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H^p SPACES FOR SCHRÖDINGER OPERATORS

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1. Introduction. Let $k_t(x,y)$ be the integral kernels of the semigroup of linear operators $\{T_t\}_{t>0}$ generated by a Schrödinger operator $-A = \Delta - V$ on $\mathbb{R}^d$, $d \geq 3$.

Throughout this paper we assume that V is a nonnegative potential on $\mathbb{R}^d$ that belongs to the reverse Hölder class RH^q, $q > \frac{d}{2}$, that is, there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)\,dy, \quad \text{for every ball } B. \tag{1.1}$$

Since V is nonnegative and belongs to $L^q_{\mathrm{loc}}(\mathbb{R}^d)$ the Feynman-Kac formula implies that

$$0 \leq k_t(x,y) \leq (4\pi)^{-d/2}e^{-|x-y|^2/(4t)} = p_t(x-y). \tag{1.2}$$

We say that f is an element of the space H^p_A if the maximal function

$$\mathcal{M}f(x) = \sup_{t>0}|T_tf(x)| = \sup_{t>0}\left|\int k_t(x,y)f(y)\,dy\right| \tag{1.3}$$

belongs to $L^p(\mathbb{R}^d)$.

For $0 < p \leq 1$ we define the quasi-norm $\|f\|^p_{H^p_A}$ by setting

$$\|f\|^p_{H^p_A} = \|\mathcal{M}f\|^p_{L^p}. \tag{1.4}$$

Our main result is about atomic decomposition of the elements of H^p_A for $p \leq 1$, p close to 1. The notion of H^p_A atom is determined by the following auxiliary function $m(x,V)$ which is defined by

$$m(x,V)^{-1} = \rho(x) = \sup\left\{r > 0 : \frac{1}{r^{d-2}}\int_{B(x,r)} V(y)\,dy \leq 1\right\}. \tag{1.5}$$

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A function a is an atom of the space H^p_m associated with a ball $B(y_0,r)$ if

$$\operatorname{supp}\ a\subset B(y_0,r), \tag{1.6}$$

$$\|a\|_{L^\infty}\le |B(y_0,r)|^{-1/p}, \tag{1.7}$$

$$r\le \rho(y_0), \tag{1.8}$$

$$\text{if}\ \ r<2^{-2}\rho(y_0)\ \ \text{then}\ \int a(x)\,dx=0. \tag{1.9}$$

The atomic H^p_m quasi-norm is defined by

$$\|f\|^p_{H^p_m}=\inf\sum|\lambda_j|^p, \tag{1.10}$$

where the infimum is taken over all decompositions $f=\sum_j\lambda_j a_j$, a_j being H^p_m and λ_j being scalars.

Let $\delta=2-\frac{d}{q}$, and $\delta'=\min\{1,\delta\}$. Our main result is the following

THEOREM 1.11. *Assume that* $\frac{d}{d+\delta'}<p\le 1$. *Then there exists a constant* $C>0$ *such that for every compactly supported function* $f\in L^1(\mathbb{R}^d)$ *we have*

$$C^{-1}\|f\|^p_{H^p_m}\le \|f\|^p_{H^p_A}\le C\|f\|^p_{H^p_m}. \tag{1.12}$$

In the case where $p=1$ and V satisfies (1.1) the space H^1_A was studied in [DZ2], where the atomic and Riesz transforms characterizations of the space were proved. Therefore in the present paper we shall consider the case where $\frac{d}{d+\delta'}<p<1$.

REMARK 1. The atoms for the H^p_A spaces satisfy the same size conditions as the classical $H^p(\mathbb{R}^d)$ atoms. The main difference is that the mean-value zero condition for H^p_A atoms is required only for these that are supported on small balls. Therefore, the classical Hardy space $H^p(\mathbb{R}^d)$ is always a proper subspace of H^p_A for $\frac{d}{d+\delta'}<p\le 1$.

REMARK 2. Let us recall that in the classical theory of Hardy spaces $H^p(\mathbb{R}^d)$ the condition $\int a=0$ is required for all atoms and higher order cancellation conditions are not needed provided $\frac{d}{d+1}<p\le 1$. Therefore it is natural to ask if there is an atomic decomposition of the elements of the space H^p_A for $\frac{d}{d+1}<p\le\frac{d}{d+\delta'}$. The answer is yes, however, for these values of p's different type cancellation conditions for atoms may occur. This will be studied in a forthcoming paper.

The function $m(x,V)$ appeared in [Sh] where fundamental solutions and boundedness of Riesz transforms associated with the operator A on L^p spaces, $p>1$, were investigated.

2. Useful estimates. In this section we state some result concerning properties of the function $m(x,V)$. Further we present a number of estimates of the kernels associated with the semigroup $\{T_t\}_{t>0}$.

LEMMA 2.1. *There exists a constant* $C>0$ *such that for every* $0<r<R$ *we have*

$$\frac{1}{r^{d-2}}\int_{B(x,r)}V(y)\,dy\le C\Big(\frac{r}{R}\Big)^\delta\frac{1}{R^{d-2}}\int_{B(x,R)}V(y)\,dy.$$

Proof. See [Sh, Lemma 1.2]. ∎

Corollary 2.2. *If $r < \rho(x) = m(x,V)^{-1}$ then*

$$\int_{B(x,r)} V(y)\,dy \le C(rm(x,V))^{\delta} r^{d-2}.$$

Lemma 2.3. *For every $C_1 > 0$ there exists a constant $C_2 > 0$ such that*

$$C_2^{-1} \le \frac{m(x,V)}{m(y,V)} \le C_2, \quad \text{for } |x-y| \le \frac{C_1}{m(x,V)} \tag{2.4}$$

Moreover, there exist constants $C > 0$, $k_0 > 0$ such that

$$m(y,V) \le C(1+|x-y|m(x,V))^{k_0} m(x,V), \tag{2.5}$$

$$m(y,V) \ge \frac{m(x,V)}{C(1+|x-y|m(x,V))^{k_0/(1+k_0)}}. \tag{2.6}$$

Proof. This is Lemma 1.4 of [Sh]. ∎

Lemma 2.7. *There exists a constant $C > 0$ such that if $r > \rho(x) = m(x,V)^{-1}$ then*

$$\int_{B(x,r)} V(y)\,dy \le (rm(x,V))^{C} m(x,V)^{2-d}.$$

Proof. See [Sh, Lemma 1.8]. ∎

We say that a function ω defined on $\mathbb{R}^d$ is rapidly decaying if for every $N > 0$ there exists a constant C_N such that

$$|\omega(x)| \le C_N(1+|x|)^{-N}.$$

Corollary 2.8. *If ω is a rapidly decaying nonnegative function, then there exists a constant $C > 0$ such that*

$$\int V(y)\omega_t(x-y)dy \le \begin{cases} \frac{C}{t}(m(x,V)t^{1/2})^{\delta} & \text{for } t \le m(x,V)^{-2}, \\ Ct^{-d/2}(\sqrt{t}m(x,V))^{C} m(x,V)^{2-d} & \text{for } t > m(x,V)^{-2}, \end{cases}$$

where $\omega_t(x) = t^{-d/2}\omega(t^{-1/2}x)$.

Proof. The estimate is a consequence of Corollary 2.2 and Lemma 2.7. ∎

The corollary below follows from Lemma 2.3.

Corollary 2.9. *For every rapidly decaying function ω there is a rapidly decaying function $\tilde{\omega}$ such that for every $N \ge 0$ we have*

$$\frac{\omega_t(x-y)}{(1+\sqrt{t}m(x,V))^N} \le \frac{\tilde{\omega}_t(x-y)}{(1+\sqrt{t}m(x,V)+\sqrt{t}m(y,V))^N}.$$

The Kato-Trotter formula asserts that

$$p_t(x-y) - k_t(x,y) = \int_0^t \int_{\mathbb{R}^d} p_s(x-z)V(z)k_{t-s}(z,y)\,dz\,ds = q_t(x,y). \tag{2.10}$$

Theorem 2.11. *There exists a rapidly decaying function $\omega \ge 0$ such that for every $N > 0$ there exists a constant C_N such that*

$$k_t(x,y) \le C_N\big(1+\sqrt{t}m(x,V)+\sqrt{t}m(y,V)\big)^{-N}\omega_t(x-y).$$

Proof. Let $G(x,y)$ denote the fundamental solution of the operator A. Theorem 2.7 of [Sh] asserts that for every $n \geq 0$ there exists a constant C_n such that

$$0 \leq G(x,y) \leq \frac{C_n}{\big(1+|x-y|(m(x,V)+m(y,V))\big)^n |x-y|^{d-2}}. \tag{2.12}$$

It is not difficult to check using (2.12) that for every positive integer l there exists a constant C_l such that

$$|m(x,V)^{2l} A^{-l} f(x)| \leq C_l \mathbf{M}^l f(x), \tag{2.13}$$

where $\mathbf{M}$ is the classical Hardy-Littlewood maximal operator.

Since $\{T_t\}$ is a holomorphic semigroup on $L^2(\mathbb{R}^d)$ we have

$$\|\partial_t^n T_t\|_{L^2\to L^2} \leq C_n t^{-n}. \tag{2.14}$$

Now (1.2) combined with (2.14) leads to

$$|\partial_t^n k_t(x,y)| \leq \frac{C_n}{t^{n+d/2}}.$$

Applying (2.13) we get

$$|k_t(x,y)| \leq \frac{C_n}{t^{n+d/2} m(x,V)^{2n}}. \tag{2.15}$$

Finally Theorem 2.11 follows from (1.2), (2.15) and symmetry of the kernel $k_t(x,y)$. ■

PROPOSITION 2.16. *There exists a rapidly decaying function $\omega \geq 0$ such that*

$$q_t(x,y) \leq (\sqrt{t} m(x,V))^\delta \omega_t(x-y).$$

Proof. By definition

$$q_t(x,y) = \int_0^t \int p_s(x-z)V(z)k_{t-s}(z,y)\,dz\,ds = \int_0^{t/2}\int + \int_{t/2}^t \int = I_1 + I_2.$$

Using Theorem 2.11, we have

$$I_1 \leq \int_0^{t/2} \int_{|z|\leq|x-y|/2} p_s(z)V(z+x)\omega_t(x-y)\,dz\,ds$$

$$+ \int_0^{t/2} \int_{|z|>|x-y|/2} s^{-d/2} t^{-d/2} e^{-c|x-y|/\sqrt{s}} e^{-c|z|/\sqrt{s}} V(z+x)\,dz\,ds.$$

Applying Corollary 2.8, we obtain

$$I_1 \leq \omega_t(x-y)(\sqrt{t} m(x,V))^\delta.$$

The estimates for I_2 go in the same way by using Lemma 2.3. ■

Using the same arguments as in the proof of Proposition 2.16 and the fact that $q_t(x,y) = q_t(y,x)$ we can show the following

PROPOSITION 2.17. *For every $0 < \delta'' < \delta'$ there exists a rapidly decaying function $\omega \geq 0$ such that for every $C > 0$ there exists a constant C' such that for every $h, x, y \in \mathbb{R}^d$, $|h| \leq |x-y|/4$, $|h| \leq C\rho(y)$ we have*

$$|q_t(x,y+h) - q_t(x,y)| \leq C'(|h|(m(x,V))^{\delta''} \omega_t(x-y). \ ■ \tag{2.18}$$

3. Scale of atomic H^p spaces. Fix $0 < \varepsilon \le 1$ We say that a is an atom for the space $H^p_{\varepsilon;m}$ associated to a ball $B(y_0, r)$ if

$$\text{supp } a \subset B(y_0, r), \tag{3.1}$$

$$\|a\|_{L^\infty} \le |B(y_0, r)|^{-1/p}, \tag{3.2}$$

$$r \le \varepsilon\rho(y_0), \tag{3.3}$$

$$\text{if } r < \frac{1}{4}\varepsilon\rho(y_0) \text{ then } \int a(x)\,dx = 0. \tag{3.4}$$

The atomic $H^p_{\varepsilon;m}$ quasi-norm is defined by

$$\|f\|^p_{H^p_{\varepsilon;m}} = \inf \sum |\lambda_j|^p, \tag{3.5}$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$, a_j being $H^p_{\varepsilon;m}$ atoms and λ_j being scalars. Let us note that $\|f\|^p_{H^p_{1;m}} = \|f\|^p_{H^p_m}$.

Obviously, there exists a constant C such that if $\varepsilon' \le \varepsilon$ then

$$\|f\|^p_{H^p_{\varepsilon';m}} \le C\|f\|^p_{H^p_{\varepsilon;m}}. \tag{3.6}$$

Moreover if $\varepsilon' \le \varepsilon \le 1$ then there exists a constant $C_{\varepsilon',\varepsilon}$ such that

$$\|f\|^p_{H^p_{\varepsilon;m}} \le C_{\varepsilon',\varepsilon}\|f\|^p_{H^p_{\varepsilon';m}}. \tag{3.7}$$

For fixed $0 < \varepsilon \le 1$ we define the maximal operator $\mathcal{P}^*_\varepsilon$ by the formula

$$\mathcal{P}^*_\varepsilon f(x) = \sup_{0<t\le\varepsilon^2\rho(x)^2} |f * p_t(x)|. \tag{3.8}$$

PROPOSITION 3.9. *For every $p \in (\frac{d}{d+\delta'}, 1)$ there exists a constant $C > 0$ independent of $\varepsilon \in (0, 1]$ such that for every compactly supported function $f \in L^1(\mathbb{R}^d)$ we have*

$$C^{-1}\|\mathcal{P}^*_\varepsilon f\|^p_{L^p} \le \|f\|^p_{H^p_{\varepsilon;m}} \le C\|\mathcal{P}^*_\varepsilon f\|^p_{L^p}. \tag{3.10}$$

Proof. First we prove that there is a constant $C > 0$ such that

$$\|\mathcal{P}^*_\varepsilon f\|_{L^p} \le C\|f\|_{H^p_{\varepsilon;m}}. \tag{3.11}$$

Let a be a $H^p_{\varepsilon;m}$ atom associated with a ball $B(y_0, r)$. If a has the cancellation condition $\int a = 0$, then $\|\mathcal{P}^*_\varepsilon a\|_{L^p} \le C$. If $\int a \ne 0$ then, by the definition, $\frac{1}{4}\varepsilon\rho(y_0) \le r \le \varepsilon\rho(y_0)$. Obviously $\|\mathcal{P}^*_\varepsilon a\|_{L^p(B(y_0,4r))} \le C$. If $x \notin B(y_0, 4r) = B(y_0, r)^*$ then, by Lemma 2.3, $\rho(x) \le C\max(|x - y_0|^{k_0/(k_0+1)}\rho(y_0)^{1/(k_0+1)}, \rho(y_0))$. Therefore for $0 < t < (\varepsilon\rho(x))^2$, we have

$$\begin{aligned}|p_t * a(x)| \le{}& C\|a\|_{L^1}\varepsilon^{M-d}\rho(y_0)^{(M-d)/(1+k_0)}|x - y_0|^{-(M+dk_0)/(1+k_0)}\\ &+C\|a\|_{L^1}\varepsilon^{M-d}\rho(y_0)^{M-d}|x - y_0|^{-M}.\end{aligned}$$

This leads to $\int_{|x-y_0|>4r}(\mathcal{P}^*_\varepsilon a(x))^p\,dx \le C$. Thus (3.11) is proved.

The proof of the second inequality in (3.10) is a combination of a number of lemmas.

Let $\varphi^{(\alpha)}$ be a family of C^∞ functions on $\mathbb{R}^d$ and $B_\alpha = B(y_\alpha, r_\alpha)$ be a family of balls such that there exists a constant $C > 0$ such that

$$\text{supp}\,\varphi_\alpha \subset B(y_\alpha, r_\alpha) = B_\alpha, \quad r_\alpha = \rho(y_\alpha), \tag{3.12}$$

$$\#\{\alpha' : B(y_{\alpha'}, Rr_{\alpha'}) \cap B(y_\alpha, r_\alpha) \neq \emptyset\} \leq CR^C \quad \text{for} \quad R > 2, \tag{3.13}$$

$$0 \leq \varphi_\alpha \leq 1, \quad \|\nabla\varphi_\alpha\|_{L^\infty} \leq Cm(y_\alpha, V), \tag{3.14}$$

$$\sum_\alpha \varphi_\alpha \equiv 1. \tag{3.15}$$

LEMMA 3.16. *There is a family of constants $c(\varepsilon) > 0$, $\lim_{\varepsilon\to 0} c(\varepsilon) = 0$, such that*

$$\Big\| \sup_{0<t<(\varepsilon \max(\rho(y_\alpha),\rho(x)))^2} |(f\varphi^{(\alpha)}) * p_t(x)|\Big\|^p_{L^p(B^{**c}_\alpha)} \leq c(\varepsilon)\|f\varphi^{(\alpha)}\|^p_{H^p_{\varepsilon;m}}. \tag{3.17}$$

Proof. It suffices to prove (3.17) if $f\varphi^{(\alpha)}$ is replaced by an $H^p_{\varepsilon;m}$ atom a associated with a ball $B(y_0, r)$, where $B(y_0, r) \cap B^*_\alpha \neq \emptyset$. Let us note that for $x \in B^{**c}_\alpha$, we have

$$\max(\rho(y_\alpha), \rho(x)) \leq C|x - y_0|^{k_0/(1+k_0)}\rho(y_0)^{1/(1+k_0)}.$$

Therefore, if the atom a does not satisfy the cancellation condition, then

$$|a * p_t(x)| \leq C_M\varepsilon^{M-d}\|a\|_{L^1}\rho(y_0)^{(M-d)/(1+k_0)}|x - y_0|^{-(M+dk_0)/(1+k_0)}.$$

Consequently, the left hand side of (3.17) is estimated by $C_M\varepsilon^{Mp-d}$.

If a satisfies the cancellation condition, then $r < \varepsilon\rho(y_0)/4$ and

$$|a * p_t(x)| \leq Cr^{d+1-d/p}|x - y_0|^{-d-1}.$$

Thus the left hand side of (3.17) is bounded by $C\varepsilon^{dp+p-d}$. ∎

COROLLARY 3.18. *There exists $0 < \varepsilon_0 \leq 1$ and a constant $C > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ we have*

$$\|f\varphi^{(\alpha)}\|^p_{H^p_{\varepsilon;m}} \leq C\|\mathcal{P}^*_\varepsilon(f\varphi^{(\alpha)})\|^p_{L^p}. \tag{3.19}$$

Proof. Since $f\varphi^{(\alpha)}$ is supported on $B(y_\alpha, \rho(y_\alpha))$ the theory of local Hardy spaces (cf. [G]) asserts that

$$\|f\varphi^{(\alpha)}\|^p_{H^p_{\varepsilon;m}} \leq C\Big\| \sup_{0<t<(\varepsilon\rho(y_\alpha))^2} |(f\varphi^{(\alpha)}) * p_t(x)|\Big\|^p_{L^p}$$

$$\leq C\Big\| \sup_{0<t<(\varepsilon\rho(y_\alpha))^2} |(f\varphi^{(\alpha)}) * p_t(x)|\Big\|^p_{L^p(B^{**}_\alpha)} + C\Big\| \sup_{0<t<(\varepsilon\rho(y_\alpha))^2} |(f\varphi^{(\alpha)}) * p_t(x)|\Big\|^p_{L^p(B^{**c}_\alpha)}.$$

Using Lemma 3.16 we obtain

$$\|f\varphi^{(\alpha)}\|^p_{H^p_{\varepsilon;m}} \leq C\|\mathcal{P}^*_\varepsilon(f\varphi^{(\alpha)})\|^p_{L^p} + c(\varepsilon)\|f\varphi^{(\alpha)}\|^p_{H^p_{\varepsilon;m}}.$$

This ends the proof of (3.19). ∎

For $\varepsilon \in (0, 1]$ we set

$$\mathcal{T}^*_\varepsilon f(x) = \sum_\alpha \sup_{0<t\leq\varepsilon^2\rho(x)^2} \Big| \int \big(\varphi^{(\alpha)}(x) - \varphi^{(\alpha)}(y)\big)p_t(x - y)f(y)\, dy\Big|. \tag{3.20}$$

LEMMA 3.21. *There exists a family of constants $c(\varepsilon) > 0$, $\lim_{\varepsilon\to 0} c(\varepsilon) = 0$ such that*

$$\|\mathcal{T}^*_\varepsilon f\|^p_{L^p} \leq c(\varepsilon)\|f\|^p_{H^p_{\varepsilon;m}} \tag{3.22}$$

Proof. Since $0 < p < 1$ it suffices to show that

$$\sum_\alpha \int \Big(\sup_{0<t<(\varepsilon\rho(x))^2} |\varphi^{(\alpha)}(x)P_t f(x) - P_t(\varphi^{(\alpha)} f)(x)|^p \Big)\, dx \le c(\varepsilon)\|f\|^p_{H^p_{\varepsilon;m}}, \tag{3.23}$$

where $P_t f(x) = f * p_t(x)$.

Set

$$\mathcal{J}_{\alpha,t} f(x) = \varphi^{(\alpha)}(x)P_t f(x) - P_t(\varphi^{(\alpha)} f)(x) = \int \big[\varphi^{(\alpha)}(x) - \varphi^{(\alpha)}(y)\big] p_t(x-y) f(y)\, dy.$$

Let a be an $H^p_{\varepsilon;m}$ associated with a ball $B(y_0, r)$. Let $\mathcal{I}_1 = \{\alpha :\ B(y_0,r) \cap B^{**}_\alpha = \emptyset\}$, and $\mathcal{I}_2 = \{\alpha :\ B(y_0,r) \cap B^{**}_\alpha \ne \emptyset\}$. We note that the number of elements in $\mathcal{I}_2$ is bounded by a constant independent of a. If $\alpha \in \mathcal{I}_1$, then $\mathcal{J}_{\alpha,t} a(x) = \int \varphi^{(\alpha)}(x) p_t(x-y) a(y)\, dy$. Thus, by the same arguments as in the proof of Lemma 3.16, we get

$$\sum_{\alpha\in\mathcal{I}_1} \int \sup_{0<t<(\varepsilon\rho(x))^2} |\mathcal{J}_{\alpha,t} a(x)|^p\, dx \le c(\varepsilon).$$

Let us consider α being in $\mathcal{I}_2$. If $x \notin B(y_0, \rho(y_0))^*$, then

$$\mathcal{J}_{\alpha,t} a(x) = \int p_t(x-y)\varphi^{(\alpha)}(y) a(y)\, dy.$$

Since $\|\varphi^{(\alpha)} a\|_{H^p_{\varepsilon;m}} \le C$, where the constant C is independent of ε, a and α, the same arguments as in the proof of Lemma 3.16 can be applied to obtain

$$\sum_{\alpha\in\mathcal{I}_2} \int_{B(y_0,\rho(y_0))^{*c}} \sup_{0<t<(\varepsilon\rho(x))^2} |\mathcal{J}_{\alpha,t} a(x)|^p\, dx \le c(\varepsilon).$$

If $x \in B(y_0, \rho(y_0))^*$, then $\rho(x) \sim \rho(y_0)$. Thus

$$|\mathcal{J}_{\alpha,t} a(x)| = \Big| \int \frac{\sqrt{t}}{\rho(y_0)} \Psi_t(x,y) a(y)\, dy \Big| \le C\varepsilon \Big| \int \Psi_t(x,y) a(y)\, dy \Big|,$$

where $\Psi_t(x,y) = \rho(y_0) t^{-1/2}(\varphi^{(\alpha)}(x) - \varphi^{(\alpha)}(y)) p_t(x-y)$. We note that $|\Psi_t(x,y) \le \omega_t(x-y)$ and $|\nabla_x \Psi_t(x,y)| \le t^{-1/2}\omega_t(x-y)$ for $0 < t < C\rho(y_0)^2$. Therefore the standard methods can be used in order to show that

$$\sum_{\alpha\in\mathcal{I}_2} \int_{B(y_0,\rho(y_0))^*} \sup_{0<t<(\varepsilon\rho(x))^2} |J_{\alpha,t} a(x)|^p\, dx \le c(\varepsilon). \ \blacksquare$$

Now we are in a position to finish the proof of the second inequality in (3.10). By (3.15), Corollary 3.18, and Lemma 3.21, we obtain

$$\begin{aligned} \|f\|^p_{H^p_{\varepsilon;m}} &\le C \sum_\alpha \|\varphi^{(\alpha)} f\|^p_{H^p_{\varepsilon;m}} \\ &\le C \sum_\alpha \|\mathcal{P}^*_\varepsilon(\varphi^{(\alpha)} f)\|^p_{L^p} \\ &\le C\|\mathcal{P}^*_\varepsilon f\|^p_{L^p} + C\|\mathcal{T}^*_\varepsilon f\|^p_{L^p} \le C\|\mathcal{P}^*_\varepsilon f\|^p_{L^p} + Cc(\varepsilon)\|f\|^p_{H^p_{\varepsilon;m}}. \end{aligned}$$

Taking ε_0 sufficiently small we get

$$\|f\|_{H^p_{\varepsilon;m}} \le C\|\mathcal{P}^*_\varepsilon f\|^p_{L^p}$$

provided $0 < \varepsilon \le \varepsilon_0$. From (3.6) and (3.7) we conclude that (3.22) holds for $0 < \varepsilon \le 1$. This completes the proof of Proposition 3.9. $\blacksquare$

We define the maximal function

$$\mathcal{Q}^*_\varepsilon f(x) = \sup_{0<t<\varepsilon^2\rho(x)^2} \left| \int q_t(x,y)f(y)\,dy \right|.$$

PROPOSITION 3.24. *Assume that* $\frac{d}{d+\delta'} < p \le 1$. *Then there exists a family of constants* $\gamma_\varepsilon > 0$, $\gamma_\varepsilon \to 0$ *as* $\varepsilon \to 0$, *such that*

$$\|\mathcal{Q}^*_\varepsilon f\|^p_{L^p} \le \gamma_\varepsilon \|f\|^p_{H^p_{\varepsilon;m}}.$$

Proof. Let a be an $H^p_{\varepsilon;m}$ atom associated with a ball $B(y_0,r)$. Applying Proposition 2.16, we get

$$\mathcal{Q}^*_\varepsilon a(x) \le C\varepsilon^\delta |B_r|^{1-\frac{1}{p}}.$$

Therefore

$$\int_{B(y_0,4r)} \mathcal{Q}^*_\varepsilon a(x)^p\,dx \le C\varepsilon^{p\delta}.$$

In order to estimate $\mathcal{Q}^*_\varepsilon a(x)$ outside the ball $B(y_0,4r)$ we need to consider two cases. If a satisfies the cancellation condition $\int a = 0$, then, by Proposition 2.17,

$$\left| \int q_t(x,y)a(y)\,dy \right| = \left| \int (q_t(x,y) - q_t(x,y_0))a(y)\,dy \right|$$
$$\le C(\sqrt{t}m(x,V))^{\delta''} t^{-\delta''/2} \omega_t(x-y_0) r^{\delta''+d-d/p}.$$

Thus

$$\mathcal{Q}^*_\varepsilon a(x) \le C\varepsilon^{\delta''} |x-y_0|^{-d-\delta''} r^{\delta''+d-d/p},$$

and, consequently,

$$\int_{|x-y_0|>4r} (\mathcal{Q}^*_\varepsilon a(x))^p\,dx \le C\varepsilon^{\delta'' p}.$$

If $\int a \neq 0$, then $r \sim \rho(y_0)$. Since $t < \varepsilon^2\rho(x)^2$, applying Proposition 2.16 and Lemma 2.3, we get

$$\left| \int q_t(x,y)a(y)\,dy \right| \le (\sqrt{t}m(x,V))^\delta \int |a(y)|\omega_t(x-y)\,dy$$
$$\le C_n \frac{(\sqrt{t}m(x,V))^\delta r^{d-d/p}}{m(y_0,V)^n |x-y_0|^{d+n}}.$$

Therefore

$$\int_{|x-y_0|>4r} (\mathcal{Q}^*_\varepsilon a(x))^p\,dx \le C\varepsilon^{\delta p}. \ \blacksquare$$

4. Proof of Theorem 1.11. First we prove the first inequality in (1.12), that is,

$$\|f\|^p_{H^p_m} \le C\|f\|^p_{H^p_A}. \tag{4.1}$$

By (3.6) and (3.7) it suffices to show that there is $\varepsilon > 0$ such that

$$\|f\|^p_{H^p_{\varepsilon;m}} \le C\|f\|^p_{H^p_A}. \tag{4.2}$$

Applying Proposition 3.9 we have that there exists a constant C_1 independent of ε such that

$$\|f\|^p_{H^p_{\varepsilon;m}} \le \sum_\alpha \|\varphi_\alpha f\|^p_{H^p_{\varepsilon;m}} \le C_1 \sum_\alpha \|\mathcal{P}^*_\varepsilon(\varphi_\alpha f)\|^p_{L^p}$$
$$\le C_1\|\mathcal{T}^*_\varepsilon f\|^p_{L^p} + C_1 \sum_\alpha \|\varphi_\alpha(\mathcal{P}^*_\varepsilon f)\|^p_{L^p}.$$

By Lemma 3.21, we obtain

$$\|f\|^p_{H^p_{\varepsilon;m}} \le C_1 c(\varepsilon)\|f\|^p_{H^p_{\varepsilon;m}} + C_1 \sum_\alpha \|\varphi_\alpha(\mathcal{P}^*_\varepsilon f)\|^p_{L^p}$$
$$\le C_1 c(\varepsilon)\|f\|^p_{H^p_{\varepsilon;m}} + C_1 \sum_\alpha \|\varphi_\alpha(\mathcal{Q}^*_\varepsilon f)\|^p_{L^p} + C_1 \sum_\alpha \|\varphi_\alpha(\mathcal{M} f)\|^p_{L^p}.$$

Finally, taking ε sufficiently small using (3.12)-(3.15) and Proposition 3.24, we obtain (4.2).

For proving the converse inequality it suffices to show that

$$\|\mathcal{M}a\|_{L^p} \le C$$

for every a being an H^p_m atom. So, let a be such an atom associated with a ball $B(y_0, r)$. Obviously it follows from (1.2) that

$$\|\mathcal{M}a\|^p_{L^p(B(y_0,4r))} \le C.$$

In order to estimate $\mathcal{M}a(x)$ for $x \notin B(y_0, 4r)$ we use (2.10) combined with Proposition 2.17 if a satisfies the cancellation condition, and with Proposition 2.16 otherwise. ∎

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