ON CANONICAL REPRESENTATIONS
AND BEREZIN KERNELS

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1. Introduction. These are introductory lectures on canonical representations. The lectures aim to be an advertisement for a new kind of representations. Harmonic analysis is mainly concerned with L^2 Fourier analysis, so with the decomposition of regular representations. Canonical representations are different, and are to be regarded as quantizations of regular representations. They were introduced by Berezin in or about 1975 ([2], [3], [4], [5]), as well as by Gelfand, Graev and Vershik, from a different point of view, ([50], [51]). Their study is now a hot topic, mainly because of the relation with the decomposition of the tensor product of a holomorphic and anti-holomorphic discrete series representation. We restrict because of the introductory nature to the group $SU(1,1)$. At the end of the lectures we give some literature (included in the references) for further reading.

2. Unit disc and upper half plane. Let $G = SU(1,1)$ be the group of complex 2×2-matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

satisfying $|\alpha|^2 - |\beta|^2 = 1$. It acts transitively on the unit disc $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ by

$$g \cdot \zeta = \frac{\alpha\zeta + \beta}{\overline{\beta}\zeta + \overline{\alpha}}.$$

The stabilizer of $\zeta = 0$ is the subgroup

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

so $D \simeq G/K$. We also consider the group $G_o = SL(2, \mathbb{R})$, acting on the Poincaré upper half plane $\mathbb{C}^+ = \{z = x + iy \, : \, y > 0\}$ by

$$g \cdot z = \frac{az + b}{cz + d}$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The stabilizer of $z = i$ is given by

$$K_o = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \, : \, \theta \in \mathbb{R} \right\},$$

and hence $\mathbb{C}^+ \simeq G_o/K_o$.

   The Cayley transform $c : z \to \dfrac{z - i}{z + i}$ transforms $\mathbb{C}^+$ bijectively onto D. Let C be the 2×2 matrix

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Then $G = C\,G_o\,C^{-1}$, and also, as transformation groups, $G = c\,G_0\,c^{-1}$. We call D a *bounded realization* of G/K and $\mathbb{C}^+$ a *tube realization* of G/K.

   3. Some subgroups of G and G_o. Set

$$A_o = \left\{ a_t = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix} \, : \, t \in \mathbb{R} \right\}, \quad N_o = \left\{ n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \, : \, \xi \in \mathbb{R} \right\}.$$

These are subgroups of G_o and $G_o = K_o A_o N_o = A_o N_o K_o$ (Iwasawa decomposition). The corresponding subgroups of G are:

$$A = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \, : \, t \in \mathbb{R} \right\} \text{ and } N = \left\{ \begin{pmatrix} 1 + \frac{i\xi}{2} & -\frac{i\xi}{2} \\ \frac{i\xi}{2} & 1 - \frac{i\xi}{2} \end{pmatrix} \, : \, \xi \in \mathbb{R} \right\}.$$

   4. The invariant measures and the Laplace-Beltrami operators on D and $\mathbb{C}^+$. $D \simeq G/K$ and $\mathbb{C}^+ \sim G_o/K_o$ carry invariant measures. Let us start with $\mathbb{C}^+$. Taking $dg = dt d\xi dk_o$ as Haar measure on G_o ($g = a_t n_\xi k$), dk_o being the normalized Haar measure on K_o, we get for the G_o-invariant measure $d\mu$ on $\mathbb{C}^+$:

$$d\mu(z) = \frac{1}{2} \frac{dxdy}{y^2}$$

if $z = x + iy$. Similarly on D:

$$d\nu(\zeta) = 2\,(1 - |\zeta|^2)^{-2}\,dudv$$

where $\zeta = u + iv$, is the corresponding G-invariant measure.

   The Laplace-Beltrami operator on $\mathbb{C}^+$ is given by $y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, and accordingly on $\mathcal{D}$ by the formula $(1 - |\zeta|^2)^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} = 4v^2 \, (\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})$.

   5. Berezin kernel, canonical representation. Set for $\lambda \in \mathbb{R}$, $g \in G$,

$$\psi_\lambda(g) = (1 - |\zeta|^2)^\lambda$$

if $\zeta = g \cdot 0$, $\zeta \in D$.

Clearly ψ_λ is a bi-K-invariant continuous function on G. An easy computation shows:

$$\psi_\lambda(g_1^{-1}g_2) = \left\{ \frac{(1-|\zeta|^2)(1-|\eta|^2)}{(1-\zeta\overline{\eta})(1-\eta\overline{\zeta})} \right\}^\lambda$$

if $\zeta = g_1 \cdot 0$, $\eta = g_2 \cdot 0$. Let us denote this expression by $B_\lambda(\eta, \zeta)$. B_λ is called a *Berezin kernel* on D. Clearly B_λ is G-invariant by construction: $B_\lambda(g \cdot \eta, g \cdot \zeta) = B_\lambda(\eta, \zeta)$ for all $g \in G$.

A kernel $\mathcal{K}(\eta, \zeta)$ on $\mathcal{D} \times \mathcal{D}$ is said to be positive-definite if

$$\sum_{i=1}^N \mathcal{K}(\eta_i, \eta_j)\, \lambda_i \overline{\lambda}_j \geq 0$$

for all finite sets of elements $\eta_1, \ldots, \eta_N$ in $\mathcal{D}$ and complex numbers $\lambda_1, \ldots, \lambda_N$. If $\mathcal{K}$ is continuous, this property is well-known (and easily shown by approximating with Dirac measures) to be equivalent to

$$\int_{\mathcal{D}} \int_{\mathcal{D}} \mathcal{K}(\eta, \zeta)\, f(\zeta)\, \overline{f(\eta)} d\nu(\zeta) d\nu(\eta) \geq 0$$

for all functions $f \in C_c^\infty(\mathcal{D})$, the space of C^∞-functions on $\mathcal{D}$ with compact support.

Since products and (uniform) limits of positive-definite kernels are again positive-definite, we easily get ,by expanding $(1-\zeta\overline{\eta})^{-\lambda}$ into a power series:

$$(1-\zeta\overline{\eta})^{-\lambda} = \sum_{m=0}^\infty \binom{-\lambda}{m} \zeta^m\, \overline{\eta}^m\, (-1)^m$$

that B_λ is a positive-definite kernel for $\lambda \geq 0$. Or, otherwise said, ψ_λ is a positive-definite function for $\lambda \geq 0$:

$$\int_G \int_G \psi_\lambda(g_2^{-1}g_1)\, f(g_1)\, \overline{f(g_2)} dg_1 dg_2 \geq 0$$

for all $f \in C_c^\infty(G/K)$, the space of C_c^∞-functions on G, which are right-invariant with respect to the subgroup K.

Let τ_λ denote the unitary representation, naturally associated with ψ_λ or B_λ. Its definition is straightforward. The Hilbert space is the completion of the pre-Hilbert space of functions of the form $f * \psi_\lambda$ with $f \in C_c^\infty(G/K)$ and inner product

$$\langle f_1 * \psi_\lambda,\, f_2 * \psi_\lambda \rangle = \int_G \int_G \psi_\lambda(g_2^{-1}g_1)\, f_1(g_1)\overline{f(g_2)} dg_1 dg_2$$

This is a well-defined expression, which produces a genuine inner product. The right-hand side can also be written as

$$\int_G \psi_\lambda(g)\, (\widetilde{f}_2 * f_1)(g)\, dg$$

where $\widetilde{f}_2(x) = \overline{f_2(x^{-1})}$, $x \in G$. The G-action is simply left translation.

We call the τ_λ $(\lambda \geq 0)$ *canonical representations* after Gelfand, Graev and Vershik (1975), cf. [50], [51]. We shall study these representations; in particular their behaviour for $\lambda \to \infty$ and their decomposition into irreducible representations.

6. The spherical Fourier transform of ψ_λ. For $s \in \mathbb{C}$ let

$$\varphi_s(g) = \int_K e^{(-s+1)t(kg)} dk \quad (g \in G)$$

where $t(g)$ is given by the Iwasawa decomposition $g = a_{t(g)} nk$. Then φ_s is a zonal spherical function with the parametrization of Harish-Chandra.

We also need Harish-Chandra's c-function:

$$c(s) = 2^{1-s} \frac{\Gamma(s)}{\Gamma(\frac{s+1}{2})^2}.$$

Let ψ_λ be as in Section 5. It easily follows from the explicit form of the invariant measure on D that $\psi_\lambda \in L^1(G) \cap L^2(G)$ for $\lambda > 1$ (even for $\mathrm{Re}\,\lambda > 1$, though ψ_λ was a priori defined for real λ only).

To decompose the canonical representation τ_λ into irreducible unitary representations for $\lambda > 1$, it suffices to determine the inverse spherical Fourier transform of ψ_λ:

$$\psi_\lambda(g) = \frac{1}{\pi^2} \int_0^\infty a_\lambda(i\mu)\, \varphi_{i\mu}(g)\, \frac{d\mu}{|c(i\mu)|^2}$$

in L^2-sense, where $a_\lambda(i\mu)$ is the spherical Fourier transform of ψ_λ:

$$a_\lambda(i\mu) = \int_G \psi_\lambda(g)\, \varphi_{-i\mu}(g)\, dg.$$

It will turn out that $a_\lambda(i\mu)$ is also of importance for $0 < \lambda \le 1$.

7. Computation of $a_\lambda(i\mu)$. Notice that

$$a_\lambda(i\mu) = \int_G \psi_\lambda(g)\, e^{(+i\mu+1)t(g)} dg$$

and that this is actually an integral over G/K. This integral is most easily computed in the tube realization of G/K, namely on $\mathbb{C}^+$.

Then

$$\psi_\lambda(z) = (1 - |cz|^2)^\lambda = \left(\frac{4y}{x^2 + (y+1)^2} \right)^\lambda$$

if $z = x + iy$, $y > 0$. Here c is the Cayley transform.

Set $\Delta(z) = y$. Then $\Delta(n_\xi \cdot z) = \Delta(z)$ and $\Delta(a_t \cdot z) = e^{2t}\, \Delta(z)$. Therefore

$$
\begin{aligned}
a_\lambda(i\mu) &= \int_{\mathbb{C}^+} \psi_\lambda(z)\, \Delta(z)^{\frac{1}{2}i\mu + \frac{1}{2}} d\mu(z) \\
&= \pi\, \frac{\Gamma(\lambda + \frac{i\mu-1}{2})\, \Gamma(\lambda + \frac{-i\mu-1}{2})}{\Gamma(\lambda)^2}.
\end{aligned}
$$

For a computation, applying the Cartan decomposition $G = KAK$, see [10]. This computation relies on formulae in [20].

8. Behaviour as $\lambda \to \infty$. The following is well known (see [20] p. 47, (1)):

(8.1) $\qquad \dfrac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \left\{ 1 + \dfrac{1}{2z}\, (a-b)(a+b-1) + \mathcal{O}(z^{-2}) \right\}$ as $z \to \infty$.

This leads to:

$$\frac{\lambda - 1}{\pi}\, a_\lambda(i\mu) \sim 1 + \frac{-\mu^2 - 1}{4\lambda} + \cdots \ \ (\lambda \to \infty).$$

So, up to a *right normalization*, τ_λ tends to the usual regular representation of G on $G/K \simeq D$. The normalization factor is $\frac{\lambda-1}{\pi}$.

Indeed, we also have, because $\square \varphi_s = (s^2 - 1)\varphi_s$,

$$(8.2) \qquad\qquad \frac{\lambda - 1}{\pi}\, \psi_\lambda = \delta + \frac{\square \delta}{4\lambda} + \mathcal{O}(\frac{1}{\lambda^2}),$$

where $\square$ is the Laplace-Beltrami operator, and thus obtain for all f_1, $f_2 \in C_c^\infty(G/K)$ that

$$\frac{\lambda - 1}{\pi} \int_G \psi_\lambda(g)\, (\widetilde{\widetilde{f}}_2 * f_1)(g)dg$$

tends to

$$\int_G \delta(g)\, (\widetilde{f} * f_1)((g)dg = \int_G f_1(g)\overline{f_2(g)}dg,$$

which is the usual L^2-inner product of f_1 and f_2 on G/K.
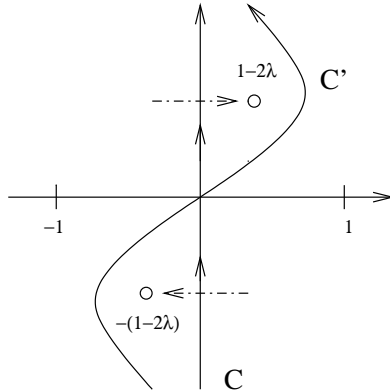
9. Decomposition of τ_λ. Set

$$a_\lambda(s) = \pi\, \frac{\Gamma(\lambda + \frac{s-1}{2})\,\Gamma(\lambda + \frac{-s-1}{2})}{\Gamma(\lambda)^2}.$$

Then we can write for $\operatorname{Re}\lambda > 1$:

$$(9.1) \qquad\qquad \psi_\lambda = \frac{1}{2i\pi^2} \int_C a_\lambda(s)\, \varphi_s\, \frac{ds}{c(s)\, c(-s)},$$

where C is the line $\operatorname{Re} s = 0$ in the complex plane, taken from below upwards. The integrand in (9.1) has, as a function of s, poles depending on λ at $s = \pm(1 - 2\lambda - 2k)$, $k = 0, 1, 2, \ldots$. So (9.1) even holds for $\operatorname{Re}\lambda > 1/2$.

For $\operatorname{Re}\lambda > 1/2$ we may shift C to C' (see figure) and the resulting integral is analytic in λ for $\operatorname{Re}\lambda \to 0$. When $\operatorname{Re}\lambda \to 0$, $\pm(1 - 2\lambda)$ pass C.

When shifting C' to C again, we meet the poles $\pm(1 - 2\lambda)$, but only for $0 < \operatorname{Re}\lambda < 1/2$.

(The case $\operatorname{Re}\lambda = 1/2$ causes trouble). The residue of $\dfrac{a_\lambda(s)}{c(s)\,c(-s)}$ at $1-2\lambda$ is equal to

$$r(\lambda) = \frac{\pi}{2}\,\frac{\Gamma(1-\lambda)^2}{\Gamma(1-2\lambda)}.$$

So we get for $0 < \operatorname{Re}\lambda < 1/2$:

$$(9.2) \qquad \psi_\lambda = \frac{1}{\pi^2}\int_0^\infty a_\lambda(i\mu)\,\varphi_{i\mu}\,\frac{d\mu}{|c(i\mu)|^2} + \frac{\Gamma(1-\lambda)^2}{\Gamma(1-2\lambda)}\,\varphi_{1-2\lambda}$$

in the sense of distributions.

Let us now consider the case $\lambda = 1/2$ (only). Then we would like to take the limit in (9.1) from the right and in (9.2) from the left. This is allowed and gives no residue. Of course, the integrand $\dfrac{a_\lambda(s)}{c(s)\,c(-s)}$ at first sight has a singularity at $s=0$; $a_\lambda(s)$ has a double pole, $\dfrac{1}{c(s)\,c(-s)}$ a double zero, so actually everything is in order.

The spherical function $\varphi_{1-2\lambda}$ is positive-definite and corresponds to the *complementary series* of G, see next section, for $\lambda \in \mathbb{R}$, $0 < \lambda < 1/2$.

Observe that (9.2) also holds for $\lambda = 0$.

A similar calculation as the above gives the analytic continuation beyond $\lambda = 0$, namely for $\lambda < 0$. If

$$-\frac{1}{2} - k < \operatorname{Re}\lambda < \frac{1}{2} - k,\ \ k = 0,1,2,\ldots$$

then we get $k+1$ extra terms $R_j(\lambda)\,\varphi_{1-2\lambda-2j}$ with

$$R_j(\lambda) = \frac{\Gamma(1-\lambda)^2 \cdot (1-2\lambda-2j)}{j!\,\Gamma(2-2\lambda-j)}$$

$(j = 0,\ldots,k)$. In other words: if $-\frac{1}{2} - k < \operatorname{Re}\lambda < \frac{1}{2} - k$, then

$$(9.3) \qquad \psi_\lambda = \frac{1}{\pi^2}\int_0^\infty a_\lambda(i\mu)\,\varphi_{i\mu}\,\frac{d\mu}{|c(i\mu)|^2} + \sum_{j=0}^{k} R_j(\lambda)\,\varphi_{1-2\lambda-2j}.$$

Of course unitarity of τ_λ fails for negative λ. It would be interesting to give a meaning to the limit for λ integer and $\lambda \to -\infty$. Observe that the integral in (9.3) disappears for λ a negative integer. It is plausible that the (normalized) limit gives the Plancherel formula for $\mathrm{SU}(2)/K$. Might we say that the τ_λ yield an interpolation between $L^2(\mathrm{SU}(2)/K)$ and $L^2(\mathrm{SU}(1,1)/K)$? [1]

10. Complementary series representations; a model. Let U denote the unit circle in $\mathbb{C}$, the boundary of D. G acts on U by fractional linear transformations:

$$g\cdot\omega = \frac{\alpha\omega + \beta}{\overline{\beta}\omega + \overline{\alpha}}$$

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[1]This question is very recently settled by Yu. A. Neretin in his preprint: *Matrix balls, radial analysis of Berezin kernels, and hypergeometric determinants*, http://xxx.lanl.gov/ math.RT/0012200. A careful analysis of the asymptotic behaviour for $\lambda \to -\infty$ shows that even the "correspondence principle", i.e. the analogue of formula (8.2) holds. See a forthcoming note by van Dijk and Molchanov. We will return to this matter in Section 14.

if $g = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$, $\omega \in U$.

For $s \in \mathbb{C}$ we define a representation π_s^c of G acting on $C(U)$:

$$\pi_s^c(g) f(\omega) = |\overline{\beta}\omega + \overline{\alpha}|^{-(s+1)} f(g^{-1} \cdot \omega)$$

if $g^{-1} = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$.

We provide $C(U)$ with the sesquilinear form

$$(f,\, g)_s = \int_U \int_U |1 - \omega\overline{\omega}'|^{s-1} f(\omega) \, \overline{g(\omega')} \, d\omega d\omega' \quad (s > 0)$$

where $d\omega$ is $\frac{1}{2\pi} d\theta$ if $\omega = e^{i\theta}$. Then $d\omega$ is quasi-invariant with respect to the action of G on U:

$$\int_U f(g \cdot \omega)\, d\omega = \int_U f(\omega) |\overline{\beta}\omega + \overline{\alpha}|^{-2} d\omega.$$

Thus, $(\,,\,)_s$ is G-invariant with respect to π_s^c. It turns out that $(\,,\,)_s$ is positive-definite for $0 < s < 1$. The corresponding series of (irreducible) unitary representations is called the *complementary series* of G.

REMARK. With respect to the usual L^2-inner product on $C(U)$, we get the *spherical principal series* π_s. For $s \in i\mathbb{R}$, π_s is unitary and irreducible.

11. The holomorphic and anti-holomorphic discrete series; canonical representations as tensor products.

For $\lambda \in \mathbb{R}$ consider the *Fock space* $\mathcal{F}_\lambda$ of holomorphic functions on D satisfying

$$\|f\|_\lambda^2 = \int_D |f(\zeta)|^2 (1 - |\zeta|^2)^\lambda \, d\nu(\zeta) < \infty.$$

This space is non-trivial for $\lambda > 1$, since $\mathcal{F}_\lambda$ contains the function which is identically 1. One has

$$\|1\|_\lambda^2 = \frac{\pi}{\lambda - 1}.$$

$\mathcal{F}_\lambda$ is a closed subspace of $L^2(D, d\nu_\lambda)$, hence a Hilbert space, where $d\nu_\lambda(\zeta) = (1 - |\zeta|^2)^\lambda \, d\nu(\zeta)$. It has a reproducing kernel, namely

$$E_\lambda(\zeta,\, \eta) = \frac{\lambda - 1}{\pi} (1 - \zeta\overline{\eta})^{-\lambda}.$$

It is also a unitary module for the action of the universal covering group $\widetilde{G}$ of G; for integer $\lambda > 1$ it is even a G-module: a *holomorphic discrete series* representation. The group G acts by

$$\pi_\lambda^d(g) f(\zeta) = f\left(\frac{\alpha\zeta + \beta}{\overline{\beta}\zeta + \overline{\alpha}}\right) (\overline{\beta}\zeta + \overline{\alpha})^{-\lambda}$$

if $g^{-1} = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$.

Let us denote by $\overline{\mathcal{F}}_\lambda$ the space of complex conjugates of elements of $\mathcal{F}_\lambda$. It consists of anti-holomorphic functions and gives rise to an obvious unitary action $\overline{\pi}_\lambda^d$ of $\widetilde{G}$ as well. So

$$\overline{\pi}_\lambda^d(g) f(\zeta) = f\left(\frac{\alpha\zeta + \beta}{\overline{\beta}\zeta + \overline{\alpha}}\right) (\beta\overline{\zeta} + \alpha)^{-\lambda}$$

if $g^{-1} = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix} \right)$, $f \in \overline{\mathcal{F}}_\lambda$, $\lambda > 1$. For $\lambda \in \mathbb{N}$, $\lambda > 1$ we get the *anti-holomorphic discrete series* of G.

Consider the Hilbert space tensor product $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ with $\lambda > 1$. The group $\widetilde{G}$ acts diagonally. It turns out that we actually have a G-action, which for integer λ is given by

$$g_o \cdot F(\zeta, \eta) = F(g_o^{-1} \cdot \zeta, g_o^{-1} \cdot \eta) \, (\overline{\alpha} + \overline{\beta}\zeta)^{-\lambda} \, (\alpha + \beta\overline{\eta})^{-\lambda}$$

if $g_o^{-1} = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix} \right)$.

Let A_λ denote the linear map, defined on $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ by

$$F(\zeta, \eta) \to F(\zeta, \zeta) \, (1 - |\zeta|^2)^\lambda.$$

A_λ is a bounded linear operator $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda \to L^2(D, d\nu)$ where $L^2(D, d\nu)$ is considered as a G-space in the natural way. A_λ intertwines the G-actions. Let us show that A_λ is bounded.

Let $F(\zeta, \eta) \in \mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$. Then, according to the reproducing property,

$$F(\zeta, \eta) = \int_D E_\lambda(w, \eta) \, F(\zeta, w) \, d\nu_\lambda(w),$$

therefore

$$A_\lambda F(\zeta) = \int_D E_\lambda(w, \zeta) \, F(\zeta, w) \, d\nu_\lambda(w) \cdot (1 - |\zeta|^2)^\lambda.$$

Hence

$$|A_\lambda F(\zeta)|^2 \leq \int_D |E_\lambda(w, \zeta)|^2 \, d\nu_\lambda(w) \cdot \int_D |F(\zeta, w)|^2 \, d\nu_\lambda(w) \cdot (1 - |\zeta|^2)^{2\lambda}.$$

So

$$\|A_\lambda F\|^2 = \int_D |A_\lambda F(\zeta)|^2 \, d\nu(\zeta) \leq \frac{1}{\|1\|_\lambda^2} \cdot \|F\|_2^2.$$

We now determine A_λ^*.

Let $F(\zeta, \eta)$ be holomorphic in ζ, anti-holomorphic in η, and belonging to $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$. Let $h \in L^2(D, d\nu)$. We shall determine an explicit expression for $A_\lambda^* h$. It is clear that $A_\lambda^* h(\zeta, \eta)$ is holomorphic in ζ and anti-holomorphic in η, since it belongs to $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$.

One has:

$$\begin{aligned} (A_\lambda^* h, F) &= (h, A_\lambda F) \\ &= \int_D \int_D h(z) \, E_\lambda(z, w) \, \overline{F}(z, w) \, (1 - |z|^2)^\lambda \, d\nu_\lambda(w) \, d\nu(z). \end{aligned}$$

So $A_\lambda^* h$ is the projection of the function

$$(z, w) \to h(z) \, E_\lambda(z, w)$$

onto $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$. This function is in $L^2(D \times D, d\nu_\lambda \otimes d\nu_\lambda)$. The orthogonal projection, call it E, is given by

$$EF(z, w) = \int_D \int_D E_\lambda(w', w) \, E_\lambda(z, z') \, F(z', w') \, d\nu_\lambda(z') \, d\nu_\lambda(w').$$

Hence

$$A_\lambda^* h(\zeta, \eta) = \int_D E_\lambda(\zeta, z') \, E_\lambda(z', \eta) \, h(z') \, d\nu_\lambda(z').$$

Finally,

$$A_\lambda A_\lambda^* f(\zeta) = \int_D E_\lambda(\zeta, \eta) \, E_\lambda(\eta, \zeta) \, f(\eta) \, d\nu_\lambda(\eta) \cdot (1 - |\zeta|^2)^\lambda.$$

So $A_\lambda A_\lambda^*$ is a kernel operator on $L^2(D, d\nu)$ with kernel:

$$
\begin{aligned}
B_\lambda(\zeta, \eta) &= (\|1\|_\lambda^2)^{-2} \, \frac{E_\lambda(\zeta, \eta) \, E_\lambda(\eta, \zeta)}{E_\lambda(\zeta, \zeta) \, E_\lambda(\eta, \eta)} \\
&= (\|1\|_\lambda^2)^{-2} \left\{ \frac{(1 - |\zeta|^2) \, (1 - |\eta|^2)}{(1 - \zeta\overline{\eta}) \, (1 - \overline{\zeta}\eta)} \right\}^\lambda.
\end{aligned}
$$

This is again the Berezin kernel up to a positive constant.

Consequently, applying the well-known theory of L. Schwartz on invariant Hilbert subspaces (cf. [7]), we get $\tau_\lambda \simeq$ *restriction* of $\pi_\lambda^d \widehat{\otimes}_2 \overline{\pi}_\lambda^d$ to G.

12. Principal series of $SL(2, \mathbf{C})$; connection with canonical representations.
Let $G_c = SL(2, \mathbb{C})$, a complexification of $G = SU(1, 1)$. Let P_c^- be the Borel subgroup of G_c consisting of the lower triangular matrices

$$P_c^- : \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}$$

with $c \in \mathbb{C}$, $a \in \mathbb{C}^*$ and let P_c^+ be the group of upper triangular matrices

$$P_c^- : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

with $b \in \mathbb{C}$, $a \in \mathbb{C}^*$. G_c acts on $S = \{s \in \mathbb{C}^2 : \|s\|^2 = 1\}$ and on $\widetilde{S} = S/\sim$, where $s \sim s'$ iff $s = \lambda s'$, $|\lambda| = 1$, by

$$g \cdot s = \frac{g(s)}{\|g(s)\|}$$

transitively. Clearly $\mathrm{Stab}\,\widetilde{(0, 1)} = P_c^-$. So, $\widetilde{S} \simeq G_c/P_c^-$. Similarly $\widetilde{S} \simeq G_c/P_c^+$. If ds is the usual normalized surface measure on S, then

$$d(g \cdot s) = \|g(s)\|^{-4} ds.$$

For $\mu \in \mathbb{C}$, define the character ω_μ of $P_c^\pm$ by

$$\omega_\mu(p) = |a|^\mu.$$

Consider $\pi_\mu^\pm = \mathrm{Ind}_{P_c^\pm \uparrow G_c} \omega_{\mp\mu}$.

π_μ^+ and π_μ^- can both be realized on $C^\infty(\widetilde{S})$: the space of C^∞-functions φ on S satisfying $\varphi(\lambda s) = \varphi(s)$ for $s \in S$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

The formula for π_μ^- is:

$$\pi_\mu^-(g) \, \varphi(s) = \varphi(g^{-1} \cdot s) \, \|g^{-1}(s)\|^\mu.$$

Let θ be the Cartan involution of G_c given by $\theta(g) = (g^*)^{-1}$. Then

$$\pi_\mu^+(g) \, \varphi(s) = \varphi(\theta(g^{-1}) \cdot s) \, \|\theta(g^{-1}(s))\|^\mu.$$

Let $(\ ,\ )$ denote the standard inner product on $L^2(S)$:

$$(\varphi,\ \psi) = \int_S \varphi(s)\,\overline{\psi}(s)ds.$$

Then clearly this form is invariant with respect to the pairs

$$(\pi_\mu^-,\ \pi_{-\overline{\mu}-4}^-) \quad \text{and} \quad (\pi_\mu^+,\ \pi_{-\overline{\mu}-4}^+).$$

Therefore, if $\mathrm{Re}\,\mu = -2$, then the representations $\pi_\mu^\pm$ are unitarizable, the inner product being $(\ ,\ )$.

Irreducibility and intertwining operators. It is known that:

- $\pi_\mu^\pm$ is irreducible if $\mu \notin (-4 - 2\mathbb{N}_0) \cup (2\mathbb{N}_0)$;

- if $\mu = 2k$ $(k = 0, 1, 2, \ldots)$, then there is one irreducible finite-dimensional subspace for $\pi_\mu^\pm$, namely $\oplus_{a=0}^k \mathcal{H}_a$ ($\mathcal{H}_a$: space of harmonic polynomials on $\mathbb{C}^2$, homogeneous of degree a in z and $\overline{z}$). The quotient representation is irreducible.

- if $\mu = -2k - 4$, then there is one infinite-dimensional irreducible subspace for $\pi_\mu^\pm$, namely $\oplus_{a=k+1}^\infty \mathcal{H}_a$. The quotient representation is irreducible and finite-dimensional.

- $\pi_\mu^+ \sim \pi_{-\mu-4}^-$ (if π_μ^+ is irreducible), $\pi_\mu^+ \sim \pi_\mu^-$.

- $\pi_\mu^\pm$ is unitarizable if and only if
  - $\mathrm{Re}\,\mu = -2$: (spherical) principal series
  - $\mu \in (-2, 0)$: complementary series.

Define the operator A_μ on $C^\infty(\widetilde{S})$ by the formula

$$A_\mu\,\varphi(s) = \int_S |(s,\ t)|^{-\mu-4}\varphi(t)\,dt.$$

The integral is absolutely convergent for $\mathrm{Re}\,\mu < -3$ and can be analytically extended to the whole complex plane as a meromorphic function. It is easily checked that A_μ is an intertwining operator

$$A_\mu\,\pi_\mu^\pm(g) = \pi_{-\mu-4}^\mp\,A_\mu.$$

The operator $A_{-\mu-4} \cdot A_\mu$ intertwines $\pi_\mu^\pm$ with itself, and is therefore a scalar $c(\mu)$. It can be computed using K-types:

$$c(\mu) = -\left(\frac{\mu}{2} + 1\right)^2.$$

G acts on S ($G = SU(1,1)$, seen as the group of 2×2 matrices, with determinant 1, leaving $|z_1|^2 - |z_2|^2$ invariant; set $[z,\ w] = z_1\overline{w}_1 - z_2\overline{w}_2$ $(z, w \in \mathbb{C}^2)$).

There are 3 orbits, given by

$$[s, s] > 0, \qquad [s, s] = 0, \qquad [s, s] < 0.$$

All three orbits are invariant under $s \to \lambda s$ ($|\lambda| = 1$). Call the corresponding orbits on $\widetilde{S}$: $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$. Then

$$\mathcal{O}_1 \simeq G/K \quad \text{via} \quad g \to g \cdot e_1$$

$$\mathcal{O}_2 \simeq G/MAN \quad \text{via} \quad g \to g \cdot (e_1 + e_2)$$

$$\mathcal{O}_3 \simeq G/K \quad \text{via} \quad g \to g \cdot e_2.$$

Here $M = \{\pm I\}$, MAN is a minimal parabolic subgroup of G and e_i is the i-th unit vector in $\mathbb{C}^2$.

Let φ be a C^∞-function on $\mathcal{O}_1$. Then

$$\psi(s) = [s, s]^{-\mu/2}\, \varphi(s)$$

again lives on $\mathcal{O}_1$ and satisfies

$$
\begin{aligned}
\psi(g \cdot s) &= [g \cdot s,\, g \cdot s]^{-\mu/2}\, \varphi(g \cdot s) \\
&= \pi_\mu^-(g)\varphi(s)\, [s, s]^{-\mu/2} \quad (g \in G).
\end{aligned}
$$

So $\varphi \to \psi$ is a linear intertwiner of $\pi_\mu^- | G$ with the left regular action of G on $C_c^\infty(G/K) = C_c^\infty(\mathcal{O}_1) = \mathcal{D}(\mathcal{O}_1)$. Since $\theta(g) = J g J$ $(g \in G)$, $J = diag\,(1, -1)$, A_μ becomes

$$A_\mu \varphi(s) = \int_S |[s, t]|^{-\mu-4} \varphi(t)\, dt \quad (s \in \mathcal{O}_1)$$

and is defined on $C_c^\infty(\mathcal{O}_1)$ for all μ, and intertwines π_μ^- and $\pi_{-\overline{\mu}-4}^-$ restricted to G.

For $\varphi_1, \varphi_2 \in C_c^\infty(\mathcal{O}_1)$ and $\mu \in \mathbb{R}$ consider the Hermitian form

$$\langle \varphi_1,\, A_\mu \varphi_2 \rangle = \int_S \int_S |[s, t]|^{-\mu-4} \varphi_1(s)\, \overline{\varphi_2(t)}\, ds dt.$$

This form is clearly $\pi_\mu^-(G)$-invariant. A G-invariant measure on $\mathcal{O}_1$ is given by $dm(s) = \dfrac{ds}{[s, s]^2}$. Applying the above linear map $\varphi \to \psi$ we get instead of (12.13),

$$\langle \psi_1,\, \psi_2 \rangle_\mu = \int_S \int_S \psi_1(s) \overline{\psi_2(t)} \left[\frac{[s, s]\, [t, t]}{[s, t]\, [t, s]} \right]^{\frac{\mu}{2}+2} dm(s) dm(t).$$

Thus

$$B_\lambda(s, t) = \left\{ \frac{[s, s]\, [t, t]}{[s, t]\, [t, s]} \right\}^\lambda$$

is again the Berezin kernel, now living on $[s, s] > 0$ $(s \in \widetilde{S}) \simeq \mathcal{O}_1$. Indeed, $B_\lambda(g \cdot e_1, e_1) = (\cosh t)^{-2\lambda}$ if $g = a_t$.

This approach was firstly observed by Molchanov in [33].

13. Some generalizations and remarks

• Let us restrict the holomorphic discrete series π_λ^d to the real group $SO_o(1, 1)$ and the functions in $\mathcal{F}_\lambda$ to the real "unit disc" $(x \in \mathbb{R}, |x| < 1)$. More precisely, let us define the analogue of A_λ by

$$F(\zeta) \to F(x)\, (1 - |x|^2)^{\lambda/2}.$$

Then we get "the" canonical representation of $SO_0(1, 1)$ with Berezin kernel $B_{\lambda/2}(x, y)$, which remains positive-definite for $\lambda > 0$.

Let $\psi_{\lambda/2}(x) = (1 - |x|^2)^{\lambda/2}$ be the associated positive-definite function. We can compute the spherical decomposition of this restricted representation in a similar way as before and get:

$$a_{\lambda/2}(i\mu) = 2^{\lambda-1}\, \frac{\Gamma(\frac{\lambda+i\mu}{2})\, \Gamma(\frac{\lambda-i\mu}{2})}{\Gamma(\lambda)}$$

$$c(s) = \frac{1}{2}$$

$$\psi_{\lambda/2}(g) = \frac{1}{4\pi} \int_0^\infty a_{\lambda/2}(i\mu)\, \varphi_{i\mu}(g)\, \frac{d\mu}{|c(i\mu)|^2} \quad \text{in } L^2\text{-sense}$$

If $g = a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, then $\varphi_{i\mu}(g) = e^{-i\mu t}$. Observe that this decomposition holds for $\operatorname{Re}\lambda > 0$.

The computation of $a_{\lambda/2}(i\mu)$ is again most easily done in the tube realization. Observe that $|x| < 1$ is identified with the positive imaginary axis. The integral becomes:

$$2^{\lambda-1} \int_0^\infty \frac{y^{\lambda/2+i\mu/2}}{(y+1)^\lambda}\, \frac{dy}{y}.$$

We thus finally get:

$$(\cosh t)^{-\lambda} = \psi_{\lambda/2}(a_t) = \frac{1}{2\pi} \int_0^\infty \frac{2^{\lambda-1}\Gamma(\frac{\lambda+i\mu}{2})\,\Gamma(\frac{\lambda-i\mu}{2})}{\Gamma(\lambda)}\, e^{-i\mu t}\, d\mu \quad (\operatorname{Re}\lambda > 0)$$

in L^2-sense, a "classical" Fourier transform identity. See [8] for generalization.

• A quite interesting extension of the notion of canonical representation is obtained by considering $\pi_\lambda^d \widehat{\otimes}_2 \pi_{\lambda'}^d$, with $\lambda - \lambda' = l \in \mathbb{Z}$, see Section 11 (exercise). See [8], [52].

If $|l| > 1$ we get discrete series occurring in the decomposition of "$\tau_{\lambda,l}$" (apart from principal series and complementary series if l is even).

• Extension of the results to $G = SU(1, n)$ is rather immediate. *Finitely many* complementary series occur for $\lambda > 0$, situated at the points $n - 2\lambda - 2k = s_k$, as far as $s_k > 0$. See [10].

An interesting phenomenon occurs also in the "compact" realization of τ_λ. The G-orbits on $\widetilde{S} = \{x \in \mathbb{C}^n : \|x\| < 1\}/\sim$ can be identified with G/K, G/P, G/H, where $H = S(U(1, n-1) \times U(1))$. The Berezin kernel can also be defined on the latter space and gives rise to a canonical representation on a pseudo-Riemannian space, see [11]. Unfortunately, τ_λ is not unitary anymore, so a good decomposition theory is missing! One can however try to decompose ψ_λ into H-spherical (positive-definite) distributions. (see Van Dijk-Molchanov, forthcoming publication, and thesis Charchov [6]).

• Berezin kernels are mostly studied in the context of Hermitian symmetric spaces. See e.g. [8], [30], [35], [37], [38], [39], [42], [45], [48], [52], [53].

The second model coming from the principal series representations of $SL(2, \mathbb{C})$ is well-suited for generalizing canonical representations to para-Hermitian symmetric spaces (cf. [33]).

The most general definition of canonical representation is given in S. C. Hille's thesis [29]. It is built around arbitrary so-called $\sigma\theta$-invariant parabolic subgroups.

14. Interpolation. We return to the question posed in Section 9 about the interpolation of τ_λ between the L^2-spaces of $\mathrm{SU}(2)/K$ and $\mathrm{SU}(1,1)/K$. Notice that $\mathrm{SU}(1,1)/K$ can be viewed as an open submanifold of $\mathrm{SU}(2)/K \simeq \widetilde{S}$, see Section 12. This is in our view the main reason that we can relate both spaces in question. Let us now explain Neretin's solution in our own words and add some remarks.

Let λ be a negative integer. Then

$$\psi_\lambda = \sum_{j=0}^{-\lambda} R_j(\lambda)\, \varphi_{1-2\lambda-2j}$$

according to Section 9.

Taking both sides at the point $g = a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, we get

$$(14.1) \qquad (\cosh t)^{-2\lambda} = \sum_{j=0}^{-\lambda} R_j(\lambda)\, \varphi_{1-2\lambda-2j}(a_t).$$

One has

$$\varphi_s(a_t) = {}_2F_1\left(\frac{s+1}{2}, \frac{-s+1}{2}; 1; -\sinh^2 t\right).$$

For $s = 1 - 2\lambda - 2j$, being a positive odd integer, $\varphi_s(a_t)$ is a polynomial in $\cosh^2 t$, hence (14.1) also holds if we replace $\cosh t$ by $\cos t$. We get

$$(14.2) \qquad \psi_\lambda^{SU(2)} = \sum_{j=0}^{-\lambda} R_j(\lambda)\, \varphi_{1-2\lambda-2j}^{SU(2)}$$

where

$$\psi_\lambda^{SU(2)}(u) = |(u\,e_1,\, e_1)|^{-2\lambda} \qquad (u \in SU(2))$$

and $\varphi_{1-2\lambda-2j}^{SU(2)}$ is defined on $SU(2)$ by setting its radial part equal to

$$_2F_1(1 - \lambda - j, \lambda + j; 1; \sin^2 t).$$

Observe that $\varphi_{1-2\lambda-2j}^{SU(2)}$ is a spherical function on $SU(2)$.

Set $v = 1 - 2\lambda - 2j$. Then (14.2) can be rewritten as

$$\psi_\lambda^{SU(2)} = \sum_{\substack{v=1 \\ v \text{ odd}}}^{-2\lambda+1} R_{\frac{1-2\lambda-v}{2}}(\lambda)\, \varphi_v^{SU(2)}.$$

Notice that

$$R_{\frac{1-2\lambda-v}{2}}(\lambda) = \frac{v\,\Gamma(1-\lambda)^2}{\Gamma(1-\lambda+\frac{1-v}{2})\,\Gamma(1-\lambda+\frac{1+v}{2})}.$$

Call this expression $r_v(\lambda)$. One easily computes

$$\int_{SU(2)} \psi_\lambda^{SU(2)}(u)du = \frac{1}{1-\lambda}.$$

Let us now compute the asymptotic behaviour of $(1 - \lambda)\,\psi_\lambda$ for $\lambda \to -\infty$. This comes down to determining the asymptotics of $(1 - \lambda)\,r_v(\lambda)$ for fixed v ($v = 1, 3, 5, \ldots$).

Applying formula (8.1) we obtain:

$$(14.3) \qquad (1 - \lambda)\,r_v(\lambda) = v\left\{1 + \frac{v^2 - 1}{4\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right\} \qquad (\lambda \to -\infty,\, \lambda \text{ integer}).$$

Observe that v runs over the set of dimensions of the finite-dimensional spherical representations of $SU(2)$. Clearly (14.3) is equivalent to the following identity of distributions

on $\widetilde{S}$:

$$(14.4) \qquad (1-\lambda)\,\psi_\lambda^{SU(2)} = \delta + \frac{\square\delta}{4\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \qquad (\lambda \to -\infty, \lambda \text{ integer}).$$

where $\square$ is now the Laplace-Beltrami operator on $\widetilde{S}$.

REMARK. One has for λ negative integer:

$$(14.5) \qquad\qquad \psi_\lambda^{SU(2)} = \sum_{\substack{v=1 \\ v \text{ odd}}}^{\infty} r_v(\lambda)\,\varphi_v.$$

Indeed, notice that $r_v(\lambda) = 0$ for v odd, $v > -2\lambda + 1$. It is easily checked that (14.5) remains true for all complex λ with $\operatorname{Re}\lambda < 1/2$ in L^2-sense, by computing inner products and using formulae from [21]. Neretin applies Carlson's theorem (Theorem 9.1), which is more convenient. Clearly (14.3) and (14.4) now hold without the restriction "λ integer".

A final observation is that $\psi_\lambda^{SU(2)}$, for λ a negative integer, is a positive-definite function associated with the representation $V_{-\lambda} \otimes \overline{V}_{-\lambda}$, where $V_{-\lambda}$ is the standard representation of SU(2) on holomorphic (harmonic) polynomials in z_1, z_2, homogeneous of degree $-\lambda$.

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