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A MENŠOV TYPE THEOREM

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Abstract. Any integrable function can be corrected, so that its Fourier series converges almost everywhere to the correct limit, by the addition of a measure whose singular support is small both in the algebraic and the Hausdorff sense.

1. Introduction. I should like to thank the organisers for a conference which was extraordinarily enjoyable for many reasons — one of which was the opportunity to meet mathematicians whose work I have long admired but whom I had never met. My talk at the conference was based on [1], but this paper contains new results on the topic suggested to me by conversations at the conference.

In this paper we identify $\mathbb{T}$ with $[0, 2\pi)$ in the usual way. All measures will be Borel measures and, where we do not specify otherwise, the Haar measure (that is normalised Lebesgue measure) m is intended. If E is measurable, we write $|E|$ for the measure $m(E)$ of E. If $f \in L^1(\mathbb{T})$, we write

$$S_n(f, x) = \sum_{|r| \le n} \hat{f}(r) \exp irx.$$

If μ is a measure, $S_n(\mu, x)$ is defined similarly. We write $A(\mathbb{T})$ for the set of absolutely convergent Fourier series and, if $f \in A(\mathbb{T})$, we write $\|f\|_A = \sum_{u=-\infty}^{\infty} |\hat{f}(u)|$. If μ is a measure then $\operatorname{supp} \mu$ denotes its closed support.

In [1] I proved the following result which is closely related to various results of Menšov.

THEOREM 1. *Given any $f \in L^1$, and any $\epsilon > 0$, there exists a singular measure μ with $\|\mu\| \le \epsilon$ such that*

$$S_n(f - \mu, x) \to f(x)$$

almost everywhere as $n \to \infty$.

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It is natural to ask 'how small' the measure μ must be. Here is one answer to this question.

THEOREM 2. *Given any* $f \in L^1$ *any strictly increasing continuous* $h : [0,\infty) \to [0,\infty)$ *with* $h(0) = 0$ *and any* $\epsilon > 0$*, there exists a measure* μ *with* $\|\mu\| \le \epsilon$ *and an independent Borel set* E *with zero Hausdorff* h *measure such that* $|\mu|(\mathbb{T} \setminus E) = 0$ *and*

$$S_n(f - \mu, x) \to f(x)$$

almost everywhere as $n \to \infty$.

The reader may wish to be reminded of the following definitions.

DEFINITION 3. (i) A set E is *independent* if, whenever $n \ge 1$ and $x_1, x_2, \ldots, x_n$ are distinct points in E, the only solution of the equation

$$\sum_{j=1}^{n} m_j x_j = 0$$

with the m_j integers is $m_1 = m_2 = \ldots = m_n = 0$.

(ii) If $h : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $h(0) = 0$, then a set E has *zero Hausdorff* h *measure* if, given any $\epsilon > 0$, we can find intervals I_1, $I_2, \ldots$ such that $\bigcup_{j=1}^{\infty} I_j \supseteq E$ and $\sum_{j=1}^{\infty} h(|I_j|) < \epsilon$.

The main ideas of the construction appear in the proof of the simpler result.

THEOREM 4. *Given any* $f \in L^1$ *any strictly increasing continuous* $h : [0,\infty) \to [0,\infty)$ *with* $h(0) = 0$ *and any* $\epsilon > 0$*, there exists a measure* μ *with* $\|\mu\| \le \epsilon$ *and a Borel set* E *of zero Hausdorff* h *measure such that* $|\mu|(\mathbb{T} \setminus E) = 0$ *and*

$$S_n(f - \mu, x) \to f(x)$$

almost everywhere as $n \to \infty$.

We shall therefore concentrate first on proving this result and sketch the more intricate proof towards the end of the paper.

2. The key lemma. The main step in the proof of Theorem 4 is the proof of the following lemma.

LEMMA 5. *Given any* $\eta > 0$ *we can find a* $K_1(\eta) > 1$ *with the following property. Given any strictly increasing continuous function* $h : [0,\infty) \to [0,\infty)$ *with* $h(0) = 0$ *and any* $\epsilon > 0$ *we can find a positive measure* μ *and a Borel set* G *with*

(i) $\hat{\mu}(0) = 1$,
(ii) $\hat{\mu}(n) \to 0$ *as* $|n| \to \infty$,
(iii) $|\hat{\mu}(u)| \le \epsilon$ *for all* $u \ne 0$,
(iv) $|S_n(\mu, x)| \le K_1(\eta)$ *for all* $n \ge 0$ *and all* $x \notin G$,
(v) $|G| < \eta$,
(vi) $\operatorname{supp} \mu$ *has Hausdorff* h *measure zero.*

In this section we show how to prove Theorem 4 using Lemma 5, and, in the next section, we prove Lemma 5.

LEMMA 6. *Given any $\eta > 0$ we can find a $K_2(\eta) > 1$ with the following property. Given any trigonometric polynomial Q and any strictly increasing continuous function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and any $\epsilon > 0$, we can find a measure τ and a Borel set G with*

(i) $\|\tau\| \le \|Q\|_1$,
(ii) $\hat{\tau}(n) \to 0$ *as* $|n| \to \infty$,
(iii) $|S_n(\tau, x) - S_n(Q, x)| \le K_2(\eta)|Q(x)|$ *for all* $n \ge 0$ *and all* $x \notin G$,
(iv) $|G| < \eta$,
(v) $\operatorname{supp}\tau$ *has Hausdorff* h *measure zero.*

Proof. Suppose that, for each $\epsilon > 0$ we have a positive measure μ_ϵ such that

(i) $\hat{\mu}_\epsilon(0) = 1$,
(ii) $\hat{\mu}_\epsilon(n) \to 0$ as $|n| \to \infty$,
(iii) $|\hat{\mu}_\epsilon(u)| \le \epsilon$ for all $u \ne 0$.

If we set $\tau_\epsilon = Q\mu_\epsilon$ then $\|\tau_\epsilon\| \le \|Q\|_\infty$ and

$$\int_{\mathbb{T}} P(t)\, d\tau_\epsilon(t) = \int_{\mathbb{T}} P(t)Q(t)\, d\mu_\epsilon(t) \to \int_{\mathbb{T}} P(t)Q(t)\, dm(t)$$

as $\epsilon \to 0$, so a simple density argument shows that

$$\|\tau_\epsilon\| \to \|Q\|_1$$

as $\epsilon \to 0$.

Next we observe that

$$|\widehat{\mu_\epsilon - m}(u)| \le \epsilon \text{ for all } u,$$

and so

$$(S_n(\tau_\epsilon, t) - S_n(Q, t)) - Q(t)S_n(\mu_\epsilon, t) = S_n(Q(\mu_\epsilon - m), t) - Q(t)S_n(\mu_\epsilon, t) \to 0$$

uniformly in t and $n \ge 0$ as $\epsilon \to 0$ (Similar calculations are done at length in [1].)

Thus, if we set $K_2(\eta) = K_1(\eta)+1$, take ϵ sufficiently small, take μ as in the conclusion of Lemma 6 and set

$$\tau = kQ\mu$$

with k chosen so that $\|\tau\| = \|Q\|_1$, all the conclusions required can be read off. ∎

We can now prove Theorem 4.

Proof of Theorem 4. Let K_2 be defined as in the statement of Lemma 6. We may assume $1 > \epsilon$ and we set $\epsilon_j = 2^{-2j}\epsilon K_2(2^{-j})^{-1}$. If $f \in L^1$ we can find trigonometric polynomials Q_j such that $\|Q_j\|_1 < \epsilon_j$ for $j \ge 1$ and $\sum_{j=0}^{\infty} Q_j = f$, the convergence being both L^1 and pointwise almost everywhere. In particular we can find a set B of measure zero such that $\sum_{j=0}^{N} Q_j(x) \to f(x)$ for all $x \notin B$.

By Lemma 6 we can find measures τ_j and a closed set G_j with

(i)$_j$ $\|\tau_j\| \le \|Q_j\|_1$,
(ii)$_j$ $\hat{\tau}_j(n) \to 0$ as $|n| \to \infty$,
(iii)$_j$ $|S_n(\tau_j, x) - S_n(Q_j, x)| \le K_2(2^{-j})|Q_j(x)|$ for all $n \ge 0$ and all $x \notin G_j$,

(iv)$_j$ $|G_j| < 2^{-j}$,

(v)$_j$ $\operatorname{supp}\tau_j$ has Hausdorff h measure zero.

Since $\|\tau_j\| \leq \|Q_j\|_1 < \epsilon_j \leq \epsilon 2^{-j}$, we can form the measure $\mu = \sum_{j=1}^\infty \tau_j$ with $\|\mu\| \leq \epsilon$. If we set $E = \bigcup_{j=1}^\infty \operatorname{supp}\tau_j$ then E is a Borel set such that $|\mu|(\mathbb{T}\setminus E) = 0$ and, by condition (v), E has zero Hausdorff h measure.

Since $\|Q_j\|_1 < \epsilon_j \leq 2^{-2j}K_2(2^{-j})^{-1}$, a simple Tchebychev type estimate shows that, if

$$H_j = \{x : |Q_j(x)| \geq 2^{-j}K_2(2^{-j})^{-1}\},$$

then $|H_j| \leq 2^{-j}$. Thus, setting $A_j = G_j \cup H_j$ and using (iii)$_j$, and (iv)$_j$ we have

(vi)$_j$ $|S_n(\tau_j,x) - S_n(Q_j,x)| \leq 2^{-j}$ for all $n \geq 0$ and all $x \notin A_j$,

(vii)$_j$ $|A_j| < 2^{-j+1}$.

Thus if we set

$$A = B \cup \bigcup_{j=1}^\infty \operatorname{supp}\tau_j \cup \bigcap_{j=1}^\infty \bigcup_{k=j}^\infty A_j,$$

it follows that A has Lebesgue measure zero. We complete the proof by showing that

$$S_n(f-\mu,x) \to f(x)$$

as $n \to \infty$ for all $x \in A$.

By classical localisation theorems,

$$S_n(\tau_j,x) \to 0$$

for all $x \notin \operatorname{supp}\tau_j$, and, since Q_j is a trigonometric polynomial, it is trivial that

$$S_n(Q_j,x) \to Q_j(x)$$

for all x. Thus if

$$x \notin B \cup \bigcup_{j=1}^N \operatorname{supp}\tau_j \cup \bigcup_{k=N+1}^\infty A_j$$

we have

$$\begin{aligned}
|S_n(f-\mu,x) - f(x)| &\leq \sum_{j=1}^N |S_n(\tau_j,x)| + \sum_{j=1}^N |S_n(Q_j,x) - Q_j(x)| \\
&\quad + \sum_{j=N+1}^\infty |S_n(\tau_j,x) - S_n(Q_j,x)| + \sum_{j=N+1}^\infty |Q_j(x)| \\
&\leq \sum_{j=1}^N |S_n(\tau_j,x)| + \sum_{j=1}^N |S_n(Q_j,x) - Q_j(x)| \\
&\quad + \sum_{j=N+1}^\infty 2^{-j} + \sum_{j=N+1}^\infty 2^{-j} \\
&\leq \sum_{j=1}^N |S_n(\tau_j,x)| + \sum_{j=1}^N |S_n(Q_j,x) - Q_j(x)| + 2^{-N+1}
\end{aligned}$$

and so

$$\limsup_{n\to\infty} |S_n(f-\mu, x) - f(x)| \le 2^{-N+1}.$$

Allowing $N \to \infty$, we have the result. ■

3. Proof of the key lemma. The measure μ in our key Lemma 5 is obtained as the weak star limit of a sequence of smooth functions.

LEMMA 7. *Given any $\eta > 0$ we can find a $K_1(\eta) > 1$ with the following property. Given any strictly increasing continuous function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and any $\epsilon > 0$ we can find a sequence of positive infinitely differentiable functions f_k and sets G_k such that*

(i)$_k$ $\hat{f}_k(0) = 1$,
(ii)$_k$ $|\hat{f}_k(u) - \hat{f}_{k-1}(u)| \le 2^{-k}$ for all u if $k \ge 2$,
(iii)$_k$ $|\hat{f}_k(u)| \le \epsilon(1 - 2^{-k})$ for all $u \ne 0$,
(iv)$_k$ $|S_n(f_k, x)| \le K_1(\eta)(1 - 2^{-k})$ for all $n \ge 0$ and all $x \notin G_k$,
(v)$_k$ $|G_k| < \eta(1 - 2^{-k)})$,
(vi)$_k$ If $k \ge 2$ we can find a finite set $\mathcal{I}(k)$ of closed intervals such that $\bigcup_{I\in\mathcal{I}(k)} I \supseteq \operatorname{supp} f_k$ but $\sum_{I\in\mathcal{I}(k)} h(|I|) \le 2^{-k}$,
(vii)$_k$ $G_k \supseteq G_{k-1}$ if $k \ge 2$,
(viii)$_k$ $\operatorname{supp} f_{k-1} \supseteq \operatorname{supp} f_k$ if $k \ge 2$,
(ix)$_k$ $G_k \supseteq \operatorname{supp} f_k$.

Proof of Lemma 5 from Lemma 7. Using conditions (i)$_k$ and (ii)$_k$, we see that $f_k m$ converges weakly to a positive measure μ with

(i) $\hat{\mu}(0) = 1$.

Since f_n is smooth $\hat{f}_k(u) \to 0$ as $n \to \infty$ for each k so

(ii) $\hat{\mu}(n) \to 0$ as $|n| \to \infty$.

Condition (iii)$_k$ shows that

(iii) $|\hat{\mu}(u)| \le \epsilon$ for all $u \ne 0$,

and if, we set $G = \bigcup_{k=1}^{\infty} G_k$, conditions (iv)$_k$ and (viii)$_k$ show that

(iv) $|S_n(\mu, x)| \le K_1(\eta)$ for all $n \ge 0$ and all $x \notin G$,

whilst conditions (v)$_k$ and (vii)$_k$ show that

(v) $|G| < \eta$.

Finally, conditions (viii)$_k$ and (vi)$_k$ show that

(vi) $\operatorname{supp} \mu$ has Hausdorff h measure zero. ■

It is important to observe that the first step in proving Lemma 7 is of a different nature to the others and, so far as I can see, non-trivial. To see why this might be the case observe that whilst condition (vi)$_k$ shows an improvement as we increase k, this improvement is bought at the cost of a deterioration in conditions (iv)$_k$, (v)$_k$ and (iii)$_k$.

Looking more closely, we see that our first step requires us to prove the following lemma.

LEMMA 8. *Given any $\eta > 0$ we can find a $K_1(\eta) > 1$ with the following property. Given any $\epsilon > 0$ we can find a positive infinitely differentiable function f_1 and a set G_1 such that*

(i)$_1$ $\hat{f}_1(0) = 1$,
(iii)$_1$ $|\hat{f}_1(u)| \le \epsilon/2$ *for all* $u \ne 0$,
(iv)$_1$ $|S_n(f_1, x)| \le K_1(\eta)/2$ *for all* $n \ge 0$ *and all* $x \notin G_1$,
(v)$_1$ $|G_1| < \eta/2$,
(vii)$_1$ G_1 *is the union of a finite set of closed intervals with end points rational multiples of* 2π,
(x)$_1$ $G_1 \supseteq \operatorname{supp} f_1$.

The difficult point here is that we may choose ϵ as small as we like without affecting $K_1(\eta)$. The result is proved as Lemma 2 in [1].

4. The inductive step for the key lemma. We observe that the induction required for Lemma 7 and thus the result required will be established if we can prove the following result.

LEMMA 9. *Let $h : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $h(0) = 0$. Suppose that f is a positive infinitely differentiable function with $\hat{f}(0) = 1$. Then, given any $\delta > 0$, we can find a positive infinitely differentiable function F and a closed set H such that*

(i) $\hat{F}(0) = 1$,
(ii) $|\hat{F}(u) - \hat{f}(u)| \le \delta$,
(iii) $|S_n(F, x)| \le |S_n(f, x)| + \delta$ *for all* $n \ge 0$ *and all* $x \notin H$,
(iv) We can find a finite set $\mathcal{I}$ of closed intervals such that $\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} F$ *but* $\sum_{I\in\mathcal{I}} h(|I|) \le \delta$,
(v) $\operatorname{supp} F \subseteq \operatorname{supp} f$,
(vi) $H \supseteq \operatorname{supp} f$,
(vii) $|H| < |\operatorname{supp} f| + \delta$.

Justification of the inductive step in Lemma 7. Set $f_k = f$, $f_{k+1} = F$ and $G_{k+1} = G_k \cup H$. Provided that δ is small enough, we have the desired result. Note that we need to know that $H \supseteq \operatorname{supp} f_k$ and $G_k \supseteq \operatorname{supp} f_k$ in order to obtain (v)$_{k+1}$ from (vii) and (v)$_k$. ∎

However, things are not quite as simple as a hasty inspection of Lemma 9 might lead one to believe. The inductive step itself needs (so far as I can see) to be broken up into several steps and (again so far as I can see) it is here that we need to be most careful.

We start with a preparatory step.

LEMMA 10. *Suppose that f is a positive infinitely differentiable function with $\hat{f}(0) = 1$. Then, given any $\delta > 0$, we can find an $\eta > 0$, a finite collection $\mathcal{K}$ of closed intervals such that every pair of intervals in $\mathcal{K}$ intersect at at most one point, and a closed set H such that*

(a) $H \supseteq \operatorname{supp} f$,
(b) $|H| < |\operatorname{supp} f| + \delta$,
(c) If $x \notin H$ *and* $y \in \bigcup_{K\in\mathcal{K}} K$ *then* $|x-y| > \eta$,
(d) If $K \in \mathcal{K}$ *then* $10^2\eta^{-1}\int_K f(t)\,dt < \delta$.

Proof. Since $\operatorname{supp} f$ is closed we can find a finite collection of intervals $J_1, J_2, \ldots, J_m$, say with $\bigcup_{r=1}^m J_r \supseteq \operatorname{supp} f$ and $\sum_{r=1}^m |J_r| < |\operatorname{supp} f| + \delta/2$. Choose $\eta > 0$ such that $8m\eta < \delta$ and set

$$H = \bigcup_{r=1}^{m} (J_r + [-2\eta, 2\eta])$$

where we write $[a,b] + [-2\eta, 2\eta] = [a-2\eta, b+2\eta]$. Conditions (a) and (b) are automatic.

Now choose $\mathcal{K}$ to be a finite collection of closed intervals such that

(i) $\bigcup_{K\in\mathcal{K}} K = \bigcup_{r=1}^m J_r$,
(ii) every pair of intervals in $\mathcal{K}$ intersect at at most one point,
(iii) if $K \in \mathcal{K}$ then $|K| < \|f\|_\infty^{-1} 10^{-2}\eta\delta$.

Conditions (c) and (d) are automatic. ■

We now need to apply the following lemma repeatedly.

Lemma 11. *Let* $h : [0,\infty) \to [0,\infty)$ *be a strictly increasing continuous function with* $h(0) = 0$. *Suppose that we are given* f *a positive infinitely differentiable function with* $\hat{f}(0) = 1$, $\delta > 0$ *and* $\eta > 0$, *together with a finite collection* $\mathcal{K}$ *of closed intervals such that every pair of intervals in* $\mathcal{K}$ *intersect at at most one point, and a closed set* H *such that*

(a) $H \supseteq \operatorname{supp} f$,
(b) $|H| < |\operatorname{supp} f| + \delta$,
(c) If $x \notin H$ *and* $y \in \bigcup_{K\in\mathcal{K}} K$ *then* $|x-y| > \eta$,
(d) If $K \in \mathcal{K}$ *then* $10^2\eta^{-1}\int_K f(t)\,dt < \delta$.

Then, given any $K_1 \in \mathcal{K}$ *and any* $\epsilon > 0$, *we can find a positive infinitely differentiable function* F *such that*

(d)′ If $K \in \mathcal{K}$ *then* $10^2\eta^{-1}\int_K F(t)\,dt < \delta$,

and

(i) $\hat{F}(0) = 1$,
(ii) $|\hat{F}(u) - \hat{f}(u)| \le \epsilon$ *if* $|\hat{f}(u)| > \delta/4$,
(ii)′ $|\hat{F}(u)| \le \delta/2$ *if* $|\hat{f}(u)| \le \delta/4$,
(iii) $|S_n(F,x)| \le |S_n(f,x)| + \epsilon$ *for all* $n \ge 0$ *and all* $x \notin H$ *with* $|S_n(f,x)| > \delta/4$,
(iii)′ $|S_n(F,x)| \le \delta/2$ *for all* $n \ge 0$ *and all* $x \notin H$ *with* $|S_n(f,x)| \le \delta/4$,
(iv) We can find a finite set $\mathcal{I}_{K_1}$ *of closed intervals such that* $\bigcup_{I\in\mathcal{I}_{K_1}} I \supseteq \operatorname{supp} F \cap K_1$ *but* $\sum_{I\in\mathcal{I}} h(|I|) \le \epsilon$,
(v) $\operatorname{supp} F \subseteq \operatorname{supp} f$.

Proof of Lemma 9. First apply Lemma 10 to obtain $\mathcal{K}$ with the properties given in that Lemma. We suppose that $\mathcal{K}$ contains M intervals $K(1)$, $K(2)$, …, $K(M)$. Now set

$\epsilon = \delta/(4M)$, and $F_0 = f$. By applying Lemma 11 repeatedly we obtain a sequence of M positive infinitely differentiable functions F_j with $\hat{F}_j(0) = 1$ such that if $1 \leq j \leq M$.

(i)$_j$ $\hat{F}_j(0) = 1$,
(ii)$_j$ $|\hat{F}_j(u) - \hat{F}_{j-1}(u)| \leq \epsilon$ if $|\hat{F}_{j-1}(u)| > \delta/4$,
(ii)$'_j$ $|\hat{F}_j(u)| \leq \delta/2$ if $|\hat{F}_{j-1}(u)| \leq \delta/4$,
(iii)$_j$ $|S_n(F_j, x)| \leq |S_n(F_{j-1}, x)| + \epsilon$ for all $n \geq 0$ and all $x \notin H$ with $|S_n(F_{j-1}, x)| > \delta/4$,
(iii)$'_j$ $|S_n(F_j, x)| \leq \delta/2$ for all $n \geq 0$ and all $x \notin H$ with $|S_n(F_{j-1}, x)| \leq \delta/4$,
(iv)$_j$ We can find a finite set $\mathcal{I}_{K(j)}$ of closed intervals such that $\bigcup_{I \in \mathcal{I}_{K(j)}} I \supseteq \operatorname{supp} F \cap K(j)$ but $\sum_{I \in \mathcal{I}} h(|I|) \leq \epsilon$,
(v)$_j$ $\operatorname{supp} F_j \subseteq \operatorname{supp} F_{j-1}$.

If we now set $F = F_M$ and $\mathcal{I} = \bigcup_{j=1}^{M} \mathcal{I}_{K(j)}$ then the conclusions of Lemma 9 can be read off. ■

Proof of Lemma 11. We may suppose $\epsilon < 10^{-2}\delta$ and $\eta < 1$. Choose positive infinitely differentiable functions ϕ_1 and ϕ_2 such that $\phi_1 + \phi_2 = 1$, $\phi_1(t) = 0$ for all $t \notin K_1$ and we can find two intervals J_1 and J_2 such that

$$h(|J_1|) + h(|J_2|) \leq \epsilon/2, \text{ and } \phi_2(t) = 0 \text{ for all } t \in K_1 \setminus (J_1 \cup J_2).$$

We set $f_1 = \phi_1 f$, $f_2 = \phi_2 f$.

Since f_1 and f_2 are infinitely differentiable, we can find a positive integer N such that

$$\sum_{|u| \geq N} |\hat{f}_1(u)| + |\hat{f}_2(u)| \leq \delta/16. \tag{1}$$

We now approximate f_1 in the weak star sense by a function g_1 of the form

$$g_1 = \sum_{j=1}^{M} \lambda_j \delta_{x_j} * G$$

where δ_{x_j} is the Dirac delta measure at x_j, G is an infinitely differentiable function of integral 1 and support a small interval containing 0. Provided we take M large enough and the support of G small enough and choose the λ_j and x_j appropriately, we can ensure that

$$\sum_{|u| \leq N} |\hat{f}_1(u) - \hat{g}_1(u)| \leq \epsilon/16 \tag{2}$$

$\operatorname{supp} g_1 \subseteq K_1$, and in addition

(i)$'$ $\hat{g}_1(0) = \hat{f}_1(0)$,
(iv)$'$ We can find a finite set of closed intervals $\mathcal{J}$ such that $\bigcup_{J \in \mathcal{J}} I \supseteq \operatorname{supp} g_1$ but $\sum_{J \in \mathcal{J}} h(|J|) \leq \epsilon/2$,
(v)$'$ $\operatorname{supp} g_1 \subseteq \operatorname{supp} f_1$.

Now set $F = g_1 + f_1$ and $\mathcal{I}_{K_1} = \mathcal{J} \cup \{J_1\} \cup \{J_2\}$. Conclusions (d)$'$ and (i) follow from (i)$'$ and the definitions of f_1 and f_2. Similarly conclusions (iv) and (v) follow more or less directly from (iv)$'$ and (v)$'$. To prove (ii) observe that, if $|\hat{f}(u)| > \delta/4$, then (since

$|\hat{f}_1(u)|+|\hat{f}_2(u)| \geq |\hat{f}(u)|$) equation (1) tells us that $|u|<N$ and so equation (2) gives us

$$|\hat{F}(u)-\hat{f}(u)| = |\hat{g}_1(u)-\hat{f}_1(u)| \leq \epsilon.$$

To prove (ii)$'$ observe that

$$\|f_1\|_1 + \|g_1\|_1 = \hat{f}_1(0)+\hat{g}_0(0) = 2\hat{f}_1(0) \leq 10^{-2}\eta^{-1}\delta < \delta/4$$

by condition (d) and so

$$|\hat{F}(u)| \leq |\hat{f}(u)| + |\hat{g}_1(u)-\hat{f}_1(u)| \leq |\hat{f}(u)| + |\hat{g}_1(u)| + |\hat{f}_1(u)| \leq |\hat{f}(u)|+\delta/4$$

for all u.

The proof of (iii) and (iii)$'$ follows the same pattern but constitutes a central step in the proof. Suppose $x \notin H$ and so in particular, $x \notin \operatorname{supp} f$. Since f is smooth, $\sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irx$ converges absolutely to $f(x)=0$ and so, if $|S_n(f,x)|>\delta/4$, equation (1) tells us that $|u|<N$. Equation (2) now tells us that

$$|S_n(F,x)-S_n(f,x)| = |S_n(g_1,x)-S_n(f_1,x)| \leq \sum_{|u|\leq N} |\hat{f}_1(u)-\hat{g}_1(u)| \leq \epsilon$$

and so

$$|S_n(F,x)| \leq |S_n(f,x)|+\epsilon$$

as required. To prove (iii)$'$, observe that, as we noted in the previous paragraph $\|f_1\| + \|g_1\| \leq 10^{-2}\eta^{-1}\delta$ and, using condition (c), $|x-y|>\eta$. Now, using the Dirichlet formula

$$S_n(\mu,x) = \int \frac{\sin(n+\frac{1}{2})}{\sin\frac{t}{2}}\,d\mu(t),$$

we have

$$|S_n(f_1,x)| \leq \frac{\pi}{\eta}\|f_1\|,\ S_n(g_1,x)| \leq \frac{\pi}{\eta}\|g_1\|$$

and

$$\begin{aligned}|S_n(F,x)| &\leq |S_n(f,x)| + |S_n(g_1,x)-S_n(f_1,x)|\\ &\leq |S_n(f,x)|+|S_n(g_1,x)|+|S_n(f_1,x)| \leq |S_n(f,x)|+\delta/4. \ \blacksquare\end{aligned}$$

This completes the proof of the main lemma and so of Theorem 4.

5. Algebraic independence. We now turn to the proof of the full Theorem 2. To this end we use the following modification of Lemma 5.

Lemma 12. *We can find a sequence K_j with the following property. Given any strictly increasing continuous function $h:[0,\infty)\to[0,\infty)$ with $h(0)=0$ and any sequence ϵ_j with $\epsilon_j>0$ we can find a sequence of positive measures μ_j and Borel sets G_j such that*

(i)$_j$ $\hat{\mu}_j(0)=1$,
(ii)$_j$ $\hat{\mu}_j(n)\to 0$ *as* $|n|\to\infty$,
(iii)$_j$ $|\hat{\mu}_j(u)| \leq \epsilon$ *for all* $u\neq 0$,
(iv)$_j$ $|S_n(\mu_j,x)| \leq K_j$ *for all* $n\geq 0$ *and all* $x\notin G_j$,
(v)$_j$ $|G_j|<2^{-j}$,
(vi)$_j$ $\operatorname{supp}\mu_j$ *has Hausdorff* h *measure zero,*
(vii) $\bigcup_{j=1}^{\infty} G_j$ *independent.*

The proof of Theorem 2 follows that of Theorem 4. We take

$$\mu = \sum_{j=1} Q_j \mu_j$$

for appropriate trigonometric polynomials Q_j and an appropriate ϵ_j.

To obtain Lemma 12 we need a more complicated version of Lemma 7. If $\mathbf{a} = (a_1, a_2, \dots, a_M)$ is a non-zero finite sequence of integers and $\delta > 0$, let us say that a set E is $(\mathbf{a}, \delta)$ *independent* if there is no solution of

$$\sum_{m=1}^{M} a_m x_m = 0$$

with $x_m \in E$ for $1 \le m \le M$ and $\min_{m \ne k} |x_m - x_k| > \delta$. We choose, once and for all, a sequence $\mathbf{a}(n)$ of non-zero finite sequences such that each possible sequence appears infinitely often.

LEMMA 13. *We can find a sequence K_j with the following property. Given any strictly increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and any sequence ϵ_j with $\epsilon_j > 0$ we can find positive infinitely differentiable functions $f_{j,k}$ and sets $G_{j,k}$ such that*

(i)$_k$ $\hat{f}_{j,k}(0) = 1$ for $1 \le j \le k$,
(ii)$_k$ $|\hat{f}_{j,k}(u) - \hat{f}_{j,k-1}(u)| \le 2^{-k}$ for all u if $1 \le j \le k-1$,
(iii)$_k$ $|\hat{f}_{j,k}(u)| \le \epsilon_j(1 - 2^{-k})$ for all $u \ne 0$ and $1 \le j \le k$,
(iv)$_k$ $|S_n(f_{j,k}, x)| \le K_j(1 - 2^{-k})$ for all $n \ge 0$, all $x \notin G_{j,k}$ and $1 \le j \le k$,
(v)$_k$ $|G_{j,k}| < 2^{-j}(1 - 2^{-k}))$ for all $1 \le j \le k$,
(vi)$_k$ If $1 \le j \le k-1$ we can find a finite set $\mathcal{I}(j,k)$ of closed intervals such that $\bigcup_{I \in \mathcal{I}(j,k)} \supseteq \operatorname{supp} f_k$ but $\sum_{I \in \mathcal{I}(j,k)} h(|I|) \le 2^{-k}$,
(vii)$_k$ $G_{j,k} \supseteq G_{j,k-1}$ if $1 \le j \le k-1$,
(viii)$_k$ $\operatorname{supp} f_{j,k-1} \supseteq \operatorname{supp} f_{j,k}$ if $1 \le j \le k-1$,
(ix)$_k$ $G_{j,k} \supseteq \operatorname{supp} f_{j,k}$ for $1 \le j \le k$,
(x)$_k$ $\bigcup_{j=1}^{k-1} G_{j,k}$ is $(\mathbf{a}_n, 2^{-k})$ independent.

Proof of Lemma 12 from Lemma 13. This follows the proof of Lemma 5 closely. Using conditions (i)$_k$, (ii)$_k$ and (ix)$_k$ we see that $f_{j,k}m$ converges weakly to a positive measure μ_j with support lying within $G_j = \bigcap_{j=k}^{\infty} G_{j,k}$. Conditions (i)$_j$ to (vi)$_j$ are deduced in the same way that we obtained conditions (i) to (vi) of Lemma 5.

Suppose now that $\mathbf{a} = (a_1, a_2, \dots, a_M)$ is a non-zero finite sequence of integers and $x_1, x_2, \dots, x_M$ are distinct points of $\bigcup_{j=1}^{\infty} \operatorname{supp} \mu_j$. Simple arguments show that we can find an N such that

(a) $\max_{m \ne q} |x_m - x_q| > 2^{-N}$,
(b) $x_m \in \operatorname{supp} \bigcup_{j=1}^{N-1} G_j$ for all $1 \le m \le M$,
(c) $\mathbf{a}_N = \mathbf{a}$.

Since $G_j \subseteq G_{j,N}$ for each $1 \le j \le N$ condition (x)$_N$ tells us that

$$\sum_{m=1}^{M} a_m x_m \ne 0.$$

Thus $\bigcup_{j=1}^{\infty} G_j$ is independent and we are done. ■

6. Discussion of the inductive step. If we look at Lemma 13 and compare it with Lemma 7 it is clear that (if we proceed as before) each inductive step will itself involve many substeps. The introduction of the function $f_{k,k}$ and the set $G_{k,k}$ will follow Lemma 8. The steps required to ensure $(\mathrm{vi})_k$ will follow the method given in Lemma 9. We shall therefore only deal with the steps required to obtain $(\mathrm{x})_k$.

Essentially, we need to prove the following version of Lemma 9:

LEMMA 14. *Suppose that we are given R positive infinitely differentiable functions f_k with $\hat{f}_k(0) = 1$ $[1 \leq k \leq R]$. Then, given any $\delta > 0$ and any non-zero finite sequence $\mathbf{a} = (a_1, a_2, \ldots, a_M)$ of integers, we can find positive infinitely differentiable functions G_j and a closed set H such that*

(i) $\hat{G}_k(0) = 1$ *for all* $1 \leq k \leq R$,
(ii) $|\hat{G}_k(u) - \hat{f}_k(u)| \leq \delta$ *for all* $1 \leq k \leq R$,
(iii) $|S_n(G_k, x)| \leq |S_n(f_k, x)| + \delta$ *for all* $n \geq 0$, *all* $1 \leq k \leq R$ *and all* $x \notin H$,
(iv) $\bigcup_{k=1}^{R} \operatorname{supp} G_k$ *is* $(\mathbf{a}, \delta)$ *independent.*
(v) $\operatorname{supp} G_k \subseteq \operatorname{supp} f_k$ *for all* $1 \leq k \leq R$,
(vi) $H \supseteq \bigcup_{k=1}^{R} \operatorname{supp} f_k$,
(vii) $|H| < |\bigcup_{k=1}^{R} \operatorname{supp} G_k| + \delta$.

The rest of this paper is devoted to proving this result.

We shall follow the proof of Lemma 9 with appropriate modifications. As might be expected, we start with a version of Lemma 10.

LEMMA 15. *Suppose that we are given R positive infinitely differentiable functions f_k with $\hat{f}_k(0) = 1$ $[1 \leq k \leq R]$. Then given any $\delta > 0$ and any non-zero finite sequence $\mathbf{a} = (a_1, a_2, \ldots, a_M)$ of integers we can find an $\eta > 0$, a finite collection $\mathcal{K}$ of closed intervals such that every pair of intervals in $\mathcal{K}$ intersect at at most one point, and a closed set H such that*

(a) $H \supseteq \operatorname{supp} f$,
(b) $|H| < |\operatorname{supp} f| + \delta$,
(c) If $x \notin H$ *and* $y \in \bigcup_{K \in \mathcal{K}} K$ *then* $|x - y| > \eta$,
(d) If $K \in \mathcal{K}$ *then* $10^3 \eta^{-1} \int_K f_k(t)\, dt < \delta/M$,
(e) If $K \in \mathcal{K}$ *then* $|K| < \delta/10$.

Proof. Essentially the same as that of Lemma 10. ∎

Our central step corresponds to Lemma 11.

LEMMA 16. *Suppose that we are given R positive infinitely differentiable functions f_k with $\hat{f}_k(0) = 1$ $[1 \leq k \leq R]$. together with $\delta > 0$, a non-zero finite sequence $\mathbf{a} = (a_1, a_2, \ldots, a_M)$ of integers, an $\eta > 0$, and a finite collection $\mathcal{K}$ of closed intervals such that every pair of intervals in $\mathcal{K}$ intersect at at most one point, and a closed set H such that*

(a) $H \supseteq \operatorname{supp} f$,
(b) $|H| < |\operatorname{supp} f| + \delta$,
(c) If $x \notin H$ *and* $y \in \bigcup_{K \in \mathcal{K}} K$ *then* $|x - y| > \eta$,
(d) If $K \in \mathcal{K}$ *then* $10^3 \eta^{-1} \int_K f_k(t)\, dt < \delta/M$.

(e) If $K \in \mathcal{K}$ then $|K| < \delta/10$.

Then given any $K_1,\ K_2,\ \ldots,\ K_M \in \mathcal{K}$ such that

$$\inf_{x\in K_r,\ y\in K_s} |x-y| > \delta/2 \text{ whenever } 1 \le r < s \le M$$

and any $\epsilon > 0$ we can find R positive infinitely differentiable functions F_k $[1 \le k \le R]$ such that

(d)′ If $K \in \mathcal{K}$ then $10^2\eta^{-1}\int_K F_k(t)\,dt < \delta$, for all $1 \le k \le R$

and

(i) $\hat{F}_k(0) = 1$,
(ii) $|\hat{F}_k(u) - \hat{f}_k(u)| \le \epsilon$ if $|\hat{f}_k(u)| > \delta/4$,
(ii)′ $|\hat{F}_k(u)| \le \delta/2$ if $|\hat{f}_k(u)| \le \delta/4$,
(iii) $|S_n(F_k, x)| \le |S_n(f_k, x)| + \epsilon$ for all $n \ge 0$ and all $x \notin H$ with $|S_n(f_k, x)| > \delta/4$,
(iii)′ $|S_n(F_k, x)| \le \delta/2$ for all $n \ge 0$ and all $x \notin H$ with $|S_n(f_k, x)| \le \delta/4$,
(iv) If $x_m \in \bigcup_{k=1}^R \operatorname{supp} F_k \cap K_m$ for $1 \le m \le M$ then $\sum_{j=1}^M a_j x_j \ne 0$.
(v) $\operatorname{supp} F_k \subseteq \operatorname{supp} f_k$.

Sketch proof of Lemma 14. This follows the proof of Lemma 9 but the details, if we were to give them, would require quite complex notation. We first need to apply Lemma 15 to obtain $\mathcal{K}$ with the properties given in that Lemma. We now apply Lemma 16 to each possible collection $K_1,\ K_2,\ \ldots,\ K_M \in \mathcal{K}$ such that

$$\inf_{x\in K_r,\ y\in K_s} |x-y| > \delta/2 \text{ whenever } 1 \le r < s \le R,$$

in turn. The arguments of the proof of Lemma 9 from Lemma 11 give, mutatis mutandis, all the conditions of Lemma 14 with the exception of (iv).

To prove (iv), observe that if $x_m \in \bigcup_{k=1}^R \operatorname{supp} G_k$ for $1 \le m \le M$ and $\min_{m\ne k} |x_m - x_k| > \delta$ then, automatically $x_m \in \bigcup_{K\in\mathcal{K}} K$ and there must be a collection $K_1, K_2, \ldots, K_M \in \mathcal{K}$ such that

$$\inf_{x\in K_r,\ y\in K_s} |x-y| > \delta/2 \text{ whenever } 1 \le r < s \le R$$

and $x_m \in K_m$ for $1 \le m \le M$. Thus, by looking at condition (vi) of Lemma 16, for the appropriate step we obtain

$$\sum_{j=1}^m a_j x_j \ne 0,$$

as required. ■

Sketch proof of Lemma 16. We may suppose $\epsilon < 10^{-2}\delta$ and $\eta < 1$. Choose positive infinitely differentiable functions ϕ_1 and ϕ_2 such that $\phi_1 + \phi_2 = 1$, $\phi_1(t) = 1$ for all $t \in$ and $\phi_1(t) = 0$ unless t lies in some interval $K \in \mathcal{K}$ which shares an end point with one of the K_m with $1 \le m \le M$. We set $f_{1,r} = \phi_1 f_r$, $f_2 = \phi_2 f_r$.

Since $f_{1,r}$ and $f_{2,r}$ are infinitely differentiable we can find a positive integer N such that

$$\sum_{|u|\ge N} |\hat{f}_1(u)| + |\hat{f}_2(u)| \le \delta/16 \tag{1}$$

for all $1 \le r \le R$. We now approximate the $f_{1,r}$ in the weak star sense by a function $g_{1,r}$ of the form

$$g_{1,r} = \sum_{j=1}^{J} \lambda_j \delta_{y_j} * G$$

where δ_{y_j} is the Dirac delta measure at y_j and G is an infinitely differentiable function of integral 1 and support a small interval containing 0. We demand that the y_j be independent.

Provided we take M large enough and choose the λ_j and x_j appropriately we can ensure that, provided only $\operatorname{supp} G$ lies within a sufficiently small distance of 0,

$$\sum_{|u|\le N} |\hat{f}_{1,r}(u) - \hat{g}_{1,r}(u)| \le \epsilon/16 \tag{2}$$

and in addition

(i)′ $\hat{g}_{1,r}(0) = \hat{f}_1(0)$,
(v)′ $\operatorname{supp} g_{1,r} \subseteq \operatorname{supp} f_{1,r}$,
(d)″ $\int_K g_{1,r}(t)\,dt = \int_K f_{1,r}(t)\,dt$ for all $K \in \mathcal{K}$.

Since the y_j are independent we can find a $\rho > 0$ such that, if $|x_j - y_j| \le \rho$ for $1 \le j \le M$,

$$\sum_{j=1}^{M} a_j x_j \ne 0.$$

Choose G so that $\operatorname{supp} G \subseteq [-\omega, \omega]$ and all the conclusions of the previous paragraph hold. Conclusion (iv) of our lemma is now immediate and conclusion (d)′ follows from (d)″. The remaining conclusions are proved in the same way as the corresponding conclusions in Lemma 11. ■

It would be nice to think that the correction theorems obtained in this paper had something to do with questions of pointwise convergence such as are considered in Carleson's theorem, but, so far as I can see, they do not. It would be surprising if they did, since the methods of this paper may be complicated but are certainly not deep.

References

[1] T. W. Körner, J. London Math. Soc. (2) 60 (1999), 548–560.