of $\mathrm{W}^*(G)$, $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$ coincides with the σ-topology induced by taking any $\|\cdot\|_{\mathrm{A}(G)}$-dense subset of $\mathrm{A}(G)$ in place of all of $\mathrm{A}(G)$; hence $\mathbf{B}(\mathrm{W}^*(G))$ has countable neighbourhood bases and is thereby metrizable. This means that all accumulation points are limits of convergent subsequences. ∎

Corollary 4.2.3. *Let G be countable at infinity, and let $\mu, (\mu_\ell) \subset \mathbf{B}(\mathrm{W}^*(G))$. For (μ_ℓ) to converge to μ in $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$, it is necessary and sufficient that μ be the limit of all $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$-convergent subsequences.*

Proof. Necessity is obvious. For the proof of sufficiency, let all convergent subsequences of (μ_ℓ) have μ as limit. Assume that (μ_ℓ) does not converge to μ. Then there exist $\alpha \in \mathrm{A}(G)$, $\varepsilon > 0$ and a subsequence $\gamma \prec \mathbf{N}$ such that

$$|\langle \alpha \,:\, \mu_{\gamma(\ell)} - \mu \rangle| \geq \varepsilon > 0 \quad \text{for all} \;\; \ell \in \mathbf{N},$$

i.e., no subsequence of $(\mu_{\gamma(\ell)})$ converges to μ. But by proposition 4.2.2, $\mathbf{B}(\mathrm{W}^*(G))$ is sequentially compact in $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$, so there exist convergent subsequences of $(\mu_{\gamma(\ell)})$. By assumption, they must converge to μ, a contradiction. ■

4.3. *Changing the order of limits.* We shall have to exchange limit order several times in the sequel. Grothendieck's remarkable Double Limit Criterion gives fairly general conditions under which such operations are possible. We develop an exchange theorem well-adapted to our purposes.

Notation 4.3.1. If E is a normed vector space, let $\mathbf{S}(E)$ denote its unit sphere. If E is the dual of some normed $*$-algebra, let $\mathbf{S}(E)_+$ denote the positive part of the unit sphere.

Remark 4.3.2. Recall that the ultraweak topology on a von Neumann algebra M on some Hilbert space coincides with the $\sigma(M, M_*)$-topology.

Proposition 4.3.3. *Let $K \subset G$ be compact and denote*

$$\mathrm{A}_K(G) := \{\alpha \in \mathrm{A}(G) \,\big|\, \operatorname{supp} \alpha \subset K\}.$$

If $(\mu_j) \subset \mathbf{B}(\mathrm{W}^*(G))$ and $(\alpha_i) \subset \mathbf{S}(\mathrm{A}_K(G))_+$ are sequences, then

$$\lim_i \lim_j \langle \alpha_i \,:\, \mu_j \rangle = \lim_j \lim_i \langle \alpha_i \,:\, \mu_j \rangle$$

whenever the double limits exist.

Proof. The set $E := \mathbf{B}(\mathrm{W}^*(G))$, endowed with the $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$-topology, is a compact Hausdorff space. So, by [Gro52, corollaire 2 de théorème 2], it suffices to prove that $A := \mathbf{S}(\mathrm{A}_K(G))_+$ is a relatively compact subset of $\mathcal{C}(E)$, endowed with the topology of simple convergence.

To this end, first note that since the elements of A are linear, simple convergence on points of E is equivalent to point-wise convergence on all of $\mathrm{W}^*(G)$. Furthermore, since the set $L(\mathrm{W}^*(G), \mathbf{C})$ of linear forms is closed in $\mathbf{C}^{\mathrm{W}^*(G)}$, limits of nets in A will always be linear. Note also that due to Tychonov's theorem, $\mathbf{B}(\mathbf{C})^E$ is compact, so that A is relatively compact in $\mathbf{C}^E$.

So it remains to be shown that the closure of A in $\mathbf{C}^E$ lies in $\mathcal{C}(E)$. So let $(u_\alpha) \subset A$ be a net converging point-wise on E to $u \in \mathbf{B}(\mathbf{C})^E \cap L(\mathrm{W}^*(G), \mathbf{C})$. Since (u_α) is norm bounded in $\mathrm{A}(G)$, and the $\|\cdot\|_{\mathrm{A}(G)}$-norm coincides on $\mathrm{A}(G)$ with the dual norm of $\mathrm{W}^*(G)$ because $\|\cdot\|_{\mathrm{W}^*(G)}$ is the dual norm of $\mathrm{A}(G)$ by [Eym64, (3.10) théorème], we may apply the Banach–Steinhaus theorem [Tre67, corollary to theorem 33.1] to conclude that u is norm-continuous on $\mathrm{W}^*(G)$. In particular, $u|_{\mathrm{C}^*_\#(G)}$ is norm-continuous, so $u \in \mathrm{B}_\#(G) \subset \mathrm{B}(G)$.

Now, $G \subset E$ (i.e. the ball contains the point evaluations), so for any choice of finite sequences $(z_i) \subset \mathbf{C}$ and $(g_i) \subset G$, we have

$$\sum_{i,j=0}^{n} \bar{z}_i z_j u(g_i^{-1} g_j) = \lim_\alpha \sum_{i,j=0}^{n} \bar{z}_i z_j u_\alpha(g_i^{-1} g_j) \geq 0.$$

Thus u is of positive type. Also, $\|u\|_{\mathrm{B}(G)} = u(e) = \lim_\alpha u_\alpha(e) = 1$, hence $u \in \mathbf{S}(\mathrm{B}(G))_+$. Since, on bounded subsets, the $\sigma(\mathrm{B}(G), \mathrm{C}^*(G))$-topology is weaker than the topology of point-wise convergence, and $\mathrm{A}_K(G)$ is $\sigma(\mathrm{B}(G), \mathrm{C}^*(G))$-closed by [GL81, proof of theorem B_1], we conclude $u \in A = \mathbf{S}(\mathrm{A}_K(G))_+$. This proves the assertion. ■

Remark 4.3.4. Note that although for G countable at infinity, $\mathbf{B}(\mathrm{W}^*(G))$ is metrizable by proposition 4.2.2, even in this case the proof of relative compactness in the above lemma cannot rely on the Arzelà–Ascoli theorem, since this gives relative compactness in the compact-open topology (topology of uniform convergence), and *relative* compactness is not in general hereditary to weaker topologies, although this is the case for compactness.

Lemma 4.3.5. *The set of compactly supported functions of positive type $\mathcal{K}\mathrm{A}(G)_+$ is $\|\cdot\|_{\mathrm{A}(G)}$-total in $\mathrm{A}(G)$.*

Proof. In fact, for $\varphi, \psi \in \mathcal{K}(G)$, we have the following polarisation:

$$\begin{aligned} 4 \cdot \bar{\varphi} * \psi^\vee = {} & \overline{(\varphi + \psi)} * (\varphi + \psi)^\vee - \overline{(\varphi - \psi)} * (\varphi - \psi)^\vee \\ & + i\overline{(\varphi - i\psi)} * (\varphi - i\psi)^\vee - i\overline{(\varphi + i\psi)} * (\varphi + i\psi)^\vee \end{aligned}$$

where all summands are of positive type and have compact supports. Since, by [Eym64, (3.4) proposition], the set of the $\bar{\varphi} * \psi^\vee$, $\varphi, \psi \in \mathcal{K}(G)$, is total in $\mathrm{A}(G)$, the assertion follows. ■

Corollary 4.3.6. *Let $\mu \in \mathrm{W}^*(G)$ and $(\varphi_k), (\psi_l) \subset \mathrm{A}(G)$ such that the following conditions are satisfied:*

(i) *There exists $\bar{\mu} \in \mathrm{W}^*_{\bar{G}}(G)$ such that $\lim_k \varphi_k \cdot \mu = \bar{\mu}$ in $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$.*
(ii) *There exist $\bar{\varphi}, \bar{\psi} \in \mathrm{B}(\bar{G})$ such that $\lim_k \varphi_k|_{\bar{G}} = \bar{\varphi}$ and $\lim_l \psi_l|_{\bar{G}} = \bar{\psi}$ in the strict topology.*
(iii) *Either $(\varphi_k) \subset \mathbf{S}(\mathrm{A}(G))_+$ and (ψ_l) is bounded in $\mathrm{A}(G)$ or vice versa.*

Then, for any ultraweak limit $\bar{\varrho} \in \mathrm{W}^*_{\bar{G}}(G)$ of a subsequence of $(\psi_l \cdot \mu)$, we have

$$\bar{\psi} \cdot \bar{\mu}_{\bar{G}} = \bar{\varphi} \cdot \bar{\varrho}_{\bar{G}}.$$

Proof. W.l.o.g. we may assume $\psi_l \neq 0$ for some l. Let $\gamma \prec \mathbf{N}$ such that $\bar{\varrho} = \lim_l \psi_{\gamma(l)} \cdot \mu$. For all $\alpha \in \mathrm{A}(G)$ we have by assumption

$$\langle \alpha|_{\bar{G}} \,:\, \bar{\varphi} \cdot \bar{\varrho}_{\bar{G}} \rangle = \lim_k \langle \alpha \cdot \varphi_{\gamma(k)} \,:\, \bar{\varrho} \rangle = \lim_k \lim_l \langle \alpha \,:\, \varphi_{\gamma(k)} \cdot \psi_{\gamma(l)} \cdot \mu \rangle.$$

On the other hand, for all $\alpha \in \mathrm{A}(G)$,

$$\langle \alpha|_{\bar{G}} \,:\, \bar{\psi} \cdot \bar{\varrho}_{\bar{G}} \rangle = \lim_l \langle \alpha \cdot \psi_{\gamma(l)} \,:\, \bar{\mu} \rangle = \lim_l \lim_k \langle \alpha \,:\, \varphi_{\gamma(k)} \cdot \psi_{\gamma(l)} \cdot \mu \rangle.$$

Now, let $\alpha \in \mathcal{K}\mathrm{A}(G)_+$, $\alpha \neq 0$. Then

$$\alpha_k := \|\alpha\|^{-1} \cdot \alpha \cdot \varphi_{\gamma(k)} \in \mathbf{S}(\mathrm{A}_{\operatorname{supp} \alpha}(G))_+ \quad \text{for all } k$$

since

$$\|\alpha_k\| \cdot \|\alpha\| = \alpha_k(e) = \alpha(e) \cdot \varphi_{\gamma(k)}(e) = \|\alpha\|,$$

and

$$\mu_l := (\|\mu\| \cdot \sup_l \|\psi_l\|)^{-1} \cdot \psi_l \cdot \mu \in \mathbf{B}(\mathrm{W}^*(G)) \quad \text{for all} \quad l.$$

So, by proposition 4.3.3,

$$\langle \alpha|_{\bar{G}} \,:\, \bar{\psi} \cdot \bar{\mu}_{\bar{G}} \rangle = \langle \alpha|_{\bar{G}} \,:\, \bar{\varphi} \cdot \bar{\varrho}_{\bar{G}} \rangle.$$

By lemma 4.3.5, this equality holds for any $\alpha \in \mathrm{A}(G)$, and since $\mathrm{res}_{\bar{G}}$ is surjective by theorem 4.1.1, the assertion follows. ∎

Remark 4.3.7. Note that corollary 4.3.6 requires the existence of a convergent subsequence of $(\psi_l \cdot \mu)$, so that its applications will be primarily in the case that the unit ball of $\mathrm{W}^*(G)$ is metrizable in the ultraweak operator topology, e.g. when G is countable at infinity.

4.4. *Information at ∞ and the singular set.* The following propositions give some indication that all relevant information 'at ∞' concerning $\mathcal{T}_p(G)$ should be contained in $\operatorname{sing} p$. In the second part of the paper, there is even more evidence for this intuition in the special case of $\mathcal{T}_\pm(\mathrm{SL}(2, \mathbf{R}))$.

Proposition 4.4.1. *Let G be countable at infinity, and let $(\beta_j) \subset \mathrm{B}_\#(G)$ be bounded such that $\lim_j \beta_j = 0$ a.e. on G. Then, for any $\mu \in \mathrm{W}^*(G)$ and any $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$-accumulation point $\bar{\mu}$ of $(\beta_j \cdot \mu)$, we have that*

$$\operatorname{supp} \bar{\mu} \subset \operatorname{sing} \mu.$$

Proof. First, note that by the Lebesgue dominated convergence theorem, for any $f \in \mathbf{L}^1(G)$,

$$\lim_j \langle \beta_j \,:\, f \rangle = \lim_j \int_G \beta_j(g) f(g)\, dg = 0.$$

Since (β_j) is bounded in $\mathrm{B}_\#(G)$ and $\mathbf{L}^1(G)$ is dense in $\mathrm{C}^*_\#(G)$, we find that $\beta_j \xrightarrow{j} 0$ in $\sigma(\mathrm{B}_\#(G), \mathrm{C}^*_\#(G))$. So, if $\bar{\mu} = \lim_j \beta_{\gamma(j)} \cdot \mu$ (sufficient to consider subsequences by proposition 4.2.2), then for any $\varphi \in \mathcal{K}\mathrm{A}(G)$ such that $\varphi \cdot \mu \in \mathrm{C}^*_\#(G)$, we have

$$\langle \psi \,:\, \varphi \cdot \bar{\mu} \rangle = \lim_j \langle \beta_{\gamma(j)} \,:\, \psi \cdot \varphi \cdot \mu \rangle = 0 \quad \text{for all} \quad \psi \in \mathrm{A}(G).$$

Thus $\varphi \cdot \bar{\mu} = 0$, and hence $\varphi|_{\operatorname{supp} \bar{\mu}} = 0$, whence the conclusion. ∎

Proposition 4.4.2. *Let $(\beta_j) \subset \mathrm{B}(G)$, and define*

$$\operatorname{supp}^\infty(\beta_j) := \bigcap \{E \subset G \text{ closed} \,\big|\, \forall\, \alpha \in \mathcal{K}\mathrm{A}(G) \,:\, \operatorname{supp} \alpha \subset G \setminus E \Rightarrow \lim_j \|\alpha \cdot \beta_j\| = 0\}.$$

If $\mu \in \mathrm{W}^*(G)$, then for any $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$-limit $\bar{\mu}$ of a subsequence of $(\beta_j \cdot \mu)$,

$$\operatorname{supp} \bar{\mu} \subset \operatorname{supp}^\infty(\beta_j).$$

Proof. Let $\alpha \in \mathrm{A}(G)$ such that $\lim_j \|\alpha \cdot \beta_j\| = 0$. Then

$$\|\alpha \cdot \bar{\mu}\| = \lim_j \|\alpha \cdot \beta_{\gamma(j)} \cdot \mu\| \leq \|\mu\| \cdot \lim_j \|\alpha \cdot \beta_{\gamma(j)}\| = 0,$$

so $\alpha \cdot \bar{\mu} = 0$. The assertion follows from proposition 3.1.8 (iii). ∎

4.5. *Representations of Toeplitz* C*-*algebras via convergence of Fourier coefficients.* In this subsection, we show how natural representations of Toeplitz C*-algebras can be constructed from limits of Fourier coefficients and the formalism of cocrossed products. To this end, we fix some notation.

NOTATION 4.5.1. Let $\bar{G} \sqsubset G$ be a closed subgroup. We assume that G and $\bar{G}$ are separable and of type I, so that their $\mathbf{L}^2$ spaces have a Plancherel decomposition by [DM76]. Let $p \in \mathrm{W}^*(G)$ and $\bar{p} \in \mathrm{W}^*(\bar{G})$ be orthogonal projections. For a unitary representation π weakly contained in the left regular representation $\#$, and for $a \in \mathrm{W}^*(G)$, we define $a^{\#}_{\pi} = \pi(a^*)$, so, in particular, $a^* = a^{\#}_{\#}$. We call $a^{\#}_{\pi}$ the πth *Fourier coefficient* of a. Note $\|a^{\#}_{\pi}\| \leq \|a\|$. Also, we write $\langle\, G\,\rangle_{\pi}$ for the space of the representation π, so that $a^{\#}_{\pi}$ is a bounded operator on $\langle\, G\,\rangle_{\pi}$. Finally, for all $\bar{\pi}$ weakly contained in $\#_{\bar{G}}$, we choose sequences $(\pi_\ell(\bar{\pi}))$ of unitary representations weakly contained in $\#_G$ and isometries

$$j_\ell(\bar{\pi}) : \langle\, \bar{G}\,\rangle_{\bar{\pi}} \hookrightarrow \langle\, G\,\rangle_{\pi_\ell(\bar{\pi})}.$$

We assume that for all $\bar{\pi}$, the sequence $(\pi_\ell(\bar{\pi}))$ is eventually contained in the complement of every quasi-compact subset of $\widehat{G}$, so that $((j_\ell(\bar{\pi})\,\xi \mid \pi_\ell(\bar{\pi})\,j_\ell(\bar{\pi})\,\eta))$ is a zero sequence in the $\sigma(\mathrm{B}_{\#}(G), \mathrm{C}^*_{\#}(G))$-topology for any choice of $\xi, \eta \in \langle\, \bar{G}\,\rangle_{\bar{\pi}}$ by [Dix69, 18.2.4].

We shall consider the following condition for $\alpha \in \mathrm{A}(G)$:

$$(\alpha|_{\bar{G}} \cdot \bar{p})^{\#}_{\bar{\pi}} = \lim\nolimits_\ell \mathrm{Ad}(j_\ell(\bar{\pi})^*)[(\alpha \cdot p)^{\#}_{\pi_\ell(\bar{\pi})}] \quad \text{strongly in } \mathcal{L}(\langle\, G\,\rangle_{\bar{\pi}}). \tag{4.5.1}$$

We will write $\pi_\ell = \pi_\ell(\bar{\pi})$ and $j_\ell = j_\ell(\bar{\pi})$ whenever $\bar{\pi}$ is fixed.

LEMMA 4.5.2. *If* (4.5.1) *is satisfied for all* $\alpha \in \mathrm{A}(G)$, *then*

$$\lim\nolimits_\ell \|(1 - j_\ell\, j_\ell^*)(\alpha \cdot p)^{\#}_{\pi_\ell} j_\ell\, \xi\| = 0 \quad \textit{for all } \xi \in \langle\, G\,\rangle_{\bar{\pi}}, \alpha \in \mathrm{A}(G).$$

Proof. $p^{\#}_{\pi_\ell \otimes \#}$ is an orthogonal projection on $\langle\, G\,\rangle_{\pi_\ell} \otimes \mathbf{L}^2(G)$ and $j_\ell\, j_\ell^*$ is an orthogonal projection on $\langle\, G\,\rangle_{\pi_\ell}$. Set

$$A_\ell := \mathrm{Ad}(j_\ell^* \otimes 1)[p^{\#}_{\pi_\ell \otimes \#}] \quad \text{and} \quad C_\ell := ((1 - j_\ell\, j_\ell^*) \otimes 1)p^{\#}_{\pi_\ell \otimes \#}(j_\ell \otimes 1).$$

Thus we have the relation

$$\begin{aligned} A_\ell^2 + C_\ell^* C_\ell &= (j_\ell^* \otimes 1)p^{\#}_{\pi_\ell \otimes \#}(j_\ell\, j_\ell^* \otimes 1)p^{\#}_{\pi_\ell \otimes \#}(j_\ell \otimes 1) \\ &\quad + (j_\ell^* \otimes 1)p^{\#}_{\pi_\ell \otimes \#}((1 - j_\ell\, j_\ell^*) \otimes 1)p^{\#}_{\pi_\ell \otimes \#}(j_\ell \otimes 1) \\ &= (j_\ell^* \otimes 1)p^{\#}_{\pi_\ell \otimes \#}(j_\ell \otimes 1) = A_\ell. \end{aligned}$$

Further, for all $\xi, \eta \in \langle\, G\,\rangle_{\bar{\pi}}$ and $\chi, \zeta \in \mathbf{L}^2(G)$

$$\begin{aligned} \lim\nolimits_\ell(\xi \otimes \chi \mid A_\ell\, \eta \otimes \zeta) &= \lim\nolimits_\ell(j_\ell \xi \otimes \chi \,|\, p^{\#}_{\pi_\ell \otimes \#}\, j_\ell \eta \otimes \zeta) = \lim\nolimits_\ell(j_\ell \xi \,|\, (\bar{\zeta} * \chi^\vee \cdot p)^{\#}_{\pi_\ell}\, j_\ell \eta) \\ &= (\xi \,|\, (\bar{\zeta} * \chi^\vee|_{\bar{G}} \cdot \bar{p})^{\#}_{\bar{\pi}}\, \eta) = (\xi \otimes \chi \,|\, \bar{p}^{\#}_{\bar{\pi} \otimes \#}\, \eta \otimes \zeta), \end{aligned}$$

so (A_ℓ) converges strongly, and its limit is a projection. Since multiplication of operators is strongly continuous, $C_\ell^* C_\ell = A_\ell - A_\ell^2 \xrightarrow{\ell} 0$ strongly. Since, for all $\alpha = \bar{\zeta} * \chi^\vee \in \mathrm{A}(G)$ and $\xi, \eta \in \langle\, G\,\rangle_{\bar{\pi}}$,

$$|(\xi \mid (1 - j_\ell\, j_\ell^*)(\alpha \cdot p)^{\#}_{\pi_\ell} j_\ell\, \eta)| = |(\xi \otimes \chi \mid C_\ell \zeta \otimes \eta)| \leq \|\xi\| \cdot \|\chi\| \cdot \|C_\ell \zeta \otimes \eta\|,$$

we have $\|(1 - j_\ell\, j_\ell^*)(\alpha \cdot p)^{\#}_{\pi_\ell} j_\ell\, \eta\| \leq \|\chi\| \cdot \|C_\ell \zeta \otimes \eta\| \xrightarrow{\ell} 0$. ∎

LEMMA 4.5.3. *If* (4.5.1) *is satisfied for all* $\alpha \in \mathrm{A}(G)$, *then for all finite sequences* $(\alpha_k) \subset \mathrm{A}(G)$, *we have*

$$\lim_\ell \mathrm{Ad}(j_\ell^*)\Big[\prod_k (\alpha_k \cdot p)^{\#}_{\pi_\ell}\Big] = \prod_k (\alpha_k|_{\bar G} \cdot \bar p)^{\#}_{\bar\pi} \quad \textit{strongly in } \mathcal{L}(\langle\, \bar G\,\rangle_{\bar\pi}).$$

Proof. For all k, the sequences $(j_\ell^*(\alpha_k \cdot p)^{\#}_{\pi_\ell} j_\ell)$ are bounded and strongly convergent. In particular, their product converges. By lemma 4.5.2,

$$\lim_\ell j_\ell^*(\alpha_1 \cdot p)^{\#}_{\pi_\ell}(1 - j_\ell j_\ell^*)(\alpha_2 \cdot p)^{\#}_{\pi_\ell} j_\ell = 0,$$

so we deduce

$$\begin{aligned}\lim_\ell \prod_k j_\ell^*(\alpha_k \cdot p)^{\#}_{\pi_\ell} j_\ell &= \lim_\ell j_\ell^*(\alpha_1 \cdot p)^{\#}_{\pi_\ell}[j_\ell j_\ell^* + (1 - j_\ell j_\ell^*)](\alpha_2 \cdot p)^{\#}_{\pi_\ell} j_\ell \prod_{k>2} j_\ell^*(\alpha_k \cdot p)^{\#}_{\pi_\ell} j_\ell \\ &= (\alpha_1|_{\bar G} \cdot \bar p)^{\#}_{\bar\pi} \lim_\ell j_\ell^*(\alpha_2 \cdot p)^{\#}_{\pi_\ell} j_\ell \prod_{k>2} j_\ell^*(\alpha_k \cdot p)^{\#}_{\pi_\ell} j_\ell\,.\end{aligned}$$

Inductively, the assertion follows. ∎

LEMMA 4.5.4. *Let* $\mathcal{H}$, $\mathcal{G}$ *be Hilbert spaces,* $T_j : \mathcal{H} \to \mathcal{G}$ *isometries, let* $E \subset \mathcal{L}(\mathcal{G})$, *and let* $F \subset E$ *be* $\|\cdot\|$*-dense. If* $\varphi : E \to \mathcal{L}(\mathcal{H})$ *is a contractive linear map and*

$$\varphi(a) = \lim_j T_j^* a T_j \quad \textit{strongly for all} \;\; a \in F,$$

then

$$\varphi(a) = \lim_j T_j^* a T_j \quad \textit{strongly for all} \;\; a \in E.$$

Proof. Take $a \in E$ and $(a_k) \subset F$ such that $a = \lim_k a_k$. Let $\xi \in \mathcal{H} \setminus \{0\}$ and $\varepsilon > 0$. Choose $k \in \mathbf{N}$ such that

$$\|a - a_k\| \le \frac{\varepsilon}{3\|\xi\|}.$$

Then, if $n \in \mathbf{N}$ is such that

$$\|\varphi(a_k)\xi - T_j^* a_k T_j\, \xi\| \le \frac{\varepsilon}{3} \quad \text{for all} \;\; j \ge n,$$

we have

$$\begin{aligned}\|\varphi(a)\xi - T_j^* a T_j\, \xi\| \le{}& \|\varphi(a) - \varphi(a_k)\| \cdot \|\xi\| \\ &+ \|\varphi(a_k)\xi - T_j^* a_k T_j\, \xi\| + \|T_j\|^2 \cdot \|a_k - a\| \cdot \|\xi\| \le \varepsilon\end{aligned}$$

for all $j \ge n$. ∎

PROPOSITION 4.5.5. *Let* M *be a co-zero set in* $\operatorname{supp} \#_{\bar G}$, *such that for all* $\bar\pi \in M$ *and* $\alpha \in \mathrm{A}(G)$, *the condition* (4.5.1) *is satisfied. Then*

$$\pi_{\bar G}(a)^{\#}_{\bar\pi} = \lim_\ell \mathrm{Ad}(j_\ell(\bar\pi)^*)[a^{\#}_{\pi_\ell(\bar\pi)}] \quad \textit{strongly in } \mathcal{L}(\langle\, \bar G\,\rangle_{\bar\pi}) \textit{ for all } \bar\pi \in M, a \in \mathrm{C}^*_p(G),$$

and this defines a surjective $*$*-morphism*

$$\pi_{\bar G} : \mathrm{C}^*_p(G) \to \mathrm{C}^*_{\bar p}(\bar G) : \alpha \cdot p \mapsto \alpha|_{\bar G} \cdot \bar p.$$

Proof. Let $A \subset \mathrm{C}^*_p(G)$ be the dense $*$-algebra generated by $\{\alpha \cdot p \,|\, \alpha \in \mathrm{A}(G)\}$. Since $p^* = p$, this is the linear span of

$$(\alpha_1 \cdot p) \cdots (\alpha_n \cdot p) \quad \text{where } n \in \mathbf{N}, (\alpha_j) \subset \mathrm{A}(G).$$

Define $\pi_{\bar G}((\alpha_1\cdot p)\cdots(\alpha_n\cdot p)):=(\alpha_1|_{\bar G}\cdot\bar p)\cdots(\alpha_n|_{\bar G}\cdot\bar p)$ and extend linearly. We need to see that this is well-defined. Indeed, let

$$\sum\nolimits_k(\alpha_{1,k}\cdot p)\cdots(\alpha_{n_k,k}\cdot p)=\sum\nolimits_k(\alpha'_{1,k}\cdot p)\cdots(\alpha'_{m_k,k}\cdot p).$$

By lemma 4.5.3, for $\bar\pi\in M$

$$\begin{aligned}\sum\nolimits_k\prod_{j=1}^{n_k}(\alpha_{n_k-j,k}|_{\bar G}\cdot\bar p)^{\#}_{\bar\pi}&=\lim\nolimits_\ell j_\ell(\bar\pi)^*\sum\nolimits_k\prod_{j=1}^{n_k}(\alpha_{n_k-j,k}\cdot p)^{\#}_{\pi_\ell(\bar\pi)}\,j_\ell(\bar\pi)\\&=\lim\nolimits_\ell j_\ell(\bar\pi)^*\sum\nolimits_k\prod_{j=1}^{m_k}(\alpha'_{m_k-j,k}\cdot p)^{\#}_{\pi_\ell(\bar\pi)}\,j_\ell(\bar\pi)\\&=\sum\nolimits_k\prod_{j=1}^{m_k}(\alpha'_{m_k-j,k}|_{\bar G}\cdot\bar p)^{\#}_{\bar\pi}.\end{aligned}$$

So $\pi_{\bar G}$ is well-defined, and in fact a $*$-morphism. (Note that the reversed order is due to the * in the definition of Fourier coefficients.) Moreover, for all $a\in A$ and $\bar\pi\in M$, we have by uniform boundedness

$$\|\,\pi_{\bar G}(a)^{\#}_{\bar\pi}\,\|\le\sup\nolimits_\ell\|\,a^{\#}_\ell\,\|\le\|a\|,$$

so

$$\|\pi_{\bar G}(a)\|=\operatorname{ess\,sup}_{\bar\pi}\|\,\pi_{\bar G}(a)^{\#}_{\bar\pi}\,\|\le\|a\|,$$

cf. [Tak76]. Hence $\pi_{\bar G}$ extends by continuity to a $*$-morphism of $\mathrm{C}^*_p(G)$. Clearly, $\pi_{\bar G}$ is surjective. By lemma 4.5.4, $\pi_{\bar G}$ is given by limits as stated. ∎

Remark 4.5.6. The corepresentation W associated with $\mathrm{res}_{\bar G}$ (cf. definition 3.3.2) is clearly given by

$$W\xi(s,t)=\xi(s,s^{-1}t)\quad\text{for all}\quad s\in\bar G,t\in G,\xi\in\mathbf{L}^2(\bar G\times G).$$

Recalling $\mathrm{ext}_G=\mathrm{res}'_{\bar G}$ from theorem 4.1.1, $(\mathrm{id}\otimes\mathrm{ext}_G)(W_{\bar G})=W$.

Proposition 4.5.7. *Let the conditions of proposition* 4.5.5 *be satisfied and assume further that $\bar p$ has full support and is a.e. locally contained in* $\mathrm{A}(\bar G)$*. Then $(\pi_{\bar G},\mathrm{res}_{\bar G})$ is a covariant pair of non-degenerate representations on $\mathbf{L}^2(\bar G)$ for the coaction $(\mathrm{C}^*_p(G),\delta,G)$.*

Remark 4.5.8. Note that $\pi_{\bar G}$ is surjective onto $\mathrm{C}^*_{\bar p}(\bar G)$ and in particular non-degenerate as a $*$-morphism into $\mathrm{M}(\mathrm{C}^*_{\bar p}(\bar G))$. However, the definition of a covariant pair (π,μ) requires that π and μ be non-degenerate as $*$-morphisms into the multiplier algebra of the *same* C^*-algebra B. Hence, in proposition 4.5.7, it is necessary to assume that $\mathrm{C}^*_{\bar p}(\bar G)$ acts non-degenerately on $\mathbf{L}^2(\bar G)$!

Proof of proposition 4.5.7. By proposition 3.2.4 (ii), $\pi_{\bar G}$ is a non-degenerate representation on $\mathbf{L}^2(\bar G)$. Let $\bar\pi\in M$, $\alpha\in\mathrm{A}(G)$, $a\in\mathrm{C}^*_p(G)$ and $b\in\mathrm{C}^*_{\#}(G)$. Then, writing $\pi=\pi_{\bar G}$ and omitting $\bar\pi$ where possible,

$$\begin{aligned}[(\pi\otimes\mathrm{id})(a\otimes b)\,\overline{\pi\otimes\mathrm{id}}\circ\delta(\alpha\cdot p)]^{\#}_{\bar\pi}&=(\pi\otimes\mathrm{id})[(a\otimes b)\delta(\alpha\cdot p)]^{\#}_{\bar\pi\otimes\#}\\&=\lim\nolimits_\ell\mathrm{Ad}(j_\ell^*\otimes1)[\delta(\alpha\cdot p)^{\#}_{\pi_\ell\otimes\#}(a^{\#}_{\pi_\ell}\otimes b^*)].\end{aligned}$$

Now,

$$
\begin{aligned}
(\,\xi\otimes\chi \bigm| \delta(a)^{\#}_{\pi_\ell\otimes\#}\,\eta\otimes\zeta\,) &= \langle(\pi_\ell\,\eta \mid \xi)\otimes\bar\zeta * \chi^\vee \,:\, \delta(a)\rangle \\
&= \langle(\pi_\ell\,\eta \mid \xi) \,:\, \bar\zeta * \chi^\vee\cdot a\rangle = (\xi \mid (\bar\zeta*\chi^\vee\cdot a)^{\#}_{\pi_\ell}\,\eta),
\end{aligned}
$$

so

$$
\begin{aligned}
&(\,\xi\otimes\chi \bigm| \mathrm{Ad}(j_\ell^*\otimes 1)[\delta(\alpha\cdot p)^{\#}_{\pi_\ell\otimes\#}(a^{\#}_{\pi_\ell}\otimes b^*)]\,\eta\otimes\zeta\,) \\
&\quad= (\xi \mid \mathrm{Ad}(j_\ell^*)[(\,\overline{b^*\zeta}*\chi^\vee\cdot\alpha\cdot p)^{\#}_{\pi_\ell}a^{\#}_{\pi_\ell}]\,\eta) \xrightarrow{\ell} (\,\xi \bigm| (\overline{b^*\zeta}*\chi^\vee|_{\bar G}\cdot\alpha|_{\bar G}\cdot\bar p)^{\#}_{\bar\pi}\,\pi(a)^{\#}_{\bar\pi}\,\eta\,) \\
&\quad= \langle(\,\bar\pi\,\pi(a)^{\#}_{\bar\pi}\eta \bigm| \xi\,)\otimes\overline{b^*\zeta}*\chi^\vee|_{\bar G} \,:\, \delta_{\bar G}(\alpha|_{\bar G}\cdot\bar p)\rangle \\
&\quad= \langle(\,\bar\pi\,\pi(a)^{\#}_{\bar\pi}\eta \bigm| \xi\,)\otimes\overline{b^*\zeta}*\chi^\vee \,:\, \mathrm{Ad}(W)(\alpha|_{\bar G}\cdot\bar p\otimes 1)\rangle \\
&\quad= (\,\xi\otimes\chi|[(\pi(a)\otimes b)\,\mathrm{Ad}(W)(\alpha|_{\bar G}\cdot\bar p\otimes 1)]^{\#}_{\bar\pi\otimes\#}\,\eta\otimes\zeta\,),
\end{aligned}
$$

where W is the corepresentation corresponding to $\mathrm{res}_{\bar G}$. By non-degeneracy of $\pi_{\bar G}$, we deduce

$$\overline{\pi_{\bar G}\otimes\mathrm{id}}\circ\delta(\alpha\cdot p) = \mathrm{Ad}(W)[\alpha|_{\bar G}\cdot\bar p\otimes 1],$$

and hence the assertion. ∎

Remark 4.5.9. It is easy to see that for any covariant pair (π,μ) of representations for the coaction (A,δ,G),

$$a\mapsto \mathrm{Ad}(W)(a\otimes 1) : \mathrm{C}^*(\pi,\mu)\to \mathrm{M}(\mathrm{C}^*(\pi,\mu)\otimes\mathrm{C}^*_{\#}(G)),$$

where W is the corepresentation corresponding to μ, defines a coaction on $\mathrm{C}^*(\pi,\mu)$ which restricts to a coaction of $\pi(A)$, cf. [Dei00, remark 2.2 (iv)]. Furthermore, it is straightforward to check that π, and in fact the strict extension $\overline{\pi}$, is $\mathrm{B}_{\#}(G)$-linear for the induced module structure.

Furthermore, by remark 4.5.6, if we take $(\pi,\mu)=(\pi_{\bar G},\mathrm{res}_{\bar G})$, then the $\mathrm{A}(G)$-module structure defined by the corepresentation W corresponding to $\mathrm{res}_{\bar G}$ coincides on $\mathrm{C}^*_{\bar p}(\bar G)$ with the one induced from $\mathrm{W}^*(\bar G)$, i.e.

$$\overline{\mathrm{id}\otimes\alpha}(\mathrm{Ad}(W)(a\otimes 1)) = \alpha|_{\bar G}\cdot a \quad\text{for all}\ \ a\in\mathrm{C}^*_{\bar p}(\bar G),\alpha\in\mathrm{A}(G).$$

Corollary 4.5.10. *Under the conditions of proposition* 4.5.7, *if*

$$\operatorname{sing} a\cap\bigcup_{\bar\pi\in M,\xi,\eta\in\langle\,\bar G\,\rangle_{\bar\pi}}\operatorname{supp}^\infty[(j_\ell(\bar\pi)\xi \mid \pi_\ell(\bar\pi)\,j_\ell(\bar\pi)\eta)]=\emptyset,$$

then $a\in\ker\pi_{\bar G}$. *In particular, if* p *has full support and is a.e. locally contained in* $\mathrm{A}(G)$, *then* $\mathrm{C}^*_{\#}(G)\subset\ker\pi_{\bar G}$.

Proof. Fix $\bar\pi$ and $\xi,\eta\in\langle\,\bar G\,\rangle_{\bar\pi}$. Let $a\in\mathrm{C}^*_p(G)$ be such that

$$\operatorname{sing} a\cap\operatorname{supp}^\infty[(j_\ell\xi \mid \pi_\ell\,j_\ell\eta)]=\emptyset.$$

We have for all $\alpha\in\mathrm{A}(G)$

$$
\begin{aligned}
\langle\alpha|_{\bar G} \,:\, (\bar\pi\,\eta\mid\xi)\cdot\pi_{\bar G}(a)\rangle &= (\,\xi \bigm| \pi_{\bar G}(\alpha\cdot a)^{\#}_{\bar\pi}\,\eta\,) \\
&= \lim_\ell(j_\ell\xi \mid (\alpha\cdot a)^{\#}_{\pi_\ell}\,j_\ell\eta) = \lim_\ell\langle\alpha \,:\, (\pi_\ell\,j_\ell\eta \mid j_\ell\xi)\cdot a\rangle,
\end{aligned}
$$

so

$$(\bar\pi\eta\mid\xi)\cdot\pi_{\bar G}(a) = \lim_\ell(\pi_\ell\,j_\ell\eta\mid j_\ell\xi)\cdot a \quad\text{in }\sigma(\mathrm{W}^*(G),\mathrm{A}(G))$$

where we consider $\mathrm{W}^*(\bar{G}) \subset \mathrm{W}^*(G)$. By proposition 4.4.1 and proposition 4.4.2,

$$\operatorname{supp}(\bar{\pi}\eta \mid \xi) \cdot \pi_{\bar{G}}(a) = \emptyset.$$

Since $\mathrm{C}^*_{\bar{p}}(\bar{G})$ is a non-degenerate $\mathrm{A}(G)$-module, by proposition 3.1.8 (i), we find that $(\bar{\pi}\eta \mid \xi) \cdot \pi_{\bar{G}}(a) = 0$, and hence $\pi_{\bar{G}}(\alpha \cdot a)^{\#}_{\bar{\pi}} = 0$ for all $\alpha \in \mathrm{A}(G)$. Since M is co-zero, we have

$$\alpha|_{\bar{G}} \cdot \pi_{\bar{G}}(a) = \pi_{\bar{G}}(\alpha \cdot a) = 0 \quad \text{for all } \alpha \in \mathrm{A}(G),$$

so $\pi_{\bar{G}}(a) = 0$, again by the non-degeneracy of the $\mathrm{A}(G)$-module $\mathrm{C}^*_{\bar{p}}(\bar{G})$.

The second assertion follows from proposition 3.2.4 (ii) and the fact that $\operatorname{sing} a = \emptyset$ for all $a \in \mathrm{C}^*_{\#}(G)$. ∎

THEOREM 4.5.11. *Let the conditions of proposition* 4.5.7 *be satisfied. Further, assume that p has full support and is a.e. locally contained in* $\mathrm{A}(G)$. *Then*

$$\varrho_{\bar{G}}(pap) := \bar{p}(\pi_{\bar{G}} \otimes_\delta \operatorname{res}_{\bar{G}})(a)\bar{p} \quad \textit{for all} \ \ a \in \mathrm{C}^*_p(G) \otimes_\delta \mathrm{C}^*_{\#}(G)$$

defines an irreducible $*$-representation of $\mathcal{T}_p(G)$ on $\bar{p}\,\mathbf{L}^2(\bar{G})$ with $\mathcal{LC}(p\,\mathbf{L}^2(G)) \subset \ker \varrho_{\bar{G}}$. Furthermore, $\varrho_{\bar{G}}$ satisfies the equation

$$\varrho_{\bar{G}}(\mathrm{T}_p(f)) = \mathrm{T}_{\bar{p}}(f|_{\bar{G}}) \quad \textit{for all} \ \ f \in \mathcal{C}^0(G).$$

Proof. Abbreviate $\nu = \pi_{\bar{G}} \otimes_\delta \operatorname{res}_{\bar{G}}$. Let $a, b, c \in \mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G)$ such that $pap = 0$. By proposition 3.3.12 and its proof, $bpc \in \mathrm{C}^*_p(G) \otimes \mathcal{C}^0(G)$ and $\nu(bpc) = \nu(b)\bar{p}\nu(c)$. Hence, take $a_1, a_2 \in \mathrm{C}^*_p(G) \otimes \mathcal{C}^0(G)$ such that $a = a_1 \cdot a_2$. We have

$$\nu(b)\bar{p}\nu(a)\bar{p}\nu(c) = \nu(b)\bar{p}\nu(a_1)\nu(a_2)\bar{p}\nu(c) = \nu(bpa_1)\nu(a_2pc) = \nu(bpapc) = 0.$$

Since ν is non-degenerate, $\bar{p}\nu(a)\bar{p} = 0$. Thus we have shown that $\varrho_{\bar{G}}$ is well-defined. Clearly, $\varrho_{\bar{G}}$ is linear and involutive. Furthermore, for $a, b \in \mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G)$,

$$\varrho_{\bar{G}}(papbp) = \bar{p}\nu(apb)\bar{p} = \bar{p}\nu(a)\bar{p}\nu(b)\bar{p} = \varrho_{\bar{G}}(pap)\varrho_{\bar{G}}(pbp),$$

so $\varrho_{\bar{G}}$ is a $*$-morphism. Now, for $f \in \mathcal{C}^0(G)$, clearly $\varrho_{\bar{G}}(\mathrm{T}_p(f)) = \mathcal{T}_{\bar{p}}(f|_{\bar{G}})$, so $\varrho_{\bar{G}}$ is surjective onto $\mathcal{T}_{\bar{p}}(\bar{G})$ which acts irreducibly on $\bar{p}\,\mathbf{L}^2(\bar{G})$ by proposition 3.3.11.

The statement about the kernel follows from corollary 4.5.10, proposition 3.3.11 and its proof. ∎

REMARK 4.5.12. Let p be central, i.e. $\mathrm{Ad}(G)(p) = p$. Then $\mathrm{C}^*_p(G)$ is $\mathrm{Ad}(G)$-invariant. What is more, if we set

$$d_{(g,g')} := \mathrm{Ad}((g,g)^{\#}(e,g')_{\#}) \quad \text{for all} \ \ g, g' \in G,$$

then $d : G \times G \to \mathrm{Aut}(\mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G))$ is an action of $G \times G$ on $\mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G)$.

If (π, μ) is a covariant pair of representations for $(\mathrm{C}^*_p(G), \delta, G)$ and $g, g' \in G$, then

$$\pi \otimes_\delta \mu \circ d^{-1}_{(g,g')}(\delta(a)(1 \otimes f)) = \pi(\mathrm{Ad}(g^{-1\#})(a))\mu(g * f * g') \quad \text{for all} \ \ a \in \mathrm{C}^*_p(G), f \in \mathcal{C}^0(G),$$

since $\delta \circ \mathrm{Ad}(g^{\#}) = (g,g)^{\#} \circ \delta$. Here, $g * f * g'(t) = f(g^{-1}tg')$. Moreover,

$$(\pi \circ \mathrm{Ad}(g^{-1\#}), f \mapsto \mu(g * f * g')) = (\pi', \mu')$$

is a covariant pair of representations such that $\pi' \otimes_\delta \mu' = (\pi \otimes_\delta \mu) \circ d^{-1}_{(g,g')}$, cf. [LPRS87, 5.4 lemma]. Thus, $G \times G$ acts naturally on $\operatorname{Rep} \mathrm{C}^*_p(G) \otimes_\delta \mathrm{C}^*_{\#}(G)$.

With this in mind, the proof of the following corollary is straightforward.

COROLLARY 4.5.13. *Let the conditions of theorem* 4.5.11 *be satisfied and assume further that* p *is central. Let* $g, g' \in G$ *and set*

$$\pi_{g\bar{G}g^{-1}} := \pi_{\bar{G}} \circ \mathrm{Ad}(g^{-1\#}) \quad \text{and} \quad \mu_{g\bar{G}g'^{-1}} : f \mapsto (g * f * g')|_{\bar{G}}.$$

Then

$$\varrho_{g\bar{G}g'^{-1}}(pap) := \bar{p}(\pi_{g\bar{G}g^{-1}} \otimes_\delta \mu_{g\bar{G}g'^{-1}}(a))\bar{p} \quad \text{for all} \quad a \in \mathrm{C}^*_p(G) \otimes_\delta \mathcal{C}^0(G)$$

defines an irreducible $*$*-representation of* $\mathcal{T}_p(G)$ *on* $\bar{p}\,\mathbf{L}^2(\bar{G})$ *with* $\mathcal{LC}(p\,\mathbf{L}^2(G)) \subset \ker \varrho_{\bar{G}}$*. Furthermore,* $\varrho_{\bar{G}}$ *satisfies the equation*

$$\varrho_{g\bar{G}g'^{-1}}(\mathrm{T}_p(f)) = \mathrm{T}_{\bar{p}}(g * f * g'|_{\bar{G}}) \quad \text{for all} \quad f \in \mathcal{C}^0(G).$$

5. The Hardy–Toeplitz C^***-algebras** $\mathcal{T}_\pm(\mathrm{SL}(2,\mathbf{R}))$**.** In this section, we let $G = \mathrm{SL}(2,\mathbf{R})$ and $G^{\mathbf{C}} = \mathrm{SL}(2,\mathbf{C})$. We also adopt the convention of denoting the Lie algebra of a Lie group by the corresponding lower case Fraktur letter.

5.1. *The Olshanskii domains* $G^{\mathbf{C}}_\pm$*.* The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{R})$ has the convenient basis

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

of infinitesimal generators for the respective 1-parameter subgroups

$$k_\vartheta = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix},\ h_{e^t} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

also denoted K $(= \mathrm{SO}(2))$, A and N. KAN is an Iwasawa decomposition of G. $M = \{\pm 1\} \subset K$ is the centre of G. MAN, the normalizer of N, is the set of upper triangular matrices in G. $G^{\mathbf{C}}$ has the Iwasawa decomposition $\mathrm{SU}(2) \cdot A \cdot N^{\mathbf{C}}$ where $N^{\mathbf{C}} = \left(\begin{smallmatrix} 1 & \mathbf{C} \\ 0 & 1 \end{smallmatrix}\right)$.

In the following we choose a fixed sign $\sigma \in \{+,-\}$ in order to treat the holomorphic and anti-holomorphic discrete series simultaneously.

DEFINITION 5.1.1. We introduce the (forward resp. backward) *light cone*

$$\Lambda_\sigma := \left\{ \begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} \,\middle|\, \sigma x > \sigma\sqrt{z^2 - x^2 - y^2} \right\}.$$

Also, we set $\Lambda = \Lambda_+ \cup \Lambda_-$. $\Lambda_\sigma \subset \mathfrak{g}$ is an open convex $\mathrm{Ad}(G)$-invariant cone. For such cones and $\mathfrak{g}$ with trivial centre (or $G^{\mathbf{C}}$ simply connected), Lawson's theorem on Olshanskii semigroups ensures that $G \cdot \exp i\overline{\Lambda_\sigma}$ is a closed involutive subsemigroup of $G^{\mathbf{C}}$ (a so-called *complex Olshanskii semigroup*), such that the polar decomposition

$$G \times \overline{\Lambda_\sigma} \to G \cdot \exp i\overline{\Lambda_\sigma} \subset G^{\mathbf{C}} : (g, v) \mapsto g \cdot \exp iv$$

is a homeomorphism. Furthermore, the interior $G \cdot \exp i\Lambda_\sigma$ (which we call an *Olshanskii domain*) is a $G \times G$-invariant domain in $G^{\mathbf{C}}$, cf. [Nee00].

The following proposition is easy to check with Iwasawa decomposition.

PROPOSITION 5.1.2. *The open cone* Λ_σ *decomposes as the set of orbits*

$$\Omega^\sigma_{\lambda^2} = \left\{ \begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} \,\middle|\, \sigma x > \sigma\sqrt{z^2 - x^2 - y^2} = \lambda \right\} = \mathrm{Ad}(G).\lambda Z.$$

where $\sigma\lambda > 0$. *Its boundary* $\partial\Lambda_\sigma$ *decomposes into* $\{0\}$ *and the orbit*

$$\Omega_0^\sigma = \left\{ \begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} \,\middle|\, \sigma x > \sqrt{z^2-x^2-y^2} = 0 \right\} = \mathrm{Ad}(G).\sigma X_+.$$

Moreover, if $\Pi = G/K$ *denotes the upper half plane (with base point* i*), the map*

$$\Phi : \Lambda_\sigma \to \sigma\Pi : v = \begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} \mapsto \frac{1}{v_{21}}(v_{11} - i\sqrt{\det v}) = \frac{y - i\sqrt{z^2-x^2-y^2}}{x-z}$$

is a G*-equivariant foliation. Here, the action of* G *on* $\sigma\Pi$ *is given by fractional linear transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.w := \frac{aw+b}{cw+d} \quad \textit{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, w \in \sigma\Pi.$$

REMARK 5.1.3. The map Φ is a useful tool for the explicit evaluation of the moment map for the highest weight representations of G, as we shall see below.

Corresponding to the decomposition of Λ_σ in proposition 5.1.2, the Olshanskii domains $G \cdot \exp i\Lambda_\sigma$ have the following decomposition.

THEOREM 5.1.4. *Consider the action of* $G \times G$ *on* $G^{\mathbf{C}}$ *given by*

$$(G \times G) \times G^{\mathbf{C}} \to G^{\mathbf{C}} : ((s,t),\gamma) \mapsto s\gamma t^{-1}$$

Then we have the following decomposition as a disjoint union of $G \times G$*-fibre bundles*

$$\begin{aligned} \overline{G_\sigma^{\mathbf{C}}} &= G \cup (\partial G_\sigma^{\mathbf{C}} \setminus G) \cup G_\sigma^{\mathbf{C}} \\ &= (G \times G) \times_{\mathrm{diag}(G)} \{e\} \cup (G \times G) \times_{\mathrm{diag}(MA)\cdot N \times N} N_\sigma^{\mathbf{C}} \cup (G \times G) \times_{G \times G} G_\sigma^{\mathbf{C}}. \end{aligned}$$

Here, $N_\sigma^{\mathbf{C}} = N \cdot \exp i\mathfrak{n}_\sigma$ *is the Olshanskii domain in* $N^{\mathbf{C}}$ *corresponding to the cone*

$$\mathfrak{n}_\sigma = \overline{\Lambda_\sigma} \cap \mathfrak{n} \setminus \{0\} = \{\lambda \cdot X_+ \,|\, \sigma\lambda > 0\}.$$

Proof. The first equality is clear. Also, we clearly have $G = (G \times G) \times_{\mathrm{diag}(G)} \{e\}$, since $G \times G/\mathrm{diag}(G) \cong G$ as $G \times G$-spaces. Also, $G_\sigma^{\mathbf{C}}$ is $G \times G$-invariant, so

$$G_\sigma^{\mathbf{C}} = (G \times G) \times_{G \times G} G_\sigma^{\mathbf{C}}.$$

By Lawson's theorem and proposition 5.1.2,

$$\partial G_\sigma^{\mathbf{C}} \setminus G = G \cdot \exp(i\partial\Lambda_\sigma \setminus \{0\}) = G \cdot \exp(i\Omega_0^\sigma) = G \exp(i\mathfrak{n}_\sigma)G.$$

Now, $N_\sigma^{\mathbf{C}} = N \cdot \exp(i\mathfrak{n}_\sigma) = \left(\begin{smallmatrix} 1 & \sigma\Pi \\ 0 & 1 \end{smallmatrix}\right)$, so since

$$h_\lambda n_x \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} n_{-x} h_{1/\lambda} = \begin{pmatrix} 1 & \lambda^2 w \\ 0 & 1 \end{pmatrix} \quad \text{for all } \lambda > 0, x \in \mathbf{R}, w \in \mathbf{C}$$

and M is central in G, the domain $N_\sigma^{\mathbf{C}}$ is $\mathrm{diag}(MA) \cdot N \times N$-invariant.

It remains to be shown that

$$(s,t).N_\sigma^{\mathbf{C}} \cap N_\sigma^{\mathbf{C}} \neq \emptyset \quad \Rightarrow \quad (s,t) \in \mathrm{diag}(MA) \cdot N \times N.$$

To this end, let $s\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)t$ for some $s, t \in G$ and $z, w \in \sigma\Pi$. In particular,

$$\begin{pmatrix} -s_{11}t_{21}\,\mathrm{Im}\,z & s_{11}t_{11}\,\mathrm{Im}\,z \\ -s_{21}t_{21}\,\mathrm{Im}\,z & -s_{21}t_{11}\,\mathrm{Im}\,z \end{pmatrix} = \mathrm{Im}\,s \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} t^{-1} = \mathrm{Im} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Im}\,w \\ 0 & 0 \end{pmatrix}.$$

We deduce $s_{11}t_{11} \neq 0$, so from $s_{11}t_{21} = s_{21}t_{11} = 0$ we have $s_{21} = t_{21} = 0$. Hence s, t are upper triangular, i.e. $s, t \in MAN$. Since $N_\sigma^{\mathbf{C}}$ is invariant under conjugation by this group, there exists $w' \in \sigma\Pi$ such that

$$s \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = t \begin{pmatrix} 1 & w' \\ 0 & 1 \end{pmatrix}.$$

By uniqueness of Iwasawa decomposition in $G^{\mathbf{C}}$, we conclude that s and t have the same MA-component. ∎

5.2. *The Hardy spaces* $\mathbf{H}^2(G_\pm^{\mathbf{C}})$

Remark 5.2.1. The space of holomorphic functions

$$\mathbf{H}^2(G_\sigma^{\mathbf{C}}) = \left\{ f \in \mathcal{O}(G_\sigma^{\mathbf{C}}) \;\middle|\; \sup_{\gamma \in G_\sigma^{\mathbf{C}}} \int_G |f(\gamma^* g)|^2 \, dg < \infty \right\}$$

endowed with the norm given by the square root of the integral in its definition, is called the *Hardy space of* $G_\sigma^{\mathbf{C}}$. It can be considered as a closed subspace of $\mathbf{L}^2(G)$ via the isometry

$$j : \mathbf{H}^2(G_\sigma^{\mathbf{C}}) \to \mathbf{L}^2(G), \; jf(g) = \lim_{G_\sigma^{\mathbf{C}} \ni \gamma \to e} f(\gamma^* g).$$

Its topology is weaker than the topology of convergence on compact subsets, so it is a reproducing kernel Hilbert space whose kernel $E_\sigma(z, w)$, the so-called *Cauchy–Szegö kernel*, is holomorphic in z and anti-holomorphic in w. Moreover, it is given by a single function as $E_\sigma(zw^*)$. $E_\sigma(z, w)$ can be extended continuously in one variable to G, cf. [Nee00], [HN93], [HÓØ91].

The map $j^* : \mathbf{L}^2(G) \to \mathbf{H}^2(G^{\mathbf{C}}) \subset \mathcal{O}(G_\sigma^{\mathbf{C}})$ is the integral operator with kernel E_σ, so jj^* has distribution kernel E_σ. Since, furthermore, the Hardy space $\mathbf{H}^2(G_\sigma^{\mathbf{C}})$ is $G \times G$-invariant, its associated orthogonal projection jj^* is a central element of $\mathrm{W}^*(G)$, so we consider $E_\sigma \in \mathrm{W}^*(G)$.

Remark 5.2.2. We recall the definition of the discrete series representations of G, cf. [War72], [Lan75], [Tay86]. The *Bergman spaces*

$$\mathcal{O}_m^2(\sigma\Pi) := \left\{ f \in \mathcal{O}(\sigma\Pi) \;\middle|\; \int_{\sigma\Pi} |f(x + iy)|^2 \, |y|^{m-2} \, dx \, dy < \infty \right\},$$

defined for $\mathbf{N} \ni m \geq 2$, and endowed with the corresponding $\mathbf{L}^2$ norm, are the reproducing kernel Hilbert spaces associated to the kernel functions

$$K_w^{m,\sigma}(z) = K^{m,\sigma}(z, w) = \frac{m!}{\pi} \cdot \left(\frac{2i\sigma}{z - \bar{w}} \right)^m.$$

We also introduce the *normalized kernel functions*

$$k_w^{m,\sigma}(z) = \frac{K^{m,\sigma}(z, w)}{K^{m,\sigma}(w, w)^{1/2}}.$$

By the reproducing property, the $k_w^{m,\sigma}$ are indeed unit vectors in $\mathcal{O}_m^2(\sigma\Pi)$. The action of G on $\mathcal{O}_m^2(\sigma\Pi)$ is given by

$$g^{\pi_m^\sigma}.f(z) = \partial g^{-1}(z)^{m/2} \cdot f(g^{-1}.z)$$

where the action of G on $\sigma\Pi$ was introduced in proposition 5.1.2. π^σ_m is an irreducible continuous unitary representation of G. In fact, it is a discrete series (i.e., square-integrable) representation.

It is well-known that the Hardy spaces $\mathbf{H}^2(G^{\mathbf{C}}_\pm)$ are associated with the discrete series of G in the following way.

PROPOSITION 5.2.3. *We have the following G-equivariant isomorphism*

$$\sum_{m\geq 2} \mathcal{O}^2_m(\sigma\Pi) \otimes \overline{\mathcal{O}^2_m(\sigma\Pi)} \cong \mathbf{H}^2(G^{\mathbf{C}}_\sigma) : (\xi_m \otimes \bar\eta_m) \mapsto \sum_{m\geq 2} d_m^{1/2} (\pi^\sigma_m\, \eta_m \mid \xi_m)$$

where d_m denotes formal dimension and $\mathbf{H}^2(G^{\mathbf{C}}_\sigma)$ is considered as a subspace of $\mathbf{L}^2(G)$.

5.3. *The Hardy–Toeplitz* C*-*algebras $\mathcal{T}_\pm(G)$ and their irreducibility*

NOTATION 5.3.1. We use the notation $\mathcal{T}_\sigma(G) = \mathcal{T}_{E_\sigma}(G)$ for the Hardy–Toeplitz C*-algebra.

In order to get more information on $\mathcal{T}_\sigma(G)$, we need to analyse E_σ in greater detail.

REMARK 5.3.2. It can be shown by elementary considerations (cf. [GG77]) that every $\gamma \in G^{\mathbf{C}}_\sigma$ has an eigenvalue $q(\gamma)$ of modulus > 1. Moreover, $q : G^{\mathbf{C}}_\sigma \to \mathbf{C}$ is holomorphic, and

$$E_\sigma = \frac{q^2}{(q-1)^3(q+1)},$$

cf. [GG77], [Ols95].

PROPOSITION 5.3.3. *The function q has a smooth extension q_σ to*

$$G_{reg} = \{\, g \in G \;\big|\; \operatorname{tr} g - 4 \neq 0\},$$

the set of $g \in G$ where g has distinct eigenvalues. Moreover, writing

$$N^K = \bigcup_{k\in K} kNk^{-1},$$

we have $G \setminus G_{reg} = M \cdot N^K$.

Furthermore, $(q_\sigma + 1)^{-1}$ is locally integrable near mN^K, $m = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence

$$\operatorname{sing\,supp} E_\sigma \subset M \cdot N^K \quad \textit{and} \quad \operatorname{sing} E_\sigma \subset N^K.$$

Proof. The first part follows by considering the quadratic equation defining q and noting that $\operatorname{tr} g - 4$ is the discriminant of the characteristic polynomial χ_g. The local integrability of $(q_\sigma + 1)^{-1}$ is seen by explicit integration. The remaining assertions now follow from remark 5.3.2. ∎

COROLLARY 5.3.4. *The Hardy–Toeplitz* C*-*algebra $\mathcal{T}_\pm(G)$ contains $\mathcal{LC}(\mathbf{H}^2(G^{\mathbf{C}}_\pm))$, and hence acts irreducibly on $\mathbf{H}^2(G^{\mathbf{C}}_\pm)$.*

Proof. Clearly, $\operatorname{supp} E_\sigma = G$. Moreover, $\operatorname{sing\,supp} E_\sigma \subset M \cdot N^K$ is negligible. So the assertion follows from theorem 3.3.11. ∎

So far, we have established the irreducibility of the 'identity' representation of $\mathcal{T}_\pm(G)$ on $\mathbf{H}^2(G^{\mathbf{C}}_\pm)$, associated with the interior $G^{\mathbf{C}}_\sigma$ of $\overline{G^{\mathbf{C}}_\sigma}$.

6. The representation theory of $\mathcal{T}_\pm(\mathrm{SL}(2,\mathbf{R}))$. In this section $G = \mathrm{SL}(2,\mathbf{R})$, and we use the notation from section 5.

Following the general procedure outlined at the end of the previous section, we construct representations of $\mathcal{T}_\pm(G)$ corresponding to the boundary faces of $G_\pm^{\mathbf{C}}$ given by theorem 5.1.4. They correspond to information 'at ∞' (in $\widehat{G}$), and consequently vanish on $\mathcal{LC}(\mathbf{H}^2(G_\pm^{\mathbf{C}}))$.

Again, we fix a sign $\sigma \in \{+,-\}$.

6.1. *A class of pure states.* In this subsection, we define functions of positive type Δ_v for $v \in \Lambda$ contained in an 'integral orbit' $\Omega^\sigma_{m^2}$. Of course, this can be considered as a pure state of $\mathrm{C}^*_\#(G)$, indeed of $\mathrm{C}^*_{E_\sigma}(G)$. It also proves useful to consider this function as an element of both $\mathrm{A}(G)$ and $\mathbf{H}^2(G^{\mathbf{C}}_{-\sigma})$. What is more, as we shall see, although all v contained in the same orbit define the same representation of G via the GNS construction, different 'directions' in which we consider limits to infinity in Λ_σ give rise to different representations of $\mathrm{C}^*_{E_\sigma}(G)$.

Definition 6.1.1. Let $\mathbf{N} \ni m \geq 2$ and $v \in \Omega^\sigma_{m^2}$. For all $s \in G$, set

$$\Delta_v(s) = \left(k^{m,\sigma}_{\Phi(v)} \,\middle|\, s^{\pi^\sigma_m}.k^{m,\sigma}_{\Phi(v)}\right).$$

Remark 6.1.2.

(i) We have $-\Omega^\sigma_{m^2} = \Omega^{-\sigma}_{m^2}$ and $\overline{\Delta_v} = \Delta_{-v}$, so $\Delta_v \in \mathbf{H}^2(G^{\mathbf{C}}_{-\sigma})$ by proposition 5.2.3.

(ii) π^σ_m is square-integrable; so by Schur orthogonality [War72, corollary 4.5.9.4], it is easy to see that $\Delta_v \in \mathrm{A}(G)$. Indeed,

$$\Delta_v = \left(k^{m,\sigma}_{\Phi(v)} \,\middle|\, k^{m,\sigma}_{\Phi(v)}\right) \cdot \Delta_v = d_{\pi^\sigma_m} \cdot \Delta_v * \Delta_v \in \mathrm{A}(G).$$

We shall consider the asymptotic behaviour of Δ_v for certain sequences of vs.

Notation 6.1.3. Let $\bar\nu \in \mathbf{R} \setminus \{0\}$ and $m \geq 2$. Define

$$\beta_e(\sigma,m) := \sigma m Z = \begin{pmatrix} 0 & \sigma m \\ -\sigma m & 0 \end{pmatrix} \text{ and } \beta_N(\bar\nu,m) := \bar\nu X_- + \frac{m^2}{\bar\nu} X_+ = \begin{pmatrix} 0 & m^2/\bar\nu \\ -\bar\nu & 0 \end{pmatrix}$$

where $X_- = -X_+^t$. Thus $\beta_e(\sigma,m) \in \Omega^\sigma_{m^2}$ and $\beta_N(\bar\nu,m) \in \Omega^{\operatorname{sgn}\bar\nu}_{m^2}$.

Proposition 6.1.4. *For all $m \geq 2$, let $v_m \in \mathfrak{n} \oplus \bar{\mathfrak{n}} \cap \Omega^\sigma_{m^2}$, so that*

$$v_m = \begin{pmatrix} 0 & * \\ -x_m & 0 \end{pmatrix} \quad \textit{for some } \sigma x_m > 0.$$

Then

$$\Delta_{v_m}(t) = \left(\frac{2}{t_{11} + t_{22} + i\left(\frac{x_m}{m} t_{12} - \frac{m}{x_m} t_{21}\right)}\right)^m.$$

In particular,

$$\Delta_{\beta_e(\sigma,m)}(t) = \left(\frac{2}{t_{11} + t_{22} + i\sigma(t_{12} - t_{21})}\right)^m \tag{6.1.1}$$

$$\Delta_{\beta_N(\bar\nu,m)}(t) = \left(\frac{2}{t_{11} + t_{22} + i(t_{12}\bar\nu/m - t_{21}m/\bar\nu)}\right)^m \tag{6.1.2}$$

Proof. Since $\Delta_{-v} = \overline{\Delta_v}$, w.l.o.g. we may assume $\sigma = +$. We have $\Phi(v_m) = i\frac{m}{x_m}$, hence

$$\begin{aligned}
\Delta_{v_m}(t) &= (k^{m,+}_{im/x_m} \mid t^{\pi^+_m}.k^{m}_{im/x_m}) = K^{m,+}(i\frac{m}{x_m}, i\frac{m}{x_m}) \cdot [t^{\pi^+_m}.k^{m,+}_{im/x_m}](im/x_m) \\
&= \left(\frac{m}{x_m}\right)^m (\partial t^{-1})^{m/2}(im/x_m)\left(\frac{2i}{(t^{-1}.im/x_m) + im/x_m}\right)^m \\
&= \left(\frac{m}{x_m}\right)^m \left(t_{11} - i\frac{m}{x_m}t_{21}\right)^{-m} \cdot \left(\frac{2i}{(t^{-1}.im/x_m) + im/x_m}\right)^m \\
&= \left(\frac{m}{x_m}\right)^m \left(\frac{2i}{it_{22}m/x_m - t_{12} + im/x_m \cdot (t_{11} - it_{21}m/x_m)}\right)^m \\
&= \left(\frac{2}{t_{11} + t_{22} + i(\frac{x_m}{m}t_{12} - \frac{m}{x_m}t_{21})}\right)^m .
\end{aligned}$$

The other equations follow immediately. ∎

Remark 6.1.5. Since the kernel of $\mathcal{O}^2_m(\Pi)$ at $i = \Phi(\beta_e(+,m))$ is K-invariant (because i is the base point of $\Pi = G/K$), it is easy see to that the $\Delta_{\beta_e(\sigma,m)}$ are highest weight vectors of the lowest K-type in π_m. So, in a sense, taking the whole orbit picture generalizes the highest weight theory.

Notation 6.1.6. For $\varepsilon \geq 0$, define

$$K_\varepsilon := \{s \in G \,\big|\, |s_{11} - s_{22}|^2 + |s_{12} + s_{21}|^2 \leq 4\varepsilon\},$$

and

$$(MAN)_\varepsilon := \{s \in G \,\big|\, |s_{21}| \leq \varepsilon\}.$$

Lemma 6.1.7.

(i) *For all $\varepsilon > 0$, K_ε is a neighbourhood of K. Furthermore,*

$$\bigcap_{\varepsilon>0} K_\varepsilon = K_0 = K. \tag{6.1.3}$$

(ii) *For all $\varepsilon > 0$, $(MAN)_\varepsilon$ is a neighbourhood of MAN. Furthermore,*

$$\bigcap_{\varepsilon>0} (MAN)_\varepsilon = (MAN)_0 = MAN. \tag{6.1.4}$$

Proof. (i) Let $s \in G$. Manifestly, $s \in K_0$ if and only if

$$s_{11} = s_{22} \qquad \text{und} \qquad s_{21} = -s_{12},$$

i.e. $s = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ for some $a, b \in \mathbf{R}$. Hence $K_0 = K$.

On the other hand, the family of sets $(K_\varepsilon)_{\varepsilon \in [0,\infty[}$ is increasing. From this, (6.1.3) is easily deduced. The first assertion now follows from

$$K = K_0 \subset K_{\varepsilon/2} \subset \{s \in G \,\big|\, |s_{11} - s_{22}|^2 + |s_{12} + s_{21}|^2 < 4\varepsilon\} \subset K^\circ_\varepsilon.$$

(ii) Let $s \in G$. s is an upper triangular matrix if and only if $s_{21} = 0$, i.e. $s \in (MAN)_0$. Thus $MAN = (MAN)_0$.

Trivially, the family $(MAN)_{\varepsilon \in [0,\infty[}$ is increasing, so the equation (6.1.4) follows. As in (i), we see that $(MAN)_\varepsilon$ is a neighbourhood of MAN for $\varepsilon > 0$. ∎

PROPOSITION 6.1.8. *Let $\varepsilon > 0$ and $\bar\nu \neq 0$.*

(i) *Let $t \in G \setminus K_\varepsilon$. We have*

$$|\Delta_{\beta_e(\sigma,m)}(t)| \leq (1+\varepsilon)^{-m/2} \; ; \tag{6.1.5}$$

in particular, $\Delta_{\beta_e(\sigma,m)}$ converges to 0 for $m \to \infty$ uniformly on $G \setminus K_\varepsilon$ and point-wise on $G \setminus K$.

(ii) *Let $L \subset G \setminus (MAN)_\varepsilon$ be a compact subset. There exists $C > 0$ such that for large m, and all $t \in L$*

$$|\Delta_{\beta_N(\bar\nu,m)}(t)| \leq \left(\frac{\varepsilon m}{2|\bar\nu|} - C\right)^{-m}, \tag{6.1.6}$$

in particular, $\Delta_{\beta_N(\bar\nu,m)}$ converges to 0 for $m \to \infty$ uniformly on compact subsets of $G \setminus (MAN)_\varepsilon$ and point-wise on $G \setminus MAN$.

Proof. (i) We have

$$\begin{aligned}|t_{11} + t_{22} &+ i\sigma(t_{12} - t_{21})|^2 - |t_{11} - t_{22} + i\sigma(t_{12} + t_{21})|^2 \\ &= (t_{11}+t_{22})^2 - (t_{11}-t_{22})^2 + (t_{12}-t_{21})^2 - (t_{12}+t_{21})^2 = 4\det t = 4.\end{aligned}$$

So, applying (6.1.1) from proposition 6.1.4 (i), we get

$$|\Delta_{\beta_e(\sigma,m)}(t)|^{-2/m} = 1 + \tfrac14((t_{11}-t_{22})^2 + (t_{12}+t_{21})^2) > 1+\varepsilon,$$

and hence the estimate (6.1.5). The first convergence assertion is immediate, and the second one follows from (6.1.3) in lemma 6.1.7 (i).

(ii) For all $m \geq 2$ and $t \in G$, we have

$$\frac12\left|t_{11} + t_{22} + it_{12}\frac{\bar\nu}{m}\right| \leq \frac12|t_{11}+t_{22}| + |t_{12}|\frac{|\bar\nu|}{2m} \; ;$$

by compactness of L, the right hand side is bounded for all $m \geq 2$ and $t \in L$ by some $C > 0$. Furthermore, since $|t_{21}| > \varepsilon$, there exists $2 \leq k \in \mathbf{N}$ such that for $m \geq k$ and $t \in L$,

$$|t_{21}|\frac{m}{2|\bar\nu|} - C \geq 0.$$

Hence for all $m \geq k$, $t \in L$, we have

$$|\Delta_{\beta_N(\bar\nu,m)}(t)|^{-1/m} = \left|\frac{t_{11}+t_{22}}{2} + it_{12}\frac{\bar\nu}{2m} - it_{21}\frac{m}{2\bar\nu}\right| \geq |t_{21}|\frac{m}{2|\bar\nu|} - C \geq \frac{\varepsilon m}{2|\bar\nu|} - C,$$

where (6.1.2) from proposition 6.1.4 (ii) and the inverse triangular inequality were applied. The estimate and the first convergence assertion follow. The point-wise convergence now follows from (6.1.4) in lemma 6.1.7 (ii). ∎

NOTATION 6.1.9. In the following lemma, we define for $\gamma \in \mathbf{N}^3$ the left-invariant differential operators

$$D^\gamma := X_+^{\gamma_1} Y^{\gamma_2} Z^{\gamma_3} \quad \text{on } G = \mathrm{SL}(2,\mathbf{R}).$$

Moreover, we consider for $\alpha \in \mathbf{N}^4$ the monomial

$$t^\alpha = t_{11}^{\alpha_1} t_{12}^{\alpha_2} t_{21}^{\alpha_3} t_{22}^{\alpha_4} \quad \text{in } t = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22}\end{pmatrix} \in G.$$

LEMMA 6.1.10. *Choose for all $m \geq 2$ elements $v_m \in \mathfrak{n} \oplus \bar{\mathfrak{n}} \cap \Omega^\sigma_{m^2}$ such that*

$$v_m = \begin{pmatrix} 0 & * \\ x_m & 0 \end{pmatrix}, \quad 0 < r \leq |x_m| \leq R < \infty,$$

where r and R are fixed. Then, for all $\gamma \in \mathbf{N}^3$, there are polynomials $(p_m) \subset \mathbf{C}[t_{11}, \ldots, t_{22}]$ and $p \in \mathbf{R}[x]$, the latter independent of m, such that

$$\deg p_m \leq |\gamma|, \quad \deg p \leq 2|\gamma|, \quad p_m(t) = \sum_{|\alpha| \leq |\gamma|} c^m_\alpha \cdot t^\alpha, \quad |c^m_\alpha| \leq |p(m)|$$

and

$$D^\gamma \Delta_{v_m} = \Delta_{v_m} \cdot \frac{p_m}{q_m^{|\gamma|+1}} \text{ where } q_m(t) = t_{11} + t_{22} + i\biggl(\frac{x_m}{m} \cdot t_{12} - \frac{m}{x_m} \cdot t_{22}\biggr). \tag{6.1.7}$$

Proof. First, note that for $\alpha \in \mathbf{N}^4$

$$X_+ t^\alpha = \alpha_2 \cdot t^{\alpha + (-1,1,0,0)} + \alpha_4 \cdot t^{\alpha + (0,0,-1,1)},$$
$$Y t^\alpha = (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) \cdot t^\alpha,$$
$$Z t^\alpha = -\alpha_1 \cdot t^{\alpha + (-1,1,0,0)} + \alpha_2 \cdot t^{\alpha + (1,-1,0,0)} - \alpha_3 \cdot t^{\alpha + (0,0,-1,1)} + \alpha_4 \cdot t^{\alpha + (0,0,1,-1)}.$$

So, if $V \in \{X_+, Y, Z\}$ and

$$q(t) = \sum_{|\alpha| \leq n} c_\alpha t^\alpha, \quad \text{then} \quad V q(t) = \sum_{|\alpha| \leq n} d_\alpha t^\alpha$$

where $\max_\alpha |d_\alpha| \leq (1+2n) \cdot \max_\alpha |c_\alpha|$. Furthermore, if we denote by $r_{V,m} \in \mathbf{C}[t_{11}, \ldots, t_{22}]$ the polynomial such that

$$(V\Delta_{v_m})(t) = \Delta_{v_m}(t) \cdot \frac{r_{V,m}(t)}{q_m(t)} \quad \text{for all } t \in G,$$

then $\deg r_{V,m} = 1$ and its coefficients are bounded by $\max(\frac{1}{r}, R) \cdot m^2$. So, let $m \in \mathbf{N}$, and let p_m and p satisfy the assertions of the lemma with $|\gamma|$ replaced by k. Then, applying the product and quotient rules, we get

$$V\biggl[\Delta_{v_m} \cdot \frac{p_m}{q_m^k}\biggr](t) = \Delta_{v_m}(t) \cdot \frac{r_{V,m}(t) \cdot p_m(t) + V p_m(t) \cdot q_m(t) - k \cdot p_m(t) \cdot V q_m(t)}{q_m(t)^{k+1}}.$$

The numerator of the fraction on the right hand side is a polynomial in $\mathbf{C}[t_{11}, \ldots, t_{22}]$ of degree $\leq k+1$. Its coefficients are bounded by

$$[\max(\tfrac{1}{r}, R) \cdot m^2 + (1+2k) \cdot \max(\tfrac{1}{r}, R) \cdot m + 5k \cdot \max(\tfrac{1}{r}, R)] \cdot |p(m)|$$

since the coefficients of q_m are bounded by $\max(\frac{1}{r}, R) \cdot m$. This proves the assertion. ∎

COROLLARY 6.1.11. *Let $\bar{\nu} \neq 0$.*

(i) *We have $\Delta_{\beta_e(\sigma,m)}|_{G \setminus K} \xrightarrow{m} 0$ in $\mathcal{E}(G \setminus K)$, i.e.* $\operatorname{supp}^\infty(\Delta_{\beta_e(\sigma,m)}) \subset K$.

(ii) *We have $\Delta_{\beta_N(\bar{\nu},m)}|_{G \setminus MAN} \xrightarrow{m} 0$ in $\mathcal{E}(G \setminus MAN)$, i.e.* $\operatorname{supp}^\infty(\Delta_{\beta_N(\bar{\nu},m)}) \subset MAN$.

Proof. (i) The topology of $\mathcal{E}(G \setminus K)$ is the topology of uniform convergence of all derivatives on compact subsets. Since G is locally generated by its Lie algebra $\mathfrak{g}$, it suffices to restrict attention to left invariant differential operators. By the Poincaré–Birkhoff–Witt theorem, a basis for these is given by the ordered monomials in the basis X_+, Y, Z. Set $v_m = \beta_e(\sigma, m)$.

Let $\gamma \in \mathbf{N}^3$. By (6.1.7) in lemma 6.1.10, there exist $(p_m) \subset \mathbf{C}[t_{11}, \ldots, t_{22}]$ and $p \in \mathbf{R}[x]$ such that $\deg p_m \leq |\gamma|$, $\deg p \leq 2|\gamma|$, the coefficients of p_m are bounded by $p(m)$, and

$$(D^\gamma \Delta_{v_m})(t) = \Delta_{v_m}(t) \cdot \frac{p_m(t)}{q_m(t)^{|\gamma|+1}} \quad \text{for all } t \in G, m \geq 2.$$

Let $L \subset G \setminus K$ be a compact subset, $\varepsilon > 0$ such that $L \subset G \setminus K_\varepsilon$ and $M \geq 1$ such that $\max_{k,l=1,2} |t_{kl}| \leq M$ for all $t \in L$. Then

$$\left| \frac{1}{q_m(s)} \right| = |\Delta_{\beta_e(\sigma,m)}|^{1/m} \leq 1,$$

and for all $m \geq 2$ and $t \in L$, by (6.1.5) from proposition 6.1.8 (i), we have

$$|(D^\gamma \Delta_{v_m})(t)| \leq M^{|\gamma|} \cdot |p(m)| \cdot (1+\varepsilon)^{-m/2} \xrightarrow{m} 0,$$

proving the assertion.

(ii) The proof is analogous to that of (i), using (6.1.6) from proposition 6.1.8 (i). ■

6.2. *Convergence of states defined by the* 0*-dimensional faces.* In this section, we shall study the behaviour of $\Delta_{v_m} \cdot E_\sigma$ at infinity, where

$$v_m = \beta_e(\sigma', m) = \begin{pmatrix} 0 & \sigma' m \\ -\sigma' m & 0 \end{pmatrix}.$$

Here, $\sigma' \in \{+,-\}$ is a sign possibly distinct from σ.

These sequences are precisely the points of intersection of $\mathfrak{k}$ with the 'integral orbits' $\Omega^{\sigma'}_{m^2}$. As we shall see, in the limit, they give rise the to Fourier transform of the projection of $\mathbf{C} = \mathbf{L}^2(\{e\})$ onto the Hardy space $\mathbf{C} = \mathbf{H}^2(\{e\})$.

NOTATION 6.2.1. For the sake of brevity, let us write $\Delta^{\sigma'}_m := \Delta_{\beta_e(\sigma',m)}$.

THEOREM 6.2.2. *For all $\alpha \in \mathrm{A}(G)$,*

$$\lim_m (k^{m,-\sigma'}_{\Phi(\beta_e(-\sigma',m))} \,|\, (\alpha \cdot E_\sigma)^{\#}_{\pi_m^{-\sigma'}} k^{m,-\sigma'}_{\Phi(\beta_e(-\sigma',m))}) = \delta_{\sigma',\sigma} \cdot \alpha(e) = \delta_{\sigma',\sigma} \cdot (\alpha|_{\{e\}} \cdot \delta_e)^{\#}_1.$$

Proof. By corollary 4.2.3, let μ be a subsequential limit of $(\Delta^{\sigma'}_m \cdot E_\sigma)$. By proposition 5.3.3, corollary 6.1.11 (i), proposition 4.4.1 and proposition 4.4.2,

$$\operatorname{supp} \mu \subset \operatorname{sing} E_\sigma \cap \operatorname{supp}^\infty(\Delta^{\sigma'}_m) \subset N^K \cap K = \{e\}.$$

So, by [Eym64, (4.9) théorème],

$$\mu = \zeta \cdot \delta_e \quad \text{for some } \zeta \in \mathbf{C}.$$

Thus

$$\zeta = \zeta \cdot \Delta^{\sigma'}_2(e) = \lim_m \langle \Delta^{\sigma'}_2 \cdot \Delta^{\sigma'}_{\gamma(m)} \,:\, E_\sigma \rangle = \lim_m E^{\#}_\sigma (\Delta^{\sigma'}_2 \cdot \Delta^{\sigma'}_{\gamma(m)})^\vee(e) = \delta_{\sigma',\sigma},$$

proving the assertion. ■

COROLLARY 6.2.3. *For every $g \in G$,*

$$\varrho_g : \mathcal{T}_\pm(G) \to \mathbf{C} : \mathrm{T}_\pm(f) \mapsto f(g)$$

defines a non-trivial character vanishing on $\mathcal{LC}(\mathbf{H}^2(G^{\mathbf{C}}_\pm))$.

Proof. This follows by applying corollary 4.5.13 to theorem 6.2.2. ■

In the proof of theorem 6.2.2, all that was needed to identify the limit of $(\Delta_m^{\sigma'} \cdot E_\sigma)$ were Eymard's version of Beurling's theorem and the decomposition of the Hardy space. As we shall see, the situation for the 1-dimensional faces is more involved.

6.3. *Convergence of states defined by the* 1*-dimensional faces.* In this section, we shall study the behaviour of $\Delta_{v_m} \cdot E_\sigma$ at infinity, where

$$v_m = \beta_N(\bar\nu, m) = \begin{pmatrix} 0 & m^2/\bar\nu \\ -\bar\nu & 0 \end{pmatrix}.$$

These sequences are precisely the points of intersection of the parallel translations of $\mathfrak{n}$ by $\bar\nu \cdot X_-$ with the 'integral orbits' $\Omega_{m^2}^{\operatorname{sgn}\bar\nu}$. As we shall see, they give rise exactly to the Fourier coefficients $(\bar E_\sigma)^\#_{\bar\nu}$ of the distribution $\bar E_\sigma \in \mathrm{W}^*(N)$ defining the projection of $\mathbf{L}^2(N)$ onto the Hardy space $\mathbf{H}^2(N_\sigma^{\mathbf{C}})$ localized at the character $e^{-i\bar\nu/2\cdot\diamond}$ of N.

Notation 6.3.1. For the sake of brevity, we write $\Delta_m^{\bar\nu} := \Delta_{\beta_N(\bar\nu,m)}$. Also, we use the notation $\operatorname{acc}(\mu_j)$ for the set of $\sigma(\mathrm{W}^*(G),\mathrm{A}(G))$-accumulation points of $(\mu_j) \subset \mathrm{W}^*(G)$.

The proof of our convergence result requires some preparation.

Proposition 6.3.2. *Let $\bar\nu \neq 0$. If $\mu^{\bar\nu,\sigma} \in \operatorname{acc}(\Delta_m^{\bar\nu} \cdot E_\sigma)$ then*

$$\langle \alpha \,:\, \mu^{\bar\nu,\sigma}\rangle = \delta_{\sigma,\operatorname{sgn}\bar\nu} \cdot \alpha(e) \quad \textit{for any } \alpha^\vee \in \mathrm{A}(G) \cap \mathbf{H}^2(G^{\mathbf{C}}_{\operatorname{sgn}\bar\nu}).$$

In particular, this is true for $\alpha = \Delta_v$ where $v \in \Omega_{m^2}^{\operatorname{sgn}\bar\nu}$ for some $\mathbf{N} \ni m \geq 2$.

Proof. Choose a subsequence $\gamma \prec \mathbf{N}$ associated to $\mu^{\bar\nu,\sigma}$. We have

$$\begin{aligned}\langle \alpha \,:\, \mu^{\bar\nu,\sigma}\rangle &= \lim_m \langle \alpha \cdot \Delta^{\bar\nu}_{\gamma(m)} \,:\, E_\sigma\rangle = \lim_m E_\sigma^\#(\alpha\cdot\Delta^{\bar\nu}_{\gamma(m)})^\vee(e) \\ &= \delta_{\sigma,\bar\nu}\cdot\alpha(e)\cdot \lim_m \Delta^{\bar\nu}_{\gamma(m)}(e) = \delta_{\sigma,\bar\nu}\cdot\alpha(e),\end{aligned}$$

since $(\alpha\cdot\Delta^{\bar\nu}_{\gamma(m)})^\vee \in \mathbf{H}^2(G^{\mathbf{C}}_{\operatorname{sgn}\bar\nu})$, and $E_\sigma^\#$ is the orthogonal projection onto $\mathbf{H}^2(G^{\mathbf{C}}_\sigma)$. ∎

Proposition 6.3.3. *Let $y, (y_m) \subset \mathbf{R}\backslash\{0\}$ such that $y = \lim_m y_m$ and $v_m \in \Omega^\sigma_{m^2} \cap \mathfrak{n}\oplus\bar{\mathfrak{n}}$ such that $\operatorname{pr}_{\bar{\mathfrak{n}}} v_m = y_m$. (In particular, $\sigma\cdot y > 0, \sigma\cdot y_m > 0$.)*

(i) *For all $x \in \mathbf{R}$, we have $\lim_m \|n_x^{\pi_m^\sigma} k^{m,\sigma}_{\Phi(v_m)} - e^{-iyx/2}k^{m,\sigma}_{\Phi(v_m)}\|_{\mathcal{O}^2_m(\sigma\Pi)} = 0$.*

(ii) *For all $x \in \mathbf{R}$, we have $\lim_m \|\Delta_{v_m}(n_x\diamond) - e^{-iyx/2}\Delta_{v_m}\|_{\mathrm{A}(G)} = 0$.*

(iii) *We have $e^{-iy/2\cdot\diamond} = \lim_m \Delta_{v_m}|_N$ strictly in $\mathrm{B}(N)$.*

Proof. (i) Since π_m^σ is a unitary representation of G, we obviously have

$$\begin{aligned}\|n_x^{\pi_m^\sigma} k^{m,\sigma}_{\Phi(v_m)} - e^{-iyx/2}k^{m,\sigma}_{\Phi(v_m)}\|^2_{\mathcal{O}^2_m(\sigma\Pi)} &= 2 - 2\operatorname{Re} e^{iyx/2}(k^{m,\sigma}_{\Phi(v_m)} \,|\, n_x^{\pi_m^\sigma} k^{m,\sigma}_{\Phi(v_m)}) \\ &= 2 - 2\operatorname{Re} e^{iyx/2}\Delta_{v_m}(n_x).\end{aligned}$$

The assertion (i) follows since $\Phi(v_m) = i\frac{m}{y_m}$, so we have

$$\Delta_{v_m}(n_x) = \left(1 + \frac{iy_m x}{2m}\right)^{-m} \xrightarrow{m} e^{-iyx/2}.$$

(ii) Since $\mathrm{A}(G)$ carries the norm induced by $\mathrm{B}(G)$ and

$$\Delta_{v_m}(n_x s) - e^{-iyx/2}\Delta_{v_m}(s) = (n_{-x}^{\pi_m^\sigma} k^{m,\sigma}_{\Phi(v_m)} - e^{iyx/2}k^{m,\sigma}_{\Phi(v_m)} \,|\, s^{\pi_m^\sigma} k^{m,\sigma}_{\Phi(v_m)}),$$

we have, by [Eym64, (2.14) lemme],

$$\|\Delta_{v_m}(n_x\diamond) - e^{-iyx/2}\Delta_{v_m}\|_{\mathrm{A}(G)} \leq \|n_{-x}^{\pi_m^\sigma} k^{m,\sigma}_{\Phi(v_m)} - e^{iyx/2}k^{m,\sigma}_{\Phi(v_m)}\|_{\mathcal{O}^2_m(\sigma\Pi)} \xrightarrow{m} 0.$$

(iii) From the proof of (i) we have point-wise convergence on N. Since $e^{-iy/2\cdot\diamond}$ is a unitary character of N, it is contained in $\mathbf{S}(\mathrm{B}(N))_+$. By theorem 4.1.1 (i), $(\Delta_{v_m}|_N) \subset \mathrm{A}(N)$. Since the inclusion $N \hookrightarrow G$ is a homomorphism, these functions are of positive type. Evaluation at the neutral element e shows their norm is 1. On norm bounded subsets of $\mathrm{B}(N)$, convergence a.e. coincides with convergence in $\sigma(\mathrm{B}(N), \mathbf{L}^1(N))$ which in turn coincides with $\sigma(\mathrm{B}(N), \mathrm{C}^*(N))$ on bounded subsets since $\mathbf{L}^1(N) \subset \mathrm{C}^*(N)$ is dense. By [GL81, theorem B$_2$], the latter topology coincides with the strict topology on the unit sphere, whence the assertion. ∎

Lemma 6.3.4.

(i) *Let $m \in \mathbf{N}$ and $w \in \sigma\Pi$. For any $g \in G$, there is $u^\sigma_{g,w} \in \mathrm{U}(1)$ such that*

$$g^{\pi^\sigma_m} k^{m,\sigma}_w = u^\sigma_{g,w} \cdot k^{m,\sigma}_{g.w}. \tag{6.3.1}$$

If $g \in MAN$, we even have $u^\sigma_{g,w} = 1$.

(ii) *Let $v \in \Lambda$. We have*

$$\Delta_v \circ \mathrm{Int}(g) = \Delta_{\mathrm{Ad}(g^{-1})v} \qquad \textit{for all } g \in G \tag{6.3.2}$$

where $\mathrm{Int}(g)(t) = gtg^{-1}$ *for all* $t \in G$.

Proof. (i) For all $z \in \sigma\Pi$, we have

$$\begin{aligned} g^{\pi^\sigma_m} K^{m,\sigma}_w(z) &= (\partial g^{-1})^{m/2}(z) \cdot K^{m,\sigma}(g^{-1}z, w) \\ &= [\overline{(\partial g^{-1})^{m/2}(w)}]^{-1} \cdot K^{m,\sigma}(z, g.w) = [\overline{(\partial g^{-1})^{m/2}(w)}]^{-1} \cdot K^{m,\sigma}_{g.w}(z), \end{aligned}$$

hence $g^{\pi^\sigma_m} k^{m,\sigma}_w = C \cdot k^{m,\sigma}_{g.w}$ for some constant $C \in \mathbf{C}$. Since $g^{\pi^\sigma_m}$ is unitary and the ks are unit vectors, C has modulus 1.

Finally, for $g \in MAN$, one easily sees that $(\partial g)^{m/2}(w)$ is a positive number, so that C is also a positive number, and hence equals 1.

(ii) Let $m \in \mathbf{N}$, such that $v \in \Omega^\sigma_{m^2}$. Then, by (6.3.1)

$$\Delta_v(\mathrm{Int}(g^{-1})t) = (g^{\pi^\sigma_m} k^{m,\sigma}_{\Phi(v)} \,|\, t^{\pi^\sigma_m} g^{\pi^\sigma_m} k^{m,\sigma}_{\Phi(v)}) = (k^{m,\sigma}_{g.\Phi(v)} \,|\, t^{\pi^\sigma_m} k^{m,\sigma}_{g.\Phi(v)}) = \Delta_{\mathrm{Ad}(g)v}(t)$$

for all $g, t \in G$ since $g \circ \Phi = \Phi \circ \mathrm{Ad}(g)$ by proposition 5.1.2. ∎

Now we are ready to prove the following theorem.

Theorem 6.3.5. *Let $\bar\nu \neq 0$ and $\sigma' := \operatorname{sgn} \bar\nu$. Denote by $\bar E_\sigma \in \mathrm{W}^*(N)$ the orthogonal projection of $\mathbf{L}^2(N)$ onto the Hardy space $\mathbf{H}^2(N^{\mathbf{C}}_\sigma)$. Then*

$$\lim\nolimits_m (k^{m,-\sigma'}_{\Phi(\beta_N(-\bar\nu,m))} \,|\, (\alpha \cdot E_\sigma)^{\#}_{\pi^{-\sigma'}_m} k^{m,-\sigma'}_{\Phi(\beta_N(-\bar\nu,m))}) = (\alpha|_N \cdot \bar E_\sigma)^{\#}_{e^{i\bar\nu/2\cdot\diamond}} \quad \textit{for all } \ \alpha \in \mathrm{A}(G).$$

Proof. Let $\mu^{\bar\nu,\sigma} \in \mathrm{acc}(\Delta^{\bar\nu}_{\gamma(m)} \cdot E_\sigma)$ and choose a corresponding subsequence $\gamma \prec \mathbf{N}$ by proposition 4.2.3. By proposition 5.3.3, corollary 6.1.11 (ii), proposition 4.4.1 and proposition 4.4.2,

$$\operatorname{supp} \mu^{\bar\nu,\sigma} \subset \operatorname{sing} E_\sigma \cap \operatorname{supp}^\infty(\Delta^{\bar\nu}_m) \subset N^K \cap MAN = N.$$

Set $\mu := e^{i\bar\nu/2\cdot\diamond} \cdot \mu^{\bar\nu,\sigma}_N \in \mathrm{W}^*(N)$. First, assume $\sigma' = \sigma$. By proposition 6.3.3,

$$\langle \Delta^{\bar\nu}_2 \,:\, \mu^{\bar\nu,\sigma} \rangle = \lim\nolimits_m \langle \Delta^{\bar\nu}_2 \cdot \Delta^{\bar\nu}_{\gamma(m)} \,:\, E_\sigma \rangle = 1,$$

so $\|\mu\| = \|\mu^{\bar\nu,\sigma}\| = 1$. Hence there exists a unique class $f \in \mathbf{L}^\infty(\mathbf{R})$ such that

$$\mu = (n_\diamond)_\circ(\mathcal{F}^{-1})'(f).$$

Here $\mathcal{F}$ is the Fourier transform and $_\circ$ denotes image measure.

We have $f \geq 0$ a.e. and $\|f\|_\infty = 1$. In fact, f is a.e. constant on rays. Indeed, for all $\lambda > 0$, we have $\Delta_m^{\bar\nu} \circ \mathrm{Int}(h_\lambda) = \Delta_m^{\bar\nu/\lambda^2}$. By lemma 6.3.4 (ii) and centrality of E_σ,

$$\mathrm{Int}(h_\lambda)_\circ \mu^{\bar\nu,\sigma} = \lim_m \Delta_{\gamma(m)}^{\bar\nu/\lambda^2} \cdot E_\sigma.$$

Applying corollary 4.3.6, we get

$$e^{-i\bar\nu/2\cdot\diamond} \cdot \mathrm{Int}(h_\lambda)_\circ \mu_N^{\bar\nu,\sigma} = e^{-i\bar\nu/(2\lambda^2)\cdot\diamond} \cdot \mu_N^{\bar\nu,\sigma},$$

i.e. $\mathrm{Int}(h_\lambda)_\circ(\mu) = \mu$. Since, for all $\varphi \in \mathrm{A}(N)$, we have

$$\langle \varphi : \mathrm{Int}(h_\lambda)_\circ \mu\rangle = \frac{1}{2\pi}\int_{\mathbf{R}} f(y) \int_{\mathbf{R}} e^{2\pi i x y} \varphi(n_{\lambda^2 x})\, dx\, dy = \frac{1}{2\pi}\int_{\mathbf{R}} f(\lambda^2 y) \int_{\mathbf{R}} e^{2\pi i x y}\varphi(n_x)\, dx\, dy,$$

the function f is a.e. constant on rays.

Now, since $\operatorname{supp} \mathcal{F}^{-1}(\Delta_2^{\bar\nu} \circ n_\diamond) = \sigma'\mathbf{R}_+$ (cf. [Bil79, exercises 20.23 and (26.8)]) and $\|f\|_\infty = 1$, we deduce $f = 1_{\sigma'\mathbf{R}_+}$ a.e., i.e. $\mu = \bar{E}_\sigma$.

Now, let $\sigma' \neq \sigma$. By the first part of the proof,

$$e^{i\bar\nu/2\cdot\diamond} \cdot \bar{E}_\sigma = \lim_m \Delta_m^{-\bar\nu} \cdot E_\sigma \quad \text{in } \sigma(\mathrm{W}^*(G), \mathrm{A}(G)).$$

By corollary 4.3.6,

$$\mu = e^{i\bar\nu/2\cdot\diamond} \cdot \mu_N^{\bar\nu,\sigma} = e^{-i\bar\nu/2\cdot\diamond} \cdot e^{i\bar\nu/2\cdot\diamond} \cdot \bar{E}_\sigma = \bar{E}_\sigma.$$

The theorem is proven. ∎

COROLLARY 6.3.6. *For any $k, k' \in K$,*

$$\varrho_{kNk'^{-1}} : \mathcal{T}_\pm(G) \to \mathcal{T}_\pm(N) : \mathrm{T}_\pm(f) \mapsto \mathrm{T}_\pm(k * f * k'|_N)$$

defines an irreducible $*$-representation of $\mathcal{T}_\pm(G)$ on $\mathbf{H}^2(N_\pm^{\mathbf{C}})$ vanishing on $\mathcal{LC}(\mathbf{H}^2(G_\pm^{\mathbf{C}}))$.

Proof. This follows by applying corollary 4.5.13 to theorem 6.2.2. ∎

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