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SETS OF UNIQUENESS AND SETS OF MULTIPLICITY

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Rajchman, Zygmund, and Marcinkiewicz all made important contributions to the theory of sets of uniqueness and sets of multiplicity. We shall meet these results in the course of the history of the subject, which spans more than 140 years, from the work of Riemann and Cantor to recent results.

1. The prehistory: Riemann and Cantor. Riemann's habilitation thesis, "Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe," was submitted in Göttingen in 1854, but it was published only in 1867, a year after Riemann's death, by Richard Dedekind. It became famous immediately, mainly because it introduced the Riemann integral. However, the main part of the thesis deals with the following problem: to investigate and characterize functions f that are sums of everywhere-convergent trigonometric series,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

The Fourier formulas $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$ are of no use in general, since f need not be integrable in the sense of Riemann. The Fatou example (1906),

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{\log n},$$

shows that f need not be integrable in the sense of Lebesgue either [13]. Only the second totalization of Denjoy (1921) allows one to compute the coefficients a_n and b_n by using the Fourier formulas [12].

Riemann's idea was to use the "second integral" of f, specifically, the continuous function

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$$f^{(-2)}(x) = \sum_{n=1}^{\infty} \frac{-A_n(x)}{n^2}$$

and to prove that its second symmetric derivative is $f(x)$. (We neglect the linear term.) This was a bright idea, however Riemann's paper contains two weak points:

1. Riemann used the fact that the coefficients a_n and b_n are bounded, and he wrote that they tend to zero. He did not prove this, except in the case f is integrable.
2. Riemann did not consider the uniqueness question: Given f, are the coefficients a_n and b_n well defined?

Cantor filled both gaps in 1870. In his article of March 1870 entitled "Über einen die trigonometrischen Reihen betreffenden Lehrsatz," Cantor proved that the coefficients tend to zero; later in April 1870 in his paper entitled "Beweis, dass eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen lässt" he proved the uniqueness result [7]. Two years later, in 1872, he raised this question: Does the uniqueness result hold when we relax the assumption by allowing exceptional values of x? In other words, given a set E of real numbers, does the fact that $f(x)$ exists and vanishes outside of E imply that the a_n and b_n are zero?

Cantor's 1872 article, "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen," is famous because it contains his views on real numbers and the first notions of Cantor's set theory. It is also the beginning of the theory of sets of uniqueness (U-sets) and sets of multiplicity (M-sets). Referring to the question above, E is a U-set if the answer is "yes" and an M-set if the answer is "no." Cantor proved that a point, a finite set, and a reducible set (a countable compact set) are U-sets [8].

2. The new start after 1900. The subject lay stagnant for many years until the famous Comptes rendus notes by Fejér (1900) and Lebesgue (1901) gave a new impetus to the study of trigonometric series [14,31].

After Fejér proved his celebrated theorem on (C,1)-summability of Fourier series of continuous functions (in the more general form involving $f(x+0)+f(x-0)$), he turned his attention to general trigonometric series. He was able to translate the Riemann theory when ordinary convergence is replaced with (C,1)-summability. The idea in this case to is use $f^{(-4)}$ instead of $f^{(-2)}$. Fejér observed that the series

$$\frac{1}{2} + \cos x + \cos 2x + \cdots$$

is (C,1)-summable to zero at every point except the multiplicities of 2π. Based on this observation, he suggested to the young Marcel Riesz that he redo the Cantor theory within the framework of (C,1) and other summability processes. The story and results can be found in Riesz's 1907 Comptes rendus note entitled "Sur les séries trigonométriques," which was published when he was 20, and in his thesis, which was translated into English by J. Horváth and published in his collected papers [43,44].

Lebesgue first applied his results on measure and integration to geometrical questions (lengths and areas) and to the "primitive" problem, that is, to find a function f whose derivative f' is given. He later turned to the study of trigonometric series. His first publication was a Comptes rendus note in 1902 entitled "Un théorème sur les séries trigonométriques." This was followed in 1903 by an important article entitled "Sur les séries trigonométriques." His book, "Leçons sur les séries trigonométriques," appeared in 1906. The first note concerns bounded functions that are sums of everywhere convergent trigonometric series. According to Cantor, the coefficients are well defined when the function is given. Lebesgue showed that they are given by the Fourier formulas $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$, where the integral is the Lebesgue integral. He then added the following comment: "Le résultat précédent subsiste si les valeurs où $f(x)$ n'est pas définie forment un ensemble fermé de mesure nulle." This is a mistake, and Lebesgue corrected it immediately in his 1903 article: The result holds when $f(x)$ is defined except on a reducible set and *not* a null set [32,33]. However, the question on null sets was left open. Lebesgue knew that a set of positive Lebesgue measure is an M-set, since this is implicit in his book [34]. W. H. Young extended Cantor's theorem in 1909 by proving that every countable set is a U-set [49].

It was only in 1916 that D. Menšov (Menchoff) constructed a closed M-set of zero Lebesgue measure in the form of a Cantor set whose dissection ratio is $1/k$ at the k^{th} step [37]. (This is the standard Cantor construction, except that one removes the middle k^{th} at the k^{th} step rather than removing the middle 3^{rd} at each step. Of course one must start with $k = 2$.)

A. Rajchman provided the first result on perfect sets in the opposite direction in 1922: He showed that the ordinary Cantor set (with dissection ratio 1/3) is a U-set. His article entitled "Sur l'unicité du développement trigonométrique" introduced the notion of H-sets, which were first called "ensembles du type de Hardy–Littlewood–Steinhaus." Here is Rajchman's definition: Given $E \subset \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, let d_k be the length of the largest interval contained in $\mathbb{T} \setminus kE$, where $k = 1, 2, \ldots$; E is called an H-set if $\overline{\lim}\, d_k > 0$. All H-sets are U-sets [40].

3. Zygmund and Bari. Antoni Zygmund was a student of Rajchman, and Nina Bari was a student of Menšov. In 1923 both Zygmund and Bari were working on the same subject, namely, the countable unions of U-sets. They arrived at the same result: Every countable union of closed U-sets is a U-set. In Zygmund's book, Trigonometric Series, Vol. I, p. 349, this is called "Theorem of N. Bary" [56]. Zygmund liked this result very much, and he could have claimed priority. But he did not, and here is the story.

The Comptes rendus dated 1 October 1923 contains Zygmund's note entitled "Sur les séries trigonométriques," where one finds the following statement: "Si les ensembles $E_1, E_2, \ldots, E_n, \ldots$ sont tous de type U, l'ensemble $E_1 + E_2 + \cdots$ est aussi du type U." (When the E_n are H-sets, the result was already established by Nina Bari.) There was a mistake in this statement, and Zygmund corrected it immediately: The sets $E_1, E_2, \ldots, E_n, \ldots$ should be *closed* U-sets. The correction was published in the Comptes rendus dated 22 October 1923 [50].

On 3 December 1923, Nina Bari submitted a note entitled "Sur l'unicité du développement trigonométrique" that contained the correct statement [3]. She mentioned Zygmund's note of 1 October without being aware of the correction published on 22 October.

The final step was taken by Zygmund in a note dated 1 January 1924 entitled "Sur les séries de Fourier restreintes." He related the whole story, which established his priority, and then concluded that "la priorité de ce théorème revient à Melle Bary" [51].

This is a little more than fair play. Once I asked Professor Zygmund why he gave up that way. He told me that he had been really puzzled, consulted Rajchman, and followed his advice: "Never compete with a girl."

4. Zygmund and Marcinkiewicz. Zygmund wrote a series of important articles on U-sets and their generalizations between 1926 and 1937. I shall return to some of these in a moment, but first I am going to jump to the joint paper from 1937 by Zygmund and Marcinkiewicz entitled "Two theorems on trigonometric series" [36].

The first theorem in this paper says that the class of U-sets is invariant under homotheties. This is a simple consequence of the fact that compact U-sets are the same for trigonometric series (the usual case) and for trigonometric integrals, as Zygmund had already observed in 1928.

The second theorem is actually an example of a summation process for trigonometric series for which the analysis of Cantor's theorem fails: There exists a nonzero trigonometric series that is summable to zero everywhere. This is very much in the spirit of the joint work by Zygmund and Marcinkiewicz on generalized derivatives.

The first theorem by Zygmund and Marcinkiewicz is constantly used in the theory of U-sets. For example, given that a special triadic Cantor set is an H-set, hence a U-set, all triadic Cantor sets are U-sets. The same holds for all Cantor sets of the form

$$E_\xi(a,b) = \Big\{a + b\sum_{n=1}^{\infty} \varepsilon_n \xi^n, \varepsilon_n = \pm 1\Big\},$$

where ξ^{-1} is an integer ≥ 3, a is real, and $b > 0$.

5. Zygmund and Salem. One of the pearls of the theory of U-sets is that a set E_ξ (meaning $E_\xi(a,b)$ for some a and b) is a U-set if and only if ξ^{-1} is an algebraic integer whose conjugates (except itself) lie inside the unit disc of the complex plane. This famous theorem by Salem and Zygmund (1955) provides one of the most striking relations between number theory and trigonometric series.

The theorem had been guessed by Salem long before it was proved. A necessary condition for E_ξ to be a U-set is that

$$\overline{\lim_{u\to\infty}} \Big|\prod_{n=1}^{\infty} \cos \xi^n u\Big| > 0,$$

since otherwise E_ξ carries a probability measure whose Fourier transform tends to zero at infinity, which implies that E_ξ is an M-set. In particular, if ξ^{-1} is a rational > 2 but not an integer, then E_ξ is an M-set. This is a previous result by Nina Bari [4]. But there was no possibility to use Rajchman's H-sets to prove that E_ξ is a U-set in the general case when $\xi^{-1} \in PV$, the Pisot–Vijayaraghavan class defined above.

The decisive step is due to I. I. Pyateckiĭ-Šapiro (Piatetski-Shapiro), who introduced a generalization of H-sets called H^m-sets. These are obtained by a similar process involving m-dimensional space, and they enjoy the property of also being U-sets. He was able to prove (1953) that E_ξ is a U-set whenever $\xi^{-1} \in PV$ and $\xi^{-1} > 2^n$, where n is the degree of ξ^{-1} [39].

Zygmund became aware of the result and the method of Piatetski-Shapiro, he wrote to Salem, and the final result was obtained in a few days [47]. Many proofs have been written. The easiest one, which is due to Yves Meyer, can be found in the second edition of the Kahane–Salem book, pp. 203–204 [22].

6. Variations around U-sets. The first variation is to replace ordinary convergence by a summability process such as (C,1), Poisson, etc., and it goes back to Fejér and Marcel Riesz. The examples

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} n \sin nx$$

show that the Cantor uniqueness theorems cannot be extended without assumptions about the coefficients a_n, b_n. Marcel Riesz proved the following three theorems in 1907:

1. If

$$\text{(MR)} \qquad \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{n^2} < \infty$$

   and if the series $\sum_{n=1}^{\infty} A_n(x)$ is (C,1)-summable to zero everywhere, then it is the null series, that is, $a_n = b_n = 0$ for all n.
2. If $|a_n|+|b_n| = o(1)$ as $n \to \infty$ and if the series is (C,1)-summable to zero everywhere except a reducible set, then the same conclusion holds.
3. If we assume (MR) and if the series is (C,1)-summable to a bounded Lebesgue-integrable function everywhere except a reducible set, then the series is the Fourier–Lebesgue series of this function [43,44].

The second variation appears in this last theorem, and it is due to Lebesgue. It is based on considering convergence or summability to an integrable function. Of course, it depends on the notion of integral being used.

A third variation is to consider specific orthogonal series instead of trigonometric series.

All three of these variations are the subject of important papers by Rajchman, Zygmund, and Rajchman and Zygmund [5, 41, 42]. Many of the theorems can be found in Volume 1 of Zygmund's Selected Papers. Zygmund's 1930 article, "Théorie riemannienne de certains systèmes orthogonaux" contains a history of the subject and the main classification in the form of the following definitions [54].

M is a summation process stronger (in the wide sense) than ordinary convergence, and only series $\sum_{n=0}^{\infty} A_n(x)$ with $|a_n| + |b_n| = o(1)$ as $n \to \infty$ are considered.

U$_\mathrm{M}$-sets: E is a U$_\mathrm{M}$-set if, whenever $\sum_{n=0}^{\infty} A_n(x)$ is M-summable to zero on the complement of E, it is the null series.

U'_{M}-sets: E is a U'_{M}-set if, whenever $\sum_{n=0}^{\infty} A_n(x)$ is M-summable to $f(x)$ on the complement of E and f is Lebesgue integrable, it is the Fourier–Lebesgue series of f.

$\mathrm{U}''_{\mathrm{M}}$-sets: E is a $\mathrm{U}''_{\mathrm{M}}$-set if, whenever $\sum_{n=0}^{\infty} A_n(x)$ is M-summable to $f(x)$ on the complement of E and $f(x) \geq g(x)$, where g is Lebesgue integrable, it is the Fourier–Lebesgue series of f, which implies that f is necessarily Lebesgue integrable.

U-sets, U′-sets, and U″-sets correspond to the cases when M is ordinary convergence.

The inclusions between these classes are expressed by the following diagram:

$$\begin{tikzcd}
\mathrm{U} & \mathrm{U}' \arrow[l] & \mathrm{U}'' \arrow[l] \\
\mathrm{U}_{\mathrm{M}} \arrow[u] & \mathrm{U}'_{\mathrm{M}} \arrow[l] \arrow[u] & \mathrm{U}''_{\mathrm{M}} \arrow[l] \arrow[u]
\end{tikzcd}$$

The most striking results are the following (Rajchman, Steinhaus, Banach, Zygmund; see Zygmund's Selected Papers I, pp. 70, 128, 134, 215):

Every countable set is a $\mathrm{U}''_{\mathrm{P}}$-set, where P stands for Poisson summation.
Every closed U-set is a U'_{P}-set, and every closed U-set is a U″-set [53,54].

7. Another variation: Zygmund's U(ε)**-sets.** Zygmund's 1926 article, "Contribution à l'unicité du développement trigonometrique," considers the uniqueness problem for series $\sum_{n=0}^{\infty} A_n(x)$ that are subject to the condition

$$(\mathrm{Z}_\varepsilon) \qquad |a_n| + |b_n| = O(\varepsilon_n), \quad n \to \infty,$$

where $\varepsilon = (\varepsilon_n)$ is a given decreasing sequence tending to zero. Here is the definition:

A subset E of the circle is a U(ε)-set if, whenever a series $\sum_{n=0}^{\infty} A_n(x)$ satisfying (Z_ε) converges to zero on the complement of E, it is the null series.

Surprisingly, given any sequence ε, there are U(ε)-sets of positive Lebesgue measure, and there are even U(ε)-sets with Lebesgue measure arbitrarily close to 2π. After stating and proving this result, Zygmund posed two questions [52]: (1) Is every countable union of closed U(ε)-sets a U(ε)-set? (2) Do there exist U(ε)-sets of full Lebesgue measure?

The first question is still open. A positive answer would imply a positive answer to the second question. Using a different method, Kahane and Katznelson answered the second question in 1973 in a positive way [20]. Related results were obtained by Bernard Connes in 1976 on Hausdorff measures and dimensions of the complementary sets, which I denote by $\mathrm{C}U(\varepsilon)$ [9]. For example, if $\varepsilon_n = n^{-\alpha}$, $0 < \alpha < 1$, then there exists a $\mathrm{C}U(\varepsilon)$-set of Hausdorff dimension $1 - \alpha$, and there is no $\mathrm{C}U(\varepsilon)$-set of Hausdorff dimension $< 1 - \alpha$ [9].

A simplified version of the method used in [20] is given in [19]. The idea is to construct a probability measure μ, supported by a closed null set E, such that $\alpha_m = \sum_{n\in\mathbb{Z}} |\hat{\mu}_{m-n}|\varepsilon_n < \infty$, ($\varepsilon_{-n} = \varepsilon_n$) for all m and such that $\alpha_m = o(1)$, $|m| \to \infty$. Then, given a countable, dense set D on the circle, the algebraic sum $E + D$ is a $\mathrm{C}U(\varepsilon)$-set.

8. Thin sets and functional analysis. From now on we shall concentrate on closed subsets of the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For such a subset, E, $C(E)$ denotes the space of continuous complex-valued functions on E, and we assume that $C(E)$ has the usual

Banach space structure. $A(E)$ denotes the subspace of functions of $C(E)$ that can be represented as sums of absolutely convergent trigonometric series. $A(E)$ can be considered to be the quotient of $A(\mathbb{T})$ by the ideal $A(\pi)$ consisting of functions vanishing on E. $M[E]$ is the space of complex, bounded measures carried by E; it is the dual space of $C(E)$. The pseudomeasures are linear forms of $A(\mathbb{T})$ identified with the trigonometric series whose coefficients are bounded,

$$T = \sum_{n=-\infty}^{\infty} \hat{T}(n)\mathrm{e}^{int}, \quad \langle T, \mathrm{e}^{-int}\rangle = \hat{T}(n),$$

and also with the corresponding Schwartz distributions. $PM[E]$ is the space of pseudomeasures carried by E. $PF[E]$ is the space of pseudofunctions carried by E. The pseudofunctions are pseudomeasures whose coefficients tend to zero. With these notations we have the following characterizations or definitions:

E is a U-set if $PF[E] = \{0\}$.
E is an M-set if $PF[E] \neq \{0\}$.
E is a M_0-set if $M[E] \cap PF[E] \neq \{0\}$.
E is a Helson set if $A(E) = C(E)$, or, equivalently, if $M[E] = A(E)'$, where $A(E)'$ is the dual of $A(E)$.

Explanations and comments can be found in the Kahane–Salem book [23]. The following are the most important results:

(Piatetski-Shapiro, 1952–1954) There exist M-sets that are not M_0-sets [38].
(Helson, 1954) No Helson set is an M_0-set [15].
(Malliavin, 1959) There exist sets of non spectral synthesis [35].
(Körner, 1973) There exists a Helson M-set [29].

It is easy to check that Körner, together with Helson, implies both Piatetski-Shapiro and Malliavin. However, Körner's construction, even as simplified by Kaufman [27], is rather difficult. A slightly simplified version was given in the second edition of Kahane–Salem [22, pp. 213–216].

Let us consider two more notions: Kronecker sets and M_α-sets.

E is a Kronecker set if every continuous function with modulus 1 on E can be approximated uniformly on E by some sequence of imaginary exponentials, $\exp \mathrm{i}n_k t$, $n_k \in \mathbb{N}$, $n_k \to \infty$.

A Kronecker set is necessarily independent on $\mathbb{T}$, which means that all linear combinations of its elements with integer coefficients are different. Conversely, Kronecker's theorem states that every finite, independent set on $\mathbb{T}$ is a Kronecker set. There exist perfect Kronecker sets.

Kronecker set are "thin" is many respects: They are U-sets, Helson sets, and sets of spectral synthesis. On the other hand, there are Kronecker set of Hausdorff dimension 1. References to Kronecker sets can be found in [45] and [17].

E is an M_α-set if it carries a measure μ whose Fourier transform satisfies the condition $\hat{\mu}(u) = o(|u|^{-\alpha/2})$.

Here we can consider E as a subset of $\mathbb{R}$ contained in an interval of length 2π, as well as a subset of $\mathbb{T}$. The Hausdorff dimension of an M_α-set is $\geq \alpha$. We say that the Fourier dimension of a set E is the supremum of the α such that E is an M_α-set.

(Salem, 1951) Given $0 < \alpha < 1$, there exists sets E whose Hausdorff dimension and Fourier dimension are both equal to α [46].

Salem sets are defined—in $\mathbb{R}^d$, $d \geq 2$ as well as in $\mathbb{R}$—as sets whose Hausdorff and Fourier dimensions are equal. A sphere in $\mathbb{R}^d$ is a Salem set.

We close this section with a historical comment: The terms pseudomeasure and pseudofunction were coined by Kahane and Salem in 1956 [22]. However, pseudofunctions appear implicitly whenever one considers trigonometric series whose coefficients tend to zero, and thus their theory goes back to Riemann, Cantor, and Rajchman.

9. The construction of thin sets. We shall consider in turn Kronecker sets, U-sets, M_α-sets, Salem sets, and Helson U-sets. Explicit constructions are difficult, and I shall give only a few examples involving U-sets and M_α-sets. Two main tools are used in place of explicit constructions: Baire category theory and probability theory.

Robert Kaufman introduced a Baire argument for Kronecker sets in 1967. His method can be illustrated with two specific cases. First, given any Cantor set E_0, that is, a set homeomorphic to the triadic Cantor set, we consider the space $C(E_0)$ of all real-valued continuous functions on E_0. Then there exists a dense G_δ-set in $C(E_0)$ such that if f belongs to this set, then $f(E_0)$ is a Kronecker set. In short, we say that $f(E_0)$ is a Kronecker set for quasi-all f, or quasi-surely. From this point of view, Kronecker sets are the most common in the world.

We start with a particular type of Cantor set E_0 that has large gaps. This means that for arbitrarily small $\varepsilon > 0$, E_0 can be covered by intervals of lengths ε separated by intervals of length $> l$ such that l/ε is arbitrarily large. Now consider the cone $D(E_0)$ consisting of those real-valued continuous functions on E_0 that can be extended to increasing C^1 functions on $\mathbb{R}$ and whose derivative is 1 on E_0 and lies between two positive bounds on $\mathbb{R}$. $D(E_0)$ is a Baire space, and $f(E_0)$ is a Kronecker set for quasi-all $f \in D(E_0)$. Then, from a local point of view, $f(E_0)$ is obtained from E_0 by a translation and a small distortion. Starting with a "big" E_0 (its Lebesgue measure is necessarily 0, but its Hausdorff dimension can be 1), we obtain a "big" Kronecker set [24,17].

There are several explicit ways to construct U-sets. I already mentioned the H-sets of Rajchman and the H^msets of Piatetski-Shapiro in sections 2 and 5. If E is a Dirichlet set, which means that 1 is a uniform limit on E of some sequence of imaginary exponentials $\exp(\mathrm{i}n_k t)$, $n_k \to \infty$, then it is a U-set (Varopoulos, 1969) [48, 17]. Here is special class of U-sets that were discovered by Salem in 1941.

Given a compact set E, let $N(\varepsilon, E)$ be the smallest number of intervals of length ε whose union covers E. If

$$(\mathrm{S}_1) \qquad \varliminf_{\varepsilon \to 0} \frac{N(\varepsilon, E)}{\log(1/\varepsilon)},$$

then E is a U-set. This is the Salem box, or entropic, condition. Proofs of these last two results are given in [17].

There are not many explicit ways to construct M_α-sets, $\alpha > 0$. Here is an interesting example given by Robert Kaufman in 1980 [25]. Let $(a_0, a_1, \ldots, a_n, \ldots) = a_0 + (1 + 1/(a_1 + \cdots))$ be the continued fraction expansion of a positive real number x. Let F_N be the set of x's such that $a_n \leq N$ for all n. When N is large enough and α is small enough, F_N is an M_α-set.

The most powerful way to obtain M_α-sets, and even Salem sets, is to use random constructions or processes. Random constructions were introduced by Salem in 1951; they are described in chapter VIII of [23]. Lévy processes were used by Kahane and Mandelbrot in 1965, and Brownian motion was used by Kahane in 1966; in both cases the analytic tools were those used in Salem's approach [21,16]. Here is the simplest result, using Brownian motion:

Let $B(t,\omega)$ be the real Brownian motion starting from 0, $h(t)$ a positive concave function of $t > 0$, θ a probability measure such that $\theta(I) \leq h(|I|)$ for all intervals I, and μ the (random) image of θ by $B(\cdot\,,\omega)$. Then

$$(\mathrm{S}_2) \qquad \hat{\mu}(u) = O(\sqrt{\log|u|\; h(|u|^{-2})}\,), \quad |u| \to \infty, \quad \text{almost surely.}$$

As a consequence, starting from any compact subset of $\mathbb{R}$ with Hausdorff dimension $\alpha/2$, $0 < \alpha < 1$, its image under $B(\cdot\,,\omega)$ is almost surely (a. s.) a Salem set of dimension α.

Another consequence is that the Salem box condition (S_1) is best possible in the sense that the denominator $\log(1/\varepsilon)$ cannot be replaced with a substantially larger function, that is, one whose ratio with $\log(1/\varepsilon)$ tends to infinity as ε tends to zero. On the other hand, (S_1) shows that $\log|u|$ in (S_2) cannot be replaced with a substantially smaller function.

Probability methods are well suited to provide M-sets, in particular, M_α-sets and Salem sets, whereas Baire methods are well suited to provide U-sets, in particular, Kronecker sets. Random images of a given set carry measures whose Fourier transforms "decrease" as fast as possible, given their Hausdorff measures or dimensions. From the Baire point of view, the opposite is true: The images under consideration do not carry measures whose Fourier transforms tend to zero. Almost sure and quasi sure properties go in opposite directions.

10. Construction of thin sets—continued. It was a surprise to me when Tom Körner showed in 1993 how to use Baire methods to produce M-sets, M_α-sets, and Salem sets [30]. The probability method consisted of investigating Fourier properties of random sets whose Hausdorff properties were given and of proving that fast decrease of Fourier transforms of some measures (as fast as possible given the Hausdorff properties) is almost sure. Körner's method is to impose the Fourier properties by defining a convenient Baire space of sets equipped with measures or pseudomeasures and to prove (I quote) that "quasi all sets are as thin as possible," in particular, from the Hausdorff dimension point of view or from the Helson point of view. Here are two important examples ([29], see also [18, p. 174]).

Let G_α be the set of ordered pairs (E,μ), where E is a closed subset of $\mathbb{T}$ and μ is a probability measure carried by E whose Fourier coefficients satisfy $\hat{\mu}(n) = o(|n|^{-\alpha/2})$,

$|n| \to \infty$, $0 \le \alpha < 1$. When G_α is equipped with the metric

$$d((E_1,\mu_1),(E_2\mu_2)) = d(E_1,E_2) + \sup_n(|n|^{-\alpha/2}|\hat{\mu_1}(n) - \hat{\mu_2}(n)|)$$

it is a Baire space. Then E is a Salem set of dimension α quasi surely. Moreover, given any positive function $h(\varepsilon)$, $\varepsilon > 0$, such that $\varepsilon^{-\alpha}\log(1/\varepsilon) = o(h(\varepsilon))$, $\varepsilon \to 0$, then

$$\varliminf_{\varepsilon\to 0} \frac{N(\varepsilon,E)}{h(\varepsilon)} = 0$$

for quasi all (E,μ) in G_α.

Choosing $\alpha = 0$, the Salem condition S_1 for uniqueness shows that the condition on $h(\varepsilon)$ is precise in the sense that "o" cannot be replaced by "O."

Roughly speaking, quasi-all M_α-sets are Salem sets, and quasi-all M_0-sets satisfy a box condition as close as we want to the Salem uniqueness condition (S_1).

We know that no M_0-set can be a Helson set, which is Helson's theorem. However, roughly speaking, quasi-all M-sets are Helson sets; in other words, the Körner example, well considered, is generic. This is the content of the second example:

Let G be the set of ordered pairs (E,τ), where E is a closed M-set on $\mathbb{T}$ and τ is a pseudofunction carried by E such that $\hat{\tau}(0) = 1$. This is a Baire space when it is endowed with the metric

$$d((E_1,\tau_1),(E_2,\tau_2)) = d(E_1,E_2) + \sup_n |\hat{\tau_1}(n) - \hat{\tau_2}(n)|,$$

and E is a Helson set quasi surely.

In these examples the use of the Baire method is illuminating, but probability methods remain an essential tool in the proofs. The marriage of both methods is a charm of this approach.

11. The new role of set theory. Cantor's set theory appeared in 1872 in connection with the uniqueness problem for trigonometric series. In 1985 Robert Kaufman had the idea of applying the general theory of sets to the space $K(\mathbb{T})$, the space of all compact subsets of $\mathbb{T}$, to investigate U-sets and other thin sets. Important work by logicians followed. Here are a few prominent results.

Let $\mathcal{U}$ be the ensemble of U-sets in $\mathbb{T}$. Then $\mathcal{U}$ is coanalytic in $K(\mathbb{T})$, that is, $\mathcal{M}$, the ensembles of M-sets in $\mathbb{T}$, is analytic in the sense of Lusin. Moreover, neither $\mathcal{U}$ nor $\mathcal{M}$ is Borelian in $K(\mathbb{T})$, where Borelian means analytic and coanalytic. This result is by R. Kaufman and R. Solovay [26].

The theorem of Nina Bari says that $\mathcal{U}$ is a σ-ideal in $K(\mathbb{T})$. This means that a closed subset of an element of $\mathcal{U}$ belongs to $\mathcal{U}$ and that a countable union of elements of $\mathcal{U}$ belongs to $\mathcal{U}$ as soon as it belongs to $K(\mathbb{T})$. Answering a question of R. Solovay, G. Debs and J. Saint-Raymond proved that $\mathcal{U}$ as a σ-ideal has no Borelian basis. (B is a basis for the σ-ideal $\mathcal{U}$ if $\mathcal{U}$ is the smallest σ-ideal containing B.) This means that, given any system of Borelian conditions, there exists a closed U-set that is not a countable union of closed sets satisfying these conditions. The proof makes use of the existence of Helson M-sets [11].

Parallel to $\mathcal{U}$ and $\mathcal{M}$, it is possible to consider $\mathcal{U}_0$ and $\mathcal{M}_0$, complementary in $K(\mathbb{T})$, the latter being the ensemble of closed M_0-sets. Again, $\mathcal{U}_0$ is analytic, it is not Borelian, and it is a σ-ideal. But unlike $\mathcal{U}$, $\mathcal{U}_0$ has a Borelian basis [28].

12. Several variables. This field of research is relatively recent, but it is now very active. A good exposition can be found in the article by Marshall Ash and Geng Wang in [1]. The trigonometric series under consideration are

$$\sum_{m\in\mathbb{Z}^d} c_m \mathrm{e}^{\mathrm{i}(m,x)},$$

where $x = (x_1, \cdots, x_d)$, $m = (m_1, \cdots, m_d)$, and $(m, x) = m_1x_1 + \cdots + m_dx_d$. The spherical sums are

$$S_r(x) = \sum_{|m|\le r} c_m \mathrm{e}^{\mathrm{i}(m,x)},$$

where $|m| = (m_1^2 + \cdots + m_d^2)^{1/2}$, and their building blocks are the sums

$$C_r = \sum_{|m|=r} c_m \mathrm{e}^{\mathrm{i}(m,x)}.$$

Here are the main results:

(Zygmund, 1972) Suppose that $d = 2$. Then for some $\lambda > 0$, depending on E,

$$\frac{1}{\lambda}\sum_{|m|=r} |c_m|^2 \le \int_E |C_r(x)|^2\,\mathrm{d}x \le \lambda \sum_{|m|=r} |c_m|^2$$

whenever E has positive Lebesgue measure on $\mathbb{T}^2$.

Corollary. If $C_r(x)$ tends to zero as $r = (r_1^2 + r_2^2)^{1/2}$ tends to infinity on a set E of positive Lebesgue measure, then $\sum_{|m|=r} |c_m|^2$ tends to zero [55].

(B. Connes, 1976) Suppose that $d \ge 3$. Then the same conclusion holds for *open* sets E [10].

(J. Bourgain, 1996) Assume that $d \ge 2$. If $S_r(x)$ tends to zero everywhere as $r \to \infty$, then the series is the null series. This uses B. Connes's result. Finally, here is the most recent extension [6].

(M. Ash and G. Wang, 2000) If $S_r(x)$ tends to $f(x)$ for all x as $r \to \infty$ and if $f \in L^1(\mathbb{T}^d)$, then the c_m are the Fourier–Lebesgue coefficients of f [2].

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