NECESSARY AND SUFFICIENT CONDITIONS FOR A SYSTEM TO BE A QUASIBASIS IN WEIGHTED NORM SPACES

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1. Introduction. Definitions and main results. The following notation and terminology are employed in the sequel. Given a Banach space B, we denote by B^* its conjugate space. A system $X = \{x_n : n = 1, 2, \ldots\}$, of elements of B, is *complete* in B if the closure of the set of all finite linear combinations of elements of X coincides with B. A system $X = \{x_n : n = 1, 2, \ldots\}$ is a *Schauder basis* for B if for every x in B, there exists a unique series $\sum_{n=1}^{\infty} a_n(x)x_n$ that converges to x in the norm of B. The existence of a Schauder basis is equivalent to the following [1]: X is complete in B and there exists a *conjugate system*

$$X^* = \{x_n^* : n = 1, 2, \ldots\} \subset B^*$$

such that

$$x_i^*(x_j) = \delta_{ij}, \qquad \forall i \in \mathbf{N}, \qquad \forall j \in \mathbf{N}$$

and for all x in B, $\sum_{n=1}^{\infty} y_n^*(x)x_n$ converges to x in the norm of B.

Dropping the condition of the unicity of the coefficients, one gets the definition of a quasibasis. More precisely, a system $X = \{x_n : n = 1, 2, \ldots\}$ is a *quasibasis* for B if and only if there exists a system $\{y_n^* : n = 1, 2, \ldots\} \subset B^*$ such that, for all x in B, $\sum_{n=1}^{\infty} y_n^*(x)x_n$ converges to x in the norm of B. This generalization of a Schauder basis was introduced by Gelbaum [5] (see also [12]; 278,766).

The system $\{y_n^* : n = 1, 2, \ldots\} \subset B^*$ is called an *admissible system*.

Let $\Lambda(E)$ be a Banach space of real valued Lebesgue measurable functions defined on the measurable set $E \subseteq [a,b], -\infty < a < b < +\infty$ with natural linear operations; assume that the measure of E is positive. As usual, we identify functions equal almost everywhere. $\Lambda(E)$ satisfies the following postulates:

(a) if $0 \le g \le f$ a.e. on E and $f \in L(E)$, then $g \in L(E)$;

(b) if $f \in \Lambda(E)$ and χ_A is the characteristic function of the set A, then $\|f\chi_A\|$ converges to zero as the Lebesgue measure of A (denoted $|A|$) converges to zero.

Further in the paper a Lebesgue Λ-space is a Banach space $\Lambda(E)$ satisfying (a) and (b).

For any measurable subset $F \subset E$, where the measure of F is positive, $\Lambda(F)$ denotes the subspace of $\Lambda(E)$ of functions, identified almost everywhere on F, with the norm $\|f\|_{\Lambda(F)} := \|f\chi_F\|_{\Lambda(E)}$.

The *associated space* $\Lambda'(E)$ of the Lebesgue Λ-space $\Lambda(E)$ is, by definition, the collection of all Lebesgue measurable functions ϕ defined on the set E for which

$$\|\phi\|_{\Lambda(E)} := \sup_{\|f\|_{\Lambda(E)}=1} \int_E f(t)\phi(t)dt < \infty.$$

$\Lambda'(E)$, just as $\Lambda(E)$, is a Banach lattice and is isometrically isomorphic to $\Lambda^*(E)$, the dual of $\Lambda(E)$ (see [10] and also [11]).

MAIN THEOREM. *Let $\Lambda(E)$ be a Lebesgue Λ-space with a quasibasis and let $\{f_n\}_{n=1}^\infty$ be a system of measurable a.e. finite functions defined on E.*

There exists a measurable bounded function M such that $\{Mf_n\}_{n=1}^\infty$ is a quasibasis in $\Lambda(E)$ if and only if

$\Delta)$ $\forall \epsilon > 0$ $\exists E_\epsilon$ *such that $|E_\epsilon| > |E| - \epsilon$ and $\{f_n\}_{n=1}^\infty$ is a quasibasis in $\Lambda(E_\epsilon)$.*

We show that the property $\Delta)$ is stable with respect to deletion of finitely many elements of the system.

PROPOSITION 1. *Let $\{f_n\}_{n=1}^\infty$ be a system of measurable a.e. finite functions defined on $E, |E| > 0$. If $\{f_n\}_{n=1}^\infty$ satisfies the property $\Delta)$ then any system obtained from it by deleting finitely many elements also satisfies the same property.*

The possibility of obtaining quasibases in weighted $L^p, 1 \le p < \infty$ was observed in [6]. Afterwards in the joint work of the author with R. Zink [7] it was proved that if a system is a quasibasis in an L^{p_0} space then the same system is a quasibasis in weighted L^p spaces, $1 \le p \le p_0$, where the weight function that corresponds to the space L^p has the form $\omega_p = |w|^p$. Related questions in the spaces $L(E)$ which satisfy the additional postulates

(c) $L(E) \subset L_E^1$;   (d) $1 \in L(E)$

have been studied by Ben-Ami Braun [2–4].

2. The Men'shov property. Let B be a Banach space of functions defined on a measurable set $G, |G| > 0$.

DEFINITION 1. We will say that B has the *Men'shov property* if $\forall \epsilon \; \exists C(\epsilon) > 0$ such that $\forall f \in B$ and for any compact set $\Omega \subset B^*$ in the dual space there exists $f^* \in B$ that satisfies the following conditions:

$$|\{x \in G : f^*(x) \neq f(x)\}| \leq \epsilon; \qquad \operatorname{supp} f := \{x \in G : f(x) \neq 0\} = \operatorname{supp} f^*;$$

$$\|f^*\|_B \leq C(\epsilon) \cdot \|f\|_B;$$

and

$$|\phi(f^*)| < \epsilon \qquad \forall \phi \in \Omega.$$

THEOREM 1. *Let $\Lambda(E)$ be a Lebesgue Λ-space. Then $\Lambda(E)$ has the Men'shov property with $C(\epsilon) = \frac{b-a}{\epsilon}$.*

Proof. For $\eta \in \mathbf{R}, 0 < \eta < |E|$ define a periodic, of period $b - a$, function $h_\eta : \mathbf{R} \to \mathbf{R}$ such that

$$h_\eta(t) = \begin{cases} -\frac{b-a-\eta}{\eta}, & \text{if } t \in [0, \eta]; \\ 1, & \text{if } t \in (\eta, b - a). \end{cases}$$

It is easy to check that $\int_0^{b-a} h_\eta(t)dt = 0$. One may assume, without loss of generality, that $0 < \epsilon < |E|$. Let $f \in \Lambda(E)$ be an arbitrary function and let $\Omega \subset \Lambda^*(E)$ be any compact set in the dual space $\Lambda^*(E)$. There exists a finite collection of functions $\{g_i\}_{i=1}^N \subset \Lambda^*(E)$ with the following property: $\forall g \in \Omega \; \exists j \; (1 \leq j \leq N)$ such that

$$\|g - g_j\|_{\Lambda^*(E)} < \beta(\epsilon),$$

where

$$\beta(\epsilon) = \frac{\epsilon^2}{2(b-a)\|f\|_{\Lambda(E)}}.$$

Employing Fejér's lemma (see [14],49) we get that for any fixed $\eta > 0$

$$\lim_{n \to \infty} \int_E h_\eta(nt)f(t)g_i(t)dt = 0 \qquad \text{for all } i \; (1 \leq i \leq N).$$

Let $\eta = \epsilon$. Then we can find $n_0 \in \mathbf{N}$ such that

$$\left| \int_E h_\epsilon(n_0 t)f(t)g_i(t)dt \right| < \epsilon/2 \qquad \text{for all } i \; (1 \leq i \leq N).$$

Put

$$f^*(t) = h_\epsilon(n_0 t)f(t).$$

By the definition of the function h_η it follows immediately that $f^* \in \Lambda(E)$ and moreover that

$$\|f^*\|_{\Lambda(E)} \leq \frac{b-a}{\epsilon} \cdot \|f\|_{\Lambda(E)}.$$

We also have that

$$|\{t \in E : f^*(t) \neq f(t)\}| \leq \epsilon.$$

As for the third condition, one notes that, for every $g \in \Omega$,

$$\left| \int_E f^*(t)g(t)dt \right| \leq \left| \int_E f^*(t)[g(t) - g_j(t)]dt \right| + \left| \int_E f^*(t)g_j(t)dt \right|$$

$$\leq \|f^*\|_{\Lambda(E)} \cdot \|g - g_j\|_{\Lambda^*(E)} + \epsilon/2 \leq \frac{b-a}{\epsilon} \cdot \|f\|_{\Lambda(E)} \cdot \beta(\epsilon) + \epsilon/2 \leq \epsilon. \quad \blacksquare$$

3. Approximation by truncated polynomials with controlled oscillation. In this section we will show that if a system of functions $\{f_n\}_{n=1}^{\infty}$ satisfies the condition Δ) then any measurable a.e. finite function defined on the set E can be approximated by truncated polynomials on a subset with measure arbitrarily close to the measure of the set E, and the oscillation of the partial sums of the polynomial is controlled by the norm of the function on the subset. In order to approximate a function by a polynomial of the form $\sum_{i=N+1}^{m} a_i f_i$ one has to modify the function on a set of small measure in such a way that the first N functionals of the admissible system vanish on the modified function. In order to control the oscillation of the partial sums one may divide the given function into pieces with very small norms and then modify each piece. This idea goes back to Talalyan (see [13]). Similar questions in F-spaces have been studied in [8].

THEOREM 2. *Let $\Lambda(E)$ be a Lebesgue Λ-space, and let $\{f_n\}_{n=1}^{\infty}$ be a system of functions defined on E which satisfies the condition Δ). If $f \in \Lambda(E)$ then, $\forall \epsilon > 0$, and each natural number, N, there is a measurable set $G \subset E, |G| > |E| - \epsilon$ and a polynomial, $P = \sum_{k=N}^{M} a_k f_k$, such that:*

$$\|f - P\|_{L(G)} < \epsilon,$$

and

$$\sup \left\{ \Big\| \sum_{k=N}^{s} a_k f_k \Big\|_{L(G)} : N \le s \le m \right\} < \epsilon + \|f\|_{L(G)}.$$

Proof. One may assume, without loss of generality, that $0 < \epsilon < |E|$. By virtue of the property Δ) $\exists E_1$ such that $|E_1| > |E| - \epsilon/2$ and $\{f_n\}_{n=1}^{\infty}$ is a quasibasis in $\Lambda(E_1)$. Let $\{g_n^1\}_{n=1}^{\infty} \subset \Lambda^*(E_1)$ be an admissible system.

Then, for any $f \in \Lambda(E_1)$ we can define a series such that the nth partial-sum operator is given by

$$S_n(f) = \sum_{k=1}^{n} g_k(\cdot) f_k = \sum_{k=1}^{n} a_k(f) f_k,$$

where

$$a_k(f) = \int_{E_1} f(t) g_k(t) dt.$$

Since

$$\lim_n \|f - S_n(f)\|_{\Lambda(E_1)} = 0, \qquad \forall f \in \Lambda(E_1),$$

an application of the Banach–Steinhaus theorem yields a positive constant, C, such that

$$\|S_n\| \le C, \qquad \forall n \in \mathbf{N}. \tag{1}$$

There exists a natural number n such that

$$\|f \chi_{\Delta_n^i}\|_{\Lambda(E_1)} < \frac{\epsilon^2}{32 \cdot C(b-a)}, \qquad i = 1, \dots, n,$$

where

$$\Delta_n^i = E_1 \cap \left(a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n} \right), \qquad i = 1, \dots, n.$$

Among the sets $\{\Delta_n^i\}_{i=1}^n$ we choose all sets $\{E_1^j\}_{j=1}^{n_0}$ with positive measure. Evidently

$$\left| E_1 \setminus \bigcup_{j=1}^{n_0} E_1^j \right| = 0.$$

Thus we have that almost everywhere on E_1

$$f = \sum_{j=1}^{n_0} f\chi_{E_1^j} := \sum_{i=1}^{n_0} f^j.$$

Let $n_1 = N + 1$, let

$$B_1 = \max\{\|f_k\|_{\Lambda(E_1)} : 1 \le k \le n_1\},$$

and let

$$\eta_1 = \frac{\epsilon}{8nn_1B_1}.$$

By virtue of the Men'shov property for the space $\Lambda(E_1)$, there exists a function, ϕ^1, such that

$$|\{x \in E_1 : \phi^1(x) \ne f^1(x)\}| \le \epsilon/4n; \qquad \operatorname{supp} f^1 = \operatorname{supp} \phi^1;$$

$$\|\phi^1\|_{\Lambda(E_1)} \le \frac{4(b-a)}{\epsilon} \cdot \|f^1\|_{\Lambda(E_1)},$$

and

$$|b_k^1| = \left| \int_{E_1} \phi^1(t)g_k(t))dt \right| < \eta_1, \qquad \forall k \quad (1 \le k \le n_1 - 1).$$

The reader should observe that $\operatorname{supp} f^1 \subset E_1^1$ and $C(\epsilon/4n) = \frac{(b-a)}{n} \cdot \frac{4n}{\epsilon} = \frac{4(b-a)}{\epsilon}$.

Let n_2 be a natural number, greater than n_1, such that

$$\left\| \phi^1 - \sum_{k=1}^{n_2-1} a_k(\phi^1)f_k \right\|_{\Lambda(E_1)} < \epsilon \cdot 2^{-n-3},$$

and let

$$P_1 = \sum_{k=n_1}^{n_2-1} a_k(\phi^1)f_k.$$

Then,

$$\|\phi^1 - P_1\|_{\Lambda(E_1)} \le \left\| \phi^1 - \sum_{k=1}^{n_2-1} a_k(\phi^1)f_k \right\|_{\Lambda(E_1)} + \left\| \sum_{k=1}^{n_1-1} a_k(\phi^1)f_k \right\|_{\Lambda(E_1)}$$

$$< \epsilon 2^{-n-3} + \left\| \sum_{k=1}^{n_1-1} b_k^1 f_k \right\|_{\Lambda(E_1)} \le \epsilon 2^{-n-3} + n_1\eta_1 B_1 < \epsilon/4n.$$

By virtue of (1) we have

$$\sup\left\{ \left\| \sum_{k=n_1}^{m} a_k(\phi^1)f_k \right\|_{\Lambda(E_1)} : n_1 \le m < n_2 \right\} \le 2C\|\phi^1\|_{\Lambda(E_1)}$$

$$\le 8C\frac{b-a}{\epsilon} \cdot \|f^1\|_{\Lambda(E_1)} \le \epsilon/4.$$

Proceeding inductively, suppose that the polynomials associated with the functions f^j, $1 \leq j < i \leq n_0 < n$, have been determined. Let

$$B_i = \max\{\|f_k\|_{\Lambda(E_1)} : 1 \leq k \leq n_i\},$$

and let

$$\eta_i = \frac{\epsilon}{8nn_iB_i}.$$

Another application of Men'shov's property yields a function ϕ_i such that

$$|\{x \in E_i : \phi^i(x) \neq f^i(x)\}| \leq \epsilon/4n; \qquad \operatorname{supp} f^i = \operatorname{supp} \phi^i;$$

$$\|\phi^i\|_{\Lambda(E_i)} \leq \frac{4(b-a)}{\epsilon} \cdot \|f^i\|_{\Lambda(E_i)},$$

and

$$|b_k^i| = \left| \int_{E_1} \phi^i(t)g_k(t))dt \right| < \eta_i, \qquad \forall k \quad (1 \leq k \leq n_i).$$

As above, one determines a natural number n_{i+1}, greater than n_i, such that

$$\left\| \phi^i - \sum_{k=1}^{n_{i+1}-1} a_k(\phi^i)f_k \right\|_{\Lambda(E_1)} < 2^{-n-3}\epsilon.$$

Then, for

$$P_i = \sum_{k=n_i}^{n_{i+1}-1} a_k(\phi^i)f_k,$$

one has

$$\|\phi^i - P_i\|_{\Lambda(E_1)} < 2^{-n-3}\epsilon + \eta_in_iB_i < \epsilon/4n,$$

and

$$\sup\left\{ \left\| \sum_{k=n_i}^{m} a_k(\phi^i)f_k \right\|_{\Lambda(E_1)} : n_i \leq k < n_{i+1} \right\} \leq C\|\phi^i\|_{\Lambda(E_1)} < \epsilon/2.$$

Finally, let

$$P = \sum_{i=1}^{n_0} P_i, \qquad G = \left\{ t \in E_1 : f = \sum_{i=1}^{n_0} \phi^i \right\}.$$

It is easy to see that

$$|G| > |E_1| - \sum_{i=1}^{n_0} \epsilon/4n \geq |E| - \epsilon/2 - \epsilon/4 > |E| - \epsilon.$$

Since $\sum_{i=1}^{n_0} \phi^i$ agrees with f on G,

$$\|f - P\|_{L(G)} = \left\| \sum_{i=1}^{n_0} \phi^i - P \right\|_{L(G)} \leq \sum_{i=1}^{n_0} \|\phi^i - P_i\| < \epsilon.$$

As for the third condition, one notes that

$$\max\left\{ \left\| \sum_{i=1}^{j} P_i \right\|_{L(G)} : 1 \leq j \leq n_0 \right\} \leq \max\left\{ \left\| \sum_{i=1}^{j} (P_i - \phi^i) \right\|_{L(G)} : 1 \leq j \leq n_0 \right\}$$

$$+ \left\| \sum_{i=1}^{j} \left(\phi^i - f^i \right) \right\|_{L(G)} + \|f\|_{L(G)}$$

$$< \sum_{i=1}^{j} 2^{-n-2}\epsilon + \|f\|_{L(G)} \le \epsilon/2 + \|f\|_{L(G)} < \epsilon;$$

thus, for $m \ge N$, say $n_j \le m < n_{j+1}$, one has

$$\Big\| \sum_{k=N}^{m} a_k f_k \Big\|_{L(G)} \le \Big\| \sum_{i=1}^{j} P_i \Big\|_{L(G)} + \Big\| \sum_{k=n_j}^{m} a_k(\phi^i) f_k \Big\|_{L(G)} < \epsilon + \|f\|_{L(G)}. \ \blacksquare$$

4. Proof of the Main Theorem. The necessity is trivial. Let us prove the sufficiency. Let $\{\psi_i\}_{i=1}^{\infty}$ be a quasibasis in $\Lambda(E)$ with the admissible system $\{\psi_i^*\}_{i=1}^{\infty}$. Without loss in generality we can suppose that $\|\psi_i\|_{\Lambda(E)} = 1$ for all $i \in \mathbf{N}$. Let $\eta_i = 2^{-(i+3)}$ for $i \in \mathbf{N}$. The idea of the construction of the weight function M goes back to Price, Zink and Braun [11], [3]. In our case we have to modify the construction because of technical differences.

The weight function M is obtained as an infinite product of functions M_k positive almost everywhere and such that each of them is equal to one at a set $D_k \subset E$ and is less than one on the complement of D_k in E. If $\sum_{k=1}^{\infty} |E \setminus D_k| < +\infty$ then, obviously, the function $M = \prod_{k=1}^{\infty} M_k$ will be well defined and positive almost everywhere.

Let $\xi \in \Lambda(E), \|\xi\|_{L(E)} = 1$, be such that $|\{t \in E : \xi(t) \ne 0\}| = |E|$ and take positive $\delta_n < \eta_n$ so that

$$\|\psi_k\|_{L(G)} < \eta_n/4 \qquad \forall k \ (1 \le k \le n)$$

whenever $|G| < \delta_n$.

Applying Theorem 1 we find $D_1 \subset E, |D_1| > |E| - \delta_1$ and

$$P_{11} = \sum_{i=n_0(1)+1}^{n_1(1)} a_i f_i, \qquad n_0(1) = 0$$

such that

$$\|\psi_1 - P_{11}\|_{L(D_1)} < \eta_1/4$$

and

$$\sup\Big\{\Big\| \sum_{k=1}^{s} a_k f_k \Big\|_{L(D_1)} : 1 \le s \le n_1(1)\Big\} < \eta_1/2 + \|\psi_1\|_{L(D_1)}.$$

Let $f_0(t) = 1$ for $t \in E$ and define

$$[h_1(t)]^{-1} = \sup_{1 \le i \le n_1(1)} |f_i(t)| \qquad \text{for} \quad t \in E \setminus D_1.$$

Let the function M_1 be defined, on E, by setting

$$M_1(t) = d_1 \, \xi(t) \cdot h_1(t) \cdot \chi_{E \setminus D_1}(t) + \chi_{D_1}(t),$$

where

$$d_1 = \eta_1/4 \cdot \Big(1 + \sup\Big\{\Big\| \sum_{k=1}^{s} a_k f_k \Big\|_{L(D_1)} : 1 \le s \le n_1(1)\Big\}\Big)^{-1}.$$

We easily check that

$$\|\psi_1 - M_1 \cdot P_{11}\|_{L(D_1)} < \eta_1$$

and

$$\sup\left\{\left\|M_1\sum_{k=1}^{s}a_kf_k\right\|_{L(E)} : 1 \le s \le n_1(1)\right\} < \eta_1 + \|\psi_1\|_{L(E)}.$$

Suppose we have constructed polynomials $\{P_{jk}\}_{j=1,k=j}^{n,\ n}$,

$$P_{jk} = \sum_{i=n_{j-1}(k)+1}^{n_j(k)} a_if_i,$$

with

$$1 = n_0(1) < n_1(1) = n_0(2) < n_1(2) < n_2(2) = n_0(3) < \ldots$$
$$< n_{j-1}(j-1) = n_0(j) < n_1(j) < \ldots < n_j(j),$$

and measurable functions $\{M_i\}_{i=1}^{n}$, $0 < M_i(\cdot) \le 1$, almost everywhere on E, such that:

$$M_i(t) = 1 \quad \text{if} \quad t \in D_i \qquad |E \setminus D_i| < \delta_i \qquad \text{for} \quad 1 \le i \le n;$$

(2)
$$\left\|\psi_k - \prod_{i=1}^{n}M_i\sum_{j=k}^{\ell}P_{kj}\right\|_{L(E)} \le \sum_{j=\ell}^{n}\eta_j, \quad \forall k, \quad k \le \ell \le n;$$

(3)
$$\sup\left\{\left\|\prod_{i=1}^{n}M_i\sum_{i=n_{k-1}(\ell)+1}^{s}a_if_i\right\|_{L(E)} : s \le n_k(\ell)\right\} \le \sum_{j=\ell}^{n}\eta_j, \quad \text{if} \quad k < \ell \le n;$$

(4)
$$\sup\left\{\left\|\prod_{i=1}^{n}M_i\sum_{i=n_{k-1}(k)+1}^{s}a_if_i\right\|_{L(F)} : n_{k-1}(k) < s \le n_k(k)\right\} \le \sum_{j=k}^{n}\eta_j + \|\psi_k\|_{L(E)}.$$

The system $\{\prod_{i=1}^{n}M_if_j\}_{j=1}^{\infty}$ satisfies the condition Δ) hence for this system we can successively apply Theorem 2 to the functions

$$\left\{\psi_k - \prod_{i=1}^{n}M_i\sum_{j=k}^{n}P_{kj}\right\}_{k=1}^{n} \qquad \text{and} \quad \psi_{n+1}.$$

We can find $D_{n+1} \subset E, |D_{n+1}| > |E| - \delta_{n+1}$ and

$$P_{kn+1} = \sum_{i=n_{k-1}(1)+1}^{n_k(n+1)} a_if_i, \qquad n_0(n+1) = n_n(n)$$

such that

$$\left\|\psi_k - \prod_{i=1}^{n}M_i\sum_{j=k}^{n}P_{kj} - \prod_{i=1}^{n}M_iP_{kn+1}\right\|_{L(D_{n+1})} < \eta_{n+1}/4;$$

$$\left\|\psi_{n+1} - \prod_{i=1}^{n}M_iP_{n+1n+1}\right\|_{L(D_{n+1})} < \eta_{n+1}/4,$$

and

$$\sup\left\{\left\|\prod_{i=1}^{n}M_i\sum_{i=n_{k-1}(n+1)+1}^{s}a_kf_k\right\|_{L(D_{n+1})} : 1 \le s \le n_1(1)\right\} < \eta_{n+1}/2 + \sum_{j=k}^{n}\eta_j;$$

$$\sup\Big\{\Big\|\sum_{i=n_n(n+1)+1}^{s}a_i f_i\Big\|_{L(D_{n+1})} : n_n(n+1) < s \le n_{n+1}(n+1)\Big\}$$

$$< \eta_{n+1}/2 + \|\psi_{n+1}\|_{L(D_{n+1})}.$$

Define

$$[h_{n+1}(t)]^{-1} = \sup_{0 \le i \le n_{n+1}(n+1)} |f_i(t)| \qquad \text{for} \quad t \in E \setminus D_{n+1}.$$

Let the function M_{n+1} be defined, on E, by setting

$$M_{n+1}(t) = d_{n+1}\, \xi(t) \cdot h_{n+1}(t) \cdot \chi_{E \setminus D_{n+1}}(t) + \chi_{D_{n+1}}(t),$$

where

$$d_{n+1} = 1/4\eta_{n+1}\Big(1 + \sum_{\ell}^{n+1}\sum_{k=1}^{\ell}\sup\Big\{\Big\|\prod_{i=1}^{n} M_i \sum_{i=n_{k-1}(\ell)+1}^{s} a_i f_i\Big\|_{L(D_{n+1})} : 1 \le s \le n_k(\ell)\Big\}\Big)^{-1}.$$

Now we easily check that the conditions (2)–(4) hold with $n+1$ instead of n.

Let

$$M(t) = \prod_{i=1}^{\infty} M_i(t).$$

As we have noted before, the function M is well defined and, moreover, $0 < M(\cdot) \le 1$, almost everywhere on E. The functions M_i have been constructed in such a way that the sequence $\{\prod_{i=1}^{n} M_i\}_{n=1}^{\infty}$ is a fundamental sequence in the space L_E^{∞}. Thus in the inequalities (2)–(4) we can pass to the limit and obtain that

$$\Big\|\psi_k - M\sum_{j=k}^{\ell} P_{kj}\Big\|_{L(E)} \le 2^{-\ell}, \quad \forall k, \quad k \le \ell$$

$$\sup\Big\{\Big\|M\sum_{i=n_{k-1}(\ell)+1}^{s} a_i f_i\Big\|_{L(E)} : s \le n_k(\ell)\Big\} \le 2^{-\ell}, \qquad \text{if} \quad k < \ell$$

$$\sup\Big\{\Big\|M\sum_{i=n_{k-1}(k)+1}^{s} a_i f_i\Big\|_{L(F)} : n_{k-1}(k) < s \le n_k(k)\Big\} \le 2^{-k} + \|\psi_k\|_{L(E)}.$$

Let

$$f_i^* = a_i \psi_k^*, \qquad \text{if} \quad n_{k-1}(\ell) < i \le n_k(\ell), \quad \text{for some} \quad k \quad \text{and} \quad \ell \ge k.$$

Then, $\{Mf_n\}_{n=1}^{\infty}$ proves to be a quasibasis for $L(E)$ with the admissible system $\{f_n^*\}_{n=1}^{\infty}$. The rest of the proof is the same as in [7]. ∎

5. Stability of the Δ) property

Proof of Proposition 1. We may assume, without loss of generality, that the first N elements of $\{f_n\}_{n=1}^{\infty}$ are deleted.

Let $\epsilon > 0$ be any positive number. Assume that $0 < \epsilon < |E|$. By virtue of the property Δ) $\exists E_1$ such that $|E_1| > |E| - \epsilon/2$ and $\{f_n\}_{n=1}^{\infty}$ is a quasibasis in $\Lambda(E_1)$. Let $\{g_n^1\}_{n=1}^{\infty} \subset \Lambda^*(E_1)$ be an admissible system. We may assume, without loss of generality, that $\{g_n^1\}_{n=1}^{N}$ are linearly independent in $\Lambda^*(E_1)$. Otherwise, we will choose a linearly

independent collection of elements of the set $\{g_n^1\}_{n=1}^N$ such that any element from $\{g_n^1\}_{n=1}^N$ can be represented as a linear combination of the chosen elements.

Let F_1 be the (measurable) subset of E_1 which contains those points that are simultaneously points of approximate continuity for each $\{g_n^1\}_{n=1}^N$. Since each g_n^1 is approximately continuous almost everywhere in E_1, one has $|F_1| = |E_1|$. One defines, inductively, a set $T = \{t_j : 1 \leq j \leq N\} \subset F_1$, on which $\{g_n^1\}_{n=1}^N$ is linearly independent, in the following manner:

(i) Let t_1 be any point of F_1 such that $g_1^1(t_1) \neq 0$.

(ii) Suppose elements $t_1, \ldots, t_k$ of F_1 have been chosen so that $\sum_{j=1}^k a_j g_j^1(t_i) = 0$, $\forall i = 1, \ldots, k$, only if each $a_j = 0$. Let the numbers b_j be chosen (uniquely) so that $f_{k+1} = g_{k+1}^1 + \sum_{j=1}^k b_j g_j^1$ satisfies $f_{k+1}(t_j) = 0$, for $1 \leq j \leq k$. Since $\{g_n^1\}_{n=1}^N$ is linearly independent, f_{k+1} is nonzero on a subset of F_1 that has positive measure. Let t_{k+1} be any point of the latter set. It follows, at once, that $\{(g_j^1(t_1), \ldots, g_j^1(t_{k+1})) : 1 \leq j \leq k+1\}$ is linearly independent. Hence one can find $G_1 \subset F_1, 0 < |G_1| < \epsilon/2$ such that $\{g_n^1\}_{n=1}^N$ is linearly independent in $\Lambda^*(G_1)$. Thus for any collection of real numbers $\{c_n\}_{n=1}^N$ one can define a function $\phi \in \Lambda(G_1)$ such that

$$\int_{G_1} \phi(t) g_n^1(t) dt = c_n, \qquad 1 \leq n \leq N.$$

The last condition means that $\forall f \in \Lambda(E_1)$ one can modify the function f on the set G_1 so that the modified function f^* coincides with f on the set $E_1^* = E_1 \setminus G_1, |E_1^*| > |E| - \epsilon$ and

$$\int_{E_1} f^*(t) g_n^1(t) dt = 0, \qquad 1 \leq n \leq N.$$

Hence, $\{f_n\}_{n=N+1}^\infty$ is a quasibasis in $\Lambda(E_1^*)$. ∎

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