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CAPELLI IDENTITY AND RELATIVE DISCRETE SERIES OF LINE BUNDLES OVER TUBE DOMAINS

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Abstract. We use the Capelli identity to give an explicit realization of some relative discrete series of the L^2-space of sections of a line bundle over a tube domain. This amounts to a geometric construction of some Opdam shift operators.

1. Introduction. In this paper we shall study the connection between the Capelli identity for tube type Hermitian symmetric spaces, and the problem of constructing explicitly the discrete spectrum of L^2- spaces of sections of line bundles over such domains. The main result is Theorem 4.2 which gives the explicit intertwining operator between the two models of holomorphic discrete series representations of the group. It is interesting to remark (as is done after Corollary 4.4) that this differential operator is actually of a very canonical type, namely that of a generalized gradient operator. In effect, we are giving a geometric construction of a differential operator which shifts parameters in hypergeometric functions, just as is the case for the so-called Opdam shift operators, constructed for arbitrary root systems using different methods.

Let $D = G/K$ be an irreducible Hermitian symmetric space of non-compact type realized as a bounded symmetric domain in a complex vector space. The group K thus

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has one-dimensional center and the one-dimensional representations on K then induce homogeneous line bundles over D. In his paper [15] Shimeno gives the Plancherel decomposition for the L^2-space of sections of a homogeneous line bundle over D. There appear finitely many discrete parts in the decomposition; they are also called relative discrete series. It is proved in [15] that all the relative discrete parts are G-equivalent to holomorphic discrete series by identifying the infinitesimal character. For the unit ball in $\mathbb{C}^n$ this was proved also in [22] by explicit calculations. On the other hand the holomorphic discrete series have their standard module as weighted Bergman spaces of holomorphic functions on D, namely (for the so-called scalar holomorphic discrete series) the holomorphic functions square-integrable with respect to a certain probability measure on D. Thus it is of interest to find the explicit intertwining operators from the relative discrete series into the holomorphic discrete series. For the unit disk this is done in [21] via the holomorphic differential operator $(\frac{\partial}{\partial z})^l$ and the classical Bol's lemma, which asserts that the operator intertwines two actions of $G = SU(1,1)$ on certain line bundles over the unit disk. Later we realized that the those intertwining operators can also be constructed via the invariant Cauchy-Riemann operator, and we found the intertwining operators for the unit ball in [13] and a general bounded symmetric domain in [23]; see also [12] for the case of the Riemann sphere.

The L^2-space of sections of the line bundle can be realized as a functions on the domain D, this being effected by the standard parallelization of homogeneous bundles over Hermitian symmetric spaces. The corresponding L^2 space then becomes a weighted L^2-space on D, for a suitable quasi-invariant measure on D. The corresponding weighted Bergman space of holomorphic functions on D is one of the relative discrete series, whereas the other relative discrete series consist of non-holomorphic scalar-valued functions on D. It is proved in [15] that they are G-equivalent to a holomorphic discrete series with the highest weight being irreducible representations of K in the symmetric tensors of the tangent space of D. Now some of those representations of K are one-dimensional, namely those corresponding to the Jordan determinant representation. In the present note we find the intertwining operator for the corresponding relative discrete series via the Cayley type operator, which is a generalization of the differential operator $\frac{\partial}{\partial z}$. Up to a constant, we may express this intertwining operator in the form (see Section 4 for notation, in particular P_l denotes a projection)

$$S_l = P_l \mathcal{D}^{rl}$$

where $\mathcal{D}$ is the holomorphic part of the canonical Hermitian connection on the holomorphic line bundle in question on D. Acting with this on the lowest K - type, we construct explicitly the extreme weight vector (see Corollary 4.5)

$$h(z,z)^{-(\nu-l-\frac{d}{r})}\Delta(\partial)^l h(z,z)^{\nu-l-\frac{d}{r}} = \prod_{j=1}^{r}(-(\nu-\frac{d}{r})-\frac{a}{2}(j-1)+l)_l \frac{\overline{\Delta(z)}^l}{h(z,z)^l}$$

where this is an element of the space of sections of the line bundle generating a holomorphic discrete series representation.

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2. Weighted L^2-space on bounded symmetric domains. We briefly recall the bounded realization of a Hermitian symmetric space, see [5], and [9].

Let $D = G/K$ be an irreducible bounded symmetric domain of tube type in a complex vector space V of dimension d. The space V has the structure of a Jordan algebra. Let $\Delta(z)$ be the corresponding Jordan determinant function. We normalize a Hermitian inner product on V so that a minimal tripotent has norm 1, and denote $dm(z)$ the corresponding Lebesgue measure on V. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$, which is then also a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+ + \mathfrak{p}^-$ be the decomposition of the complexification of $\mathfrak{g}^{\mathbb{C}}$ under adjoint action of the center of $\mathfrak{k}$, with $\mathfrak{p}^+$ being identified with the vector space V. We fix an element Z in the center of $\mathfrak{k}$ so that it has eigenvalue ($\sqrt{-1}$ times) $\frac{2}{r}$ on the space $\mathfrak{p}^+$. Here r denotes the rank of the Jordan algebra. Let $\gamma_1, \dots, \gamma_r$ be the Harish-Chandra strongly orthogonal roots and fix a system of corresponding unit root vectors $\{e_1, e_2, \dots, e_r\}$ in V. We define a K-invariant polynomial $h(z)$ on V by

$$h(c_1 e_1 + \ldots + c_r e_r) = \prod_{j=1}^{r} (1 - |c_j|^2)$$

and let $h(z, w)$ be its polarization, which is holomorphic in z and anti-holomorphic in w. The Bergman reproducing kernel of D is then $ch(z, w)^{-p}$ for some positive constant c, where p is an integer called the genus of D. Since we are in the case where D is a tube domain we have the relation $2d = pr$. Recall that the Bergman space is the space of holomorphic functions on D square-integrable with respect to the Lebesgue measure.

Let $\alpha > -1$. Denote

$$\nu = \nu(\alpha) = p + \alpha, \tag{2.1}$$

and consider the weighted probability measure

$$d\mu_\nu(z) = C_\nu h(z)^\alpha dm(z) = C_\nu h(z)^{\nu - p} dm(z) \tag{2.2}$$

on D. Here C_ν is a normalizing constant whose precise value will not concern us.

There is a unitary representation of G on $L^2(D, d\mu_\alpha)$ given by the formula

$$U_g^{(\nu)} : f(z) \mapsto f(g^{-1}(z))(J_{g^{-1}}(z))^{\frac{\nu}{p}} \qquad (g \in G) \tag{2.3}$$

where $J_{g^{-1}}$ stands for the Jacobian of the transformation g^{-1}. Here we have parallelized the homogeneous line bundle over D induced from a character on the (covering of the) center of K.

3. Capelli identity. In this section we will use the results of Faraut-Koranyi to prove the Capelli identity, the result is by now well known and we include a proof for the sake of completeness.

Let $\mathcal{P}$ be the space of all holomorphic polynomials on V. By a well known result of Hua and Schmid, the space $\mathcal{P}$ under the action of K is decomposed into irreducible subspaces $\mathcal{P}^{\underline{\mathbf{m}}}$ of signatures (corresponding to highest weights) $\underline{\mathbf{m}} = m_1\gamma_1 + \ldots + m_r\gamma_r$, with $m_1 \geq \cdots \geq m_r \geq 0$. Consider the Fock space $\mathcal{F}$ of entire functions on V with reproducing kernel $e^{(z,w)}$, i.e. the entire functions square-integrable with respect to the Gaussian density. Let $K^{\underline{\mathbf{m}}}(z, w)$ be the reproducing kernel of the space $\mathcal{P}^{\underline{\mathbf{m}}}$ with the Fock space norm. By the decomposition $\mathcal{F}=\sum_{\underline{\mathbf{m}}}\mathcal{P}^{\underline{\mathbf{m}}}$ and definition of $K^{\underline{\mathbf{m}}}(z,w)$, we have

$$e^{(z,w)}=\sum_{\underline{\mathbf{m}}}K^{\underline{\mathbf{m}}}(z,w).$$

Recall the usual Pochhammer symbol $(x)_r=x(x+1)\cdots(x+r-1)$.

THEOREM 3.1. *The operator* $\Delta(z)^l\Delta(\partial)^l$ *acts on each* K*-space* $\mathcal{P}^{\underline{\mathbf{m}}}$ *as a scalar*

$$\Delta(z)^l\Delta(\partial)^l f=q_l(\underline{\mathbf{m}})f,\quad f\in\mathcal{P}^{\underline{\mathbf{m}}},$$

with

$$q_l(\underline{\mathbf{m}})=\prod_{k=1}^{r}(\frac{a}{2}(r-k)+1+m_k-l)_l=(-1)^{rl}\prod_{k=1}^{r}(-m_k-\frac{a}{2}(r-k))_l$$

Proof. For any $f\in\mathcal{P}^{\underline{\mathbf{m}}}\subset\mathcal{P}$,

$$f(z)=\int_{\mathfrak{p}^+}e^{(z,w)}f(w)e^{-(w,w)}dw.$$

We act on both sides by $\Delta(z)^l\Delta(\partial)^l$,

$$\begin{aligned}\Delta(z)^l\Delta(\partial)f(z)&=\int_{\mathfrak{p}^+}\Delta(z)^l\overline{\Delta(w)^l}e^{(z,w)}f(w)e^{-(w,w)}dw\\&=\int_{\mathfrak{p}^+}\Delta(z)^l\overline{\Delta(w)^l}\sum_{\underline{\mathbf{m}}'}K^{\underline{\mathbf{m}}'}(z,w)f(w)e^{-(w,w)}dw,\end{aligned}$$

where we have used the previous expansion for $e^{(z,w)}$. Now the map $g(z)\mapsto\Delta(z)^lg(z)$ is a K-intertwining map from $\mathcal{P}^{\underline{\mathbf{m}}'}$ onto $\mathcal{P}^{\underline{\mathbf{m}}'+l}$, thus

$$\Delta(z)^l\overline{\Delta(w)^l}K^{\underline{\mathbf{m}}'}(z,w)=C(\underline{\mathbf{m}}',l)K^{\underline{\mathbf{m}}'+l}(z,w)\tag{3.1}$$

for some positive constant $C(\underline{\mathbf{m}}',l)$. Taking $z=w=e$ and using Lemma 3.1 and Theorem 3.4 in [5] we find

$$C(\underline{\mathbf{m}}',l)=\frac{K^{\underline{\mathbf{m}}'}(e,e)}{K^{\underline{\mathbf{m}}'+l}(e,e)}=\frac{d_{\underline{\mathbf{m}}'}}{(n/r)_{\underline{\mathbf{m}}'}}\frac{(n/r)_{\underline{\mathbf{m}}'+l}}{d_{\underline{\mathbf{m}}'+l}}=\prod_{k=1}^{r}\left(\frac{a}{2}(r-k)+1+m_k'\right)_l.$$

Thus

$$\begin{aligned}\Delta(z)^l\Delta(\partial)^lf(z)&=\int_{\mathfrak{p}^+}\sum_{\underline{\mathbf{m}}'}C(\underline{\mathbf{m}}',l)K^{\underline{\mathbf{m}}'+l}(z,w)f(w)e^{-(w,w)}dw\\&=\int_{\mathfrak{p}^+}C(\underline{\mathbf{m}}-l,l)K^{\underline{\mathbf{m}}}(z,w)f(w)e^{-(w,w)}dw=C(\underline{\mathbf{m}}-l,l)f(z),\end{aligned}$$

where in the second last equality we use the fact that $f\in\mathcal{P}^{\underline{\mathbf{m}}}$ and Schur's lemma. Thus

$$q_l(\underline{\mathbf{m}})=C(\underline{\mathbf{m}}-l,l)=\prod_{k=1}^{r}(\frac{a}{2}(r-k)+1+m_k-l)_l.$$

This completes the proof. ∎

REMARK 3.2. Theorem A has been previously proved by Dib [3] and Arazy [1]. See also [20], [19] and [17]. Our proof above is essentially the same as that in [1].

REMARK 3.3. As is shown in [14], Proposition 1.2, the above formula is a reformulation of the main result of [7], which essentially calculates the Laplace transform of $\Delta^{\underline{\mathbf{m}}}$ by using the Gindikin Gamma function. Note that here we are still using a different ordering. (Our γ_j is Sahi's $2\varepsilon_{n-j}$ and $r = n$.)

4. Line bundles over tube domains. In this section we will use the Capelli identity to realize explicitly some relative discrete series of L^2-space of sections of a line bundle over a tube domain, namely by constructing irreducible invariant subspaces of $L^2(D, \mu_\alpha)$.

The irreducible decomposition of the space $L^2(D, \mu_\alpha)$ has been found by Shimeno [15], where the full Plancherel decomposition in terms of both continuous and discrete spectrum is given. It is proved there that all the relative discrete series (i.e. the irreducible invariant subspaces) appearing in the decomposition are holomorphic discrete series. We summarize the result there in the following.

Fix $\alpha > -1$ and let ν be as in (2.1). We define

$$k = \begin{cases} \frac{\alpha+1}{2} - 1 = \frac{\nu-p+1}{2} & \text{if } \alpha \text{ is an odd integer} \\ \left[\frac{\alpha+1}{2}\right] = \left[\frac{\nu-p+1}{2}\right] & \text{otherwise.} \end{cases} \tag{4.1}$$

Here $[t]$ stands for the integer part of $t \in \mathbb{R}$. Denote

$$D_\nu = \Big\{ \underline{\mathbf{m}} = \sum_{j=1}^r m_j \gamma_j, 0 \le m_1 \le \cdots \le m_r \le k \Big\}.$$

Shimeno proved in [15], Theorem 5.10, that all the relative discrete series in $L^2(D, \mu_\alpha)$ are indexed by the set D_ν and they are equivalent to a holomorphic discrete series. We reformulate this result in the following. Let $\mathfrak{h}^-$ be the subspace of $\mathfrak{h}$ generated by the dual elements of γ_j, namely the elements $\frac{1}{2}[e_j^+, e_j^-]$ where $e_j^+ = e_j$ and e_j^- are the root vectors of $\pm\gamma_j$, and let $\mathfrak{h} = \mathfrak{h}^- + \mathfrak{h}^+$ be the corresponding decomposition of $\mathfrak{h}$. We let $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$ be the simple component of $\mathfrak{k}$ so that $\mathfrak{k} = \mathfrak{k}_s + \mathbb{R}Z$.

THEOREM 4.1 (Shimeno [15]). *For each $\underline{\mathbf{m}}$ in D_ν there exists a relative discrete series $A^2_{\underline{\mathbf{m}}}(D, \nu)$ appearing in $L^2(D, \mu_\alpha)$, and it is equivalent to a holomorphic discrete series with highest weight (under certain ordering of the root spaces of $\mathfrak{g}^{\mathbb{C}}$)*

$$\Lambda|_{(\mathfrak{h}^-)^{\mathbb{C}}} = \underline{\mathbf{m}} - \frac{\nu}{2}\sum_{j=1}^r \gamma_j, \;\; \underline{\mathbf{m}} \in D_\nu$$

and $\Lambda(\mathfrak{h} \cap \mathfrak{k}_s) = 0$, $\Lambda(iZ) = -\nu$ in case $D = G/K$ is a non-tube domain.

When $\underline{\mathbf{m}} = (l, \ldots, l)$ we denote simply the relative discrete series $A^2_{\underline{\mathbf{m}}}(D, \nu)$ by $A^2_l(D, \nu)$. Our main result is the following, denoting by $L^2_a(D, \nu - 2l)$ the subspace of holomorphic functions, i.e. the standard module for these discrete series, namely

$$L^2_a(D, \nu - 2l) = \Big\{ f : D \mapsto \mathbb{C} \text{ holomorphic; } \int_D |f(z)|^2 d\mu_{\nu-2l}(z) < \infty \Big\},$$

where $d\mu_{\nu-2l}(z) = C_{\nu-2l} h(z,z)^{\nu-2l-p} dm(z)$ is as in (2.2).

THEOREM 4.2. *With the above notation we have that the operator*

$$S_l : f(z) \mapsto h(z,z)^{-(\nu-l-\frac{d}{r})} \Delta(\partial)^l (h(z,z)^{\nu-l-\frac{d}{r}} f(z))$$

is an intertwining operator mapping $L^2_a(D, \nu - 2l)$ onto $A^2_l(D, \nu)$ for $l = 0, 1, \ldots, k$.

To prove the theorem we need the following intertwining property of the Cayley-Capelli operator $\Delta(\partial)$, which was proved by Arazy [2], Theorem 6.4; see also [16], Lemma 7.1 and [8].

THEOREM 4.3. *The Cayley-Capelli operator $\Delta(\partial)$ intertwines the action $U^{(\frac{d}{r}-1)}$ with $U^{(\frac{d}{r}+1)}$, namely*

$$\Delta(\partial)(J_g(z))^{\frac{\frac{d}{r}-1}{p}} f(gz) = (J_g(z))^{\frac{\frac{d}{r}+1}{p}} (\Delta(\partial)f)(gz)$$

for holomorphic functions f on D and $g \in G$. Similarly, for every natural number l we have

$$\Delta(\partial)^l(J_g(z))^{\frac{\frac{d}{r}-l}{p}} f(gz) = (J_g(z))^{\frac{\frac{d}{r}+l}{p}} (\Delta(\partial)^l f)(gz)$$

for holomorphic functions f on D and $g \in G$.

Note that since the operator $\Delta(\partial)$ is a holomorphic differential operator, we see that the above result holds for all C^∞-functions f.

With this theorem we can establish the formal intertwining property of of S_l.

COROLLARY 4.4. *The operator S_l intertwines the action $U^{(\nu-2l)}$ with the action $U^{(\nu)}$ of G on C^∞-functions on D.*

Proof. In order to exhibit the intertwining property of the multiplication by $h(z,z)^c$, we introduce the notation (see [11])

$$U^{(\nu,\kappa)}(g) : f(z) \mapsto f(g(z))(J_g(z))^{\frac{\nu}{p}}(\overline{J_g(z)})^{\frac{\kappa}{p}} \qquad (g \in G)$$

Let T be the operator

$$T_\kappa\colon f(z) \mapsto f(z)h(z,z)^\kappa.$$

Then by the transformation property of $h(z,z)$ we know that T intertwines the action $U^{(\nu,\kappa)}$ with $U^{(\nu-\kappa,0)}$; see Lemma 5 in [11]. Now our operator S_l is

$$S_l = T_{-(\nu-l-\frac{d}{r})}\Delta(\partial)^l T_{\nu-l-\frac{d}{r}},$$

the results follows by the above intertwining properties of the operator T_κ and that of $\Delta(\partial)^l$ in Theorem 4.3. ∎

The operator S_l can also be constructed geometrically via the covariant holomorphic differential operator; see [18] and [24]. Indeed, consider the holomorphic line bundle on D defined via the action $U^{(\nu-2l)}$. Let ∇ be the Hermitian connection compatible with the complex structure and $\mathcal{D}$ the holomorphic part, so that $\nabla = \mathcal{D} + \bar{\partial}$. The operator $\mathcal{D}$ maps the line bundle to its tensor product with the holomorphic cotangent bundle; so that it maps a scalar-valued function to a $V' = \mathfrak{p}^-$-valued function, after trivializing the bundles. The power $\mathcal{D}^{rl}$ maps to the symmetric tensor $\otimes^{lr}V'$ of V'. However there is a distinguished K-component in the symmetric tensor, namely the one-dimensional representation with highest weight $l(\gamma_1 + \ldots + \gamma_r)$ (disregarding the action of the center of K). Let P_l be the orthogonal projection onto the component. Then we have

$$P_l\mathcal{D}^{rl} = cS_l$$

for some non-zero constant c; see Lemma 4.3 in [18] and Lemma 3.2 in [24]. As an intertwining operator between two line bundles, the operator S_l maps in particular the spherical functions for the line bundle $U^{(\nu-2l)}$ to those for $U^{(\nu)}$, and thus are the hypergeometric shift operators [6]. So our result gives a geometric construction of some of the Opdam shift operators. It has not been known before that there is a geometric interpretation of the shift operators [10].

We now prove our Theorem 4.2.

Proof. We have established the formal intertwining property of the operator S_l. We prove now that it maps into $A^2_l(D,\nu)$. Note that the condition on l implies that the weighted space $L^2_a(D,\nu-2l)$ is non-trivial. Take $f\in L^2_a(D,\nu-2l)$ to be the constant function 1. We calculate its image, namely,

$$h(z,z)^{-(\nu-l-\frac{d}{r})}\Delta(\partial)^l(h(z,z)^{\nu-l-\frac{d}{r}}).$$

We consider first $\Delta(z)^l\Delta(\partial)^l$. We use the Faraut-Koranyi expansion [5] of the reproducing kernel

$$h(z,w)^{-\kappa}=\sum_{\underline{\mathbf{m}}}(\kappa)_{\underline{\mathbf{m}}}K^{\underline{\mathbf{m}}}(z,w),$$

so that

$$\Delta(z)^l\Delta(\partial)^l h(z,z)^{\nu-l-\frac{d}{r}}=\sum_{\underline{\mathbf{m}}}(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}}\Delta(z)^l\Delta(\partial)^lK^{\underline{\mathbf{m}}}(z,z). \tag{4.2}$$

By Theorem 3.1 (with the same notation in the proof) we have

$$\Delta(z)^l\Delta(\partial)^lK^{\underline{\mathbf{m}}}(z,z)=C(\underline{\mathbf{m}}-l,l)K^{\underline{\mathbf{m}}}(z,z)$$

which vanishes whenever $m_r<l$. We write therefore $\underline{\mathbf{m}}=\underline{\mathbf{m}}'+l$ and each term in the above summation is

$$\begin{aligned}(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}}C(\underline{\mathbf{m}}-l,l)K^{\underline{\mathbf{m}}}(z,z)&=(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}'+l}C(\underline{\mathbf{m}}',l)K^{\underline{\mathbf{m}}'+l}(z,w)\\&=(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}'+l}\Delta(z)^l\overline{\Delta(w)^l}K^{\underline{\mathbf{m}}'}(z,z),\end{aligned} \tag{4.3}$$

by (3.1). However clearly,

$$(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}'+l}=(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}(-(\nu-l-\frac{d}{r})+l)_{\underline{\mathbf{m}}'},$$

the summation (4.2) is then

$$\begin{aligned}&(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}\Delta(z)^l\overline{\Delta(z)^l}\sum_{\underline{\mathbf{m}}'}(-(\nu-l-\frac{d}{r})+l)_{\underline{\mathbf{m}}'}K^{\underline{\mathbf{m}}'}(z,z)\\&=(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}\Delta(z)^l\overline{\Delta(z)^l}h(z,z)^{\nu-2l-\frac{d}{r}},\end{aligned} \tag{4.4}$$

where we have used again the Faraut-Koranyi expansion. From this it follows that

$$h(z,z)^{-(\nu-l-\frac{d}{r})}\Delta(\partial)^l(h(z,z)^{\nu-l-\frac{d}{r}})=(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}\frac{\overline{\Delta(z)^l}}{h(z,z)^l}.$$

which is nonzero, and its $L^2(D,\alpha)$-norm is dominated by

$$\int_D h(z)^{\alpha-2l}dm(z)$$

which is finite since $\alpha - 2l > -1$, by our assumption on l. That is, the function $S_l f$ is in $A_l^2(D,\nu)$ and non-zero. Now both $L_a^2(D,\nu-2l)$ and $A_l^2(D,\nu)$ are irreducible unitary representations of G (and of $\mathfrak{g}^{\mathbb{C}}$). Thus the operator S_l is a nonzero intertwining operator into $A_l^2(D,\nu)$. As both representations are unitary and irreducible there exist a unitary isomorphism between them, derived from S_l for example by polar decomposition (- the operator is closed and densely defined). Alternatively, we could say that S_l gives an equivalence of the two Harish-Chandra $(\mathfrak{g}, K)$ - modules, and hence they are also globally equivalent. ∎

The above proof actually also implies

Corollary 4.5. *The highest weight vector of $A_l^2(D,\nu)$ is*

$$h(z,z)^{-(\nu-l-\frac{d}{r})}\Delta(\partial)^l h(z,z)^{\nu-l-\frac{d}{r}} = \prod_{j=1}^{r}(-(\nu-\frac{d}{r})-\frac{a}{2}(j-1)+l)_l \frac{\overline{\Delta(z)}^l}{h(z,z)^l}.$$

This highest weight vector has also been calculated previously in [4] by using tensor product arguments.

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