FOURIER ANALYSIS AND RELATED TOPICS
BANACH CENTER PUBLICATIONS, VOLUME 56
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2002

THE HILBERT TRANSFORM, REARRANGEMENTS, AND LOGARITHMIC DETERMINANTS

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This is an extended version of notes prepared for the talk at the conference "Rajchman-Zygmund-Marcinkiewicz 2000". They are based on recent papers [13] and [15] (see also [14] and [16]). The authors thank Professor Żelazko for the invitation to participate in the conference.

1. Let g be a bounded measurable real-valued function on $\mathbb{R}$ with a compact support. We shall use the following notations:

- The Hilbert transform of g:

$$(\mathcal{H}g)(\xi) = \frac{1}{\pi} \int_{\mathbb{R}}' \frac{g(t)}{t-\xi}\, dt,$$

the prime means that the integral is understood in the principal value sense at the point $t = \xi$.

2000 *Mathematics Subject Classification*: 31A05, 42A50.
V. Matsaev and M. Sodin supported by the Israel Science Foundation of the Israel Academy of Sciences and Humanities under Grant No. 37/00-1.

• The (signed) distribution function of g:

$$N_g(s) = \begin{cases} \ \ \operatorname{meas}\{x:\ g(x) > s\}, \text{ if } \ s > 0; \\ -\operatorname{meas}\{x:\ g(x) < s\}, \text{ if } \ s < 0. \end{cases}$$

The (signed) decreasing rearrangement of g: g_d is defined as the distribution function of N_g: $g_d = N_{N_g}$.

Less formally, the functions N_g and g_d can be also defined by the following properties: they are non-negative and non-increasing for $s > 0$, non-positive and non-increasing for $s < 0$, and

$$\int_{\mathbb{R}} \Phi(g(t))\, dt = \int_{\mathbb{R}} \Phi(s)\, dN_g(s) = \int_{\mathbb{R}} \Phi(g_d(t))\, dt,$$

for any function Φ such that at least one of the three integrals is absolutely convergent.

We shall use notation $A \lesssim B$ when $A \le C \cdot B$ for a positive numerical constant C. We shall write $A \lesssim_\lambda B$ if C in the previous inequality depends on the parameter λ only.

THEOREM 1.1. *Let g be a bounded measurable real-valued function with a compact support. Then*

$$||\mathcal{H} g_d||_{L^1} \le 4 ||\mathcal{H} g||_{L^1}. \tag{1.2}$$

Hereafter, L^1 always means $L^1(\mathbb{R})$.

REMARKS.

1.3. Estimate (1.2) can be extended to a wider class of functions after an additional regularization of the Hilbert transform $\mathcal{H} g_d$ (see §3 below).

1.4. Probably, the constant 4 on the RHS is not sharp. However, Davis' discussion in [3] suggests that (1.2) ceases to hold without this factor on the RHS.

1.5. Theorem 1.1 yields a result of Tsereteli [19] and Davis [3]: *if $g \in \operatorname{Re} H^1$, then g_d is also in* $\operatorname{Re} H^1$, *and* $||\mathcal{H} g_d||_{L^1} \lesssim ||g||_{\operatorname{Re} H^1}$, where $\operatorname{Re} H^1$ is the real Hardy space on $\mathbb{R}$.

1.6. Theorem 1.1 can be extended to functions defined on the unit circle $\mathbb{T}$. Let $g(t)$ be a bounded function on $\mathbb{T}$, g_d be its signed decreasing rearrangement, and $\tilde{g}$ be the function conjugate to g:

$$\tilde{g}(t) = \frac{1}{2\pi} \int_{\mathbb{T}}' g(\xi) \cot \frac{t - \xi}{2}\, d\xi.$$

Then

$$||\widetilde{g_d}||_{L^1(\mathbb{T})} \le 4 ||\tilde{g}||_{L^1(\mathbb{T})}. \tag{1.7}$$

Juxtapose this estimate with Baernstein's inequality [1]:

$$||\tilde{g}||_{L^1(\mathbb{T})} \le ||\widetilde{g_s}||_{L^1(\mathbb{T})}, \tag{1.8}$$

where g_s is the symmetric decreasing rearrangement of g. In particular, if g_s has a conjugate in L^1, then *any* rearrangement of g has a conjugate in L^1, and if *some* rearrangement of g has a conjugate in L^1, then the conjugate of g_d is in L^1. We are not aware of a counterpart of Baernstein's inequality for the Hilbert transform and the $L^1(\mathbb{R})$-norm.

2. Here, we shall prove Theorem 1.1. WLOG, we assume that

$$\int_{\mathbb{R}} g(t)\, dt = 0, \tag{2.1}$$

otherwise

$$(\mathcal{H}g)(\xi) = -\frac{1}{\pi\xi}\int_{\mathbb{R}} g(t)\,dt + O(1/\xi^2), \qquad \xi \to \infty,$$

and the L^1-norm on the RHS of (1.2) is infinite.

The first reduction: instead of (1.2), we shall prove the inequality

$$||\mathcal{H}N_g||_{L^1} \le 2||\mathcal{H}g||_{L^1}, \tag{2.2}$$

then its iteration gives (1.2).

We introduce a (regularized) logarithmic determinant of g:

$$u_g(z) \overset{def}{=} \int_{\mathbb{R}} K(zg(t))\,dt, \qquad K(z) = \log|1-z| + \mathrm{Re}(z).$$

This function is subharmonic in $\mathbb{C}$ and harmonic outside of $\mathbb{R}$.

List of properties of u_g: Since g is a bounded function with a compact support,

$$u_g(z) = O(|z|^2), \qquad z \to 0, \tag{2.3a}$$

and by (2.1)

$$u_g(z) = \int_{\mathbb{R}} \log|1 - zg(t)|\,dt = O(\log|z|), \qquad z \to \infty. \tag{2.3b}$$

In particular,

$$\int_{\mathbb{R}} \frac{|u_g(x)|}{x^2} < \infty. \tag{2.3c}$$

Next,

$$\int_{\mathbb{R}} \frac{u_g(x)}{x^2}\,dx = 0. \tag{2.4}$$

This follows from the Poisson representation:

$$u_g(iy) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{u_g(x)}{x^2+y^2}\,dy, \qquad y > 0.$$

Dividing by y, letting $y \to 0$, and using (2.3a), we get (2.4).

Further,

$$u_g(1/t) = -\pi(\mathcal{H}N_g)(t). \tag{2.5}$$

Indeed, integrating by parts and changing variables, we obtain for real x's:

$$u_g(x) = \int_{\mathbb{R}} \log|1-xs|\,dN_g(s) = x\int_{\mathbb{R}}' \frac{N_g(s)}{1-xs}\,ds = -\pi(\mathcal{H}N_g)(1/x).$$

We have done the *second reduction:* Instead of (2.2), we shall prove the inequality

$$\int_{\mathbb{R}} \frac{u_g^-(x)}{x^2}\,dx \le \pi||\mathcal{H}g||_{L^1}. \tag{2.6}$$

Then combining (2.4) and (2.6), we get (2.2).

Now, we set

$$f(t) = g(t) + i(\mathcal{H}g)(t).$$

This function has an analytic continuation into the upper half-plane:

$$f(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{g(t)}{t-z}\, dt.$$

We define the regularized logarithmic determinant of f by the equation

$$u_f(z) = \int_{\mathbb{R}} K(zf(t))\, dt. \tag{2.7}$$

The *positivity* of this subharmonic function is central in our argument:

LEMMA 2.8. (cf. [4])

$$u_f(z) \geq 0, \qquad z \in \mathbb{C}.$$

Proof. It suffices to consider z's such that all solutions of the equation $zf(w) = 1$ are simple and not real. Then

$$\begin{aligned} u_f(z) &= \mathrm{Re}\left\{ \int_{\mathbb{R}} \left[\log(1 - zf(t)) + zf(t) \right] dt \right\} = \mathrm{Re}\left\{ z^2 \int_{\mathbb{R}} \frac{tf(t)f'(t)}{1 - zf(t)}\, dt \right\} \\ &= \mathrm{Re}\left\{ 2\pi i z^2 \sum_{\{w:\, zf(w)=1\}} \mathrm{Res}_w \left(\frac{\zeta f(\zeta) f'(\zeta)}{1 - zf(\zeta)} \right) \right\} = 2\pi \sum_{\{w:\, zf(w)=1\}} \mathrm{Im}(w) \geq 0. \end{aligned}$$

The application of the Cauchy theorem is justified since $f(\zeta) = O(1/\zeta^2)$ when $\zeta \to \infty$, $\mathrm{Im}(\zeta) \geq 0$.

To complete the proof of the theorem, we shall use an argument borrowed from the perturbation theory of compact operators [5]. We use auxiliary functions $f_1 = g + i|\mathcal{H}g|$ and

$$u_1(z) = \int_{\mathbb{R}} \log \left| \frac{1 - zg(t)}{1 - zf_1(t)} \right| dt.$$

Then on the real axis

$$u_g(x) = u_1(x) + u_f(x), \qquad x \in \mathbb{R},$$

so that $u_g(x) \geq u_1(x)$, or $u_g^-(x) \leq u_1^-(x) = -u_1(x)$, since $u_1(x) \leq 0$, $x \in \mathbb{R}$.

Next, we need an elementary inequality: if w_1, w_2 are complex numbers such that $\mathrm{Re}(w_1) = \mathrm{Re}(w_2)$ and $|\mathrm{Im}(w_1)| \leq \mathrm{Im}(w_2)$, then for all z in the upper half-plane,

$$\left| \frac{1 - zw_1}{1 - zw_2} \right| < 1.$$

Due to this inequality the function u_1 is non-positive in the upper half-plane. Since this function is harmonic in the upper half-plane, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \frac{u_g^-(x)}{x^2}\, dx &\leq - \int_{\mathbb{R}} \frac{u_1(x)}{x^2}\, dx = - \lim_{y \to 0} \int_{\mathbb{R}} \frac{u_1(x)}{x^2 + y^2}\, dx \leq -\pi \lim_{y \to 0} \frac{u_1(iy)}{y} \\ &= -\pi \lim_{y \to 0} \frac{1}{y} \int_{\mathbb{R}} \log \left| \frac{1 - iyg(t)}{1 - iyg(t) + y|(\mathcal{H}g)(t)|} \right| dt = \pi \int_{\mathbb{R}} |(\mathcal{H}g)(t)|\, dt. \end{aligned}$$

This proves (2.6) and therefore the theorem. ∎

3. Here, we will formulate a fairly complete version of estimate (2.2). The proof given in [15] follows similar lines as above, however is essentially more technical.

Now, we start with a real-valued measure $d\eta$ of finite variation on $\mathbb{R}$, and denote by $g = \mathcal{H}\eta$ its Hilbert transform. By $||\eta||$ we denote the total variation of the measure $d\eta$ on $\mathbb{R}$. Let $R_g = \mathcal{H}^{-1}N_g$ be a regularized inverse Hilbert transform of N_g:

$$R_g(t) \stackrel{def}{=} \lim_{\epsilon\to 0}\frac{1}{\pi}\int'_{|s|>\epsilon}\frac{N_g(s)}{t-s}\,ds.$$

The integral converges at infinity due to the Kolmogorov weak L^1-type estimate

$$N_g(s) \lesssim ||\eta||/s, \qquad 0 < s < \infty.$$

Existence of the limit when $\epsilon \to 0$ (and $t \neq 0$) follows from the Titchmarsh formula [18] (cf. [15]):

$$\lim_{s\to 0} sN_g(s) = \frac{\eta(\mathbb{R})}{\pi}.$$

THEOREM 3.1. *Let $d\eta$ be a real measure supported by $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} R_g^+(t)dt \le ||\eta_{\text{a.c.}}||, \tag{3.2}$$

$$\int_{\mathbb{R}} R_g^-(t)dt \le ||\eta|| - |\eta(\mathbb{R})|, \tag{3.3}$$

and

$$\int_{\mathbb{R}} R_g(t)dt = |\eta(\mathbb{R})| - ||\eta_{\text{sing}}||. \tag{3.4}$$

COROLLARY 3.5. *The function R_g always belongs to L^1 and its L^1-norm does not exceed $2||\eta||$.*

The classical Boole theorem says that if $d\eta$ is non-negative and pure singular, then $N_g(s) = \eta(\mathbb{R})/s$, and therefore R_g vanishes identically. The next two corollaries can be viewed as quantitative generalizations of this fact:

COROLLARY 3.6. *If $d\eta \ge 0$, then $R_g(t)$ is non-negative as well, and $||R_g||_{L^1} = \eta_{\text{a.c.}}(\mathbb{R})$.*

COROLLARY 3.7. *If $d\eta$ is pure singular, then $R_g(t)$ is non-positive and $||R_g||_{L^1} = ||\eta|| - |\eta(\mathbb{R})|$.*

For other recent results obtained with the help of the logarithmic determinant we refer to [8], [14] and [16].

4. In §2 we used the subharmonic function technique for proving a theorem about the Hilbert transform. The idea of logarithmic determinants also provides us with a connection which works in the opposite direction: starting with a known result about the Hilbert transform, one arrives at a plausible conjecture about a non-negative subharmonic function in $\mathbb{C}$ represented by a canonical integral of genus one. For illustration, we consider a well known inequality

$$m_f(\lambda) \lesssim \frac{1}{\lambda^2}\int_0^\lambda s m_g(s)ds + \frac{1}{\lambda}\int_\lambda^\infty m_g(s)ds, \qquad 0 < \lambda < \infty, \tag{4.1}$$

where $f = g + i\mathcal{H}g$, g is a test function on $\mathbb{R}$, $m_f(\lambda) = \text{meas}\{|f| \ge \lambda\}$, and $m_g(\lambda) = \text{meas}\{|g| \ge \lambda\} = N_g(\lambda) - N_g(-\lambda)$. Inequality (4.1) contains as special cases Kolmogorov's weak L^1-type inequality $\lambda m_f(\lambda) \lesssim ||g||_{L^1}$, and M. Riesz' inequality $||f||_{L^p} \lesssim_p ||g||_{L^p}$, $1 < p \le 2$. Inequality (4.1) can be justly attributed to Marcinkiewicz. He formulated his general interpolation theorem for sub-linear operators in [12], the proof was supplied by Zygmund in [21] with reference to Marcinkiewicz' letter. Its main ingredient is a decomposition $g = g\chi_{\{|g|<\lambda\}} + g\chi_{\{|g|\ge\lambda\}}$, where χ_E is the characteristic function of a set E. This decomposition immediately proves (4.1), see [7, Section V.C.2].

Define a logarithmic determinant u_f of genus one as in (2.7), and denote by $d\mu_f$ its Riesz measure (i.e. $1/(2\pi)$ times the distributional Laplacian Δu_f). For each Borelian subset $E \subset \mathbb{C}$, $\mu_f(E) = \mathrm{meas}(f^{-1}E^*)$, where $E^* = \{z : z^{-1} \in E\}$, and $f^{-1}E^*$ is the full preimage of E under f. Now, we can express the RHS and the LHS of inequality (4.1) in terms of μ_f. First, observe that the counting function of μ_f equals

$$\mu_f(r) \overset{def}{=} \mu_f\{|z| \le r\} = \mathrm{meas}\{|f(t)| \ge r^{-1}\} = m_f(r^{-1}).$$

In order to write down m_g in terms of μ_f, we introduce the Levin-Tsuji counting function (cf. [20], [6]):

$$\begin{aligned}\mathfrak{n}_f(r) &= \mu_f\{|z - ir/2| \le r/2\} + \mu_f\{|z + ir/2| \le r/2\} \\ &= \mu_f\{|\mathrm{Im}(z^{-1})| \ge r^{-1}\} = \mathrm{meas}\{|g| \ge r^{-1}\} = m_g(r^{-1}).\end{aligned}$$

Now, we can rewrite (4.1) in the form:

$$\mu_f(r) \lesssim r \int_0^r \frac{\mathfrak{n}_f(t)}{t^2}\, dt + r^2 \int_r^\infty \frac{\mathfrak{n}_f(t)}{t^3}\, dt, \qquad 0 < r < \infty. \tag{4.2}$$

We shall show that (4.2) persists for any subharmonic function non-negative in $\mathbb{C}$ represented by a canonical integral of genus one. In this case the operator $g \mapsto \mathcal{H}g$ disappears, and the Marcinkiewicz argument seems to be unapplicable anymore.

Let

$$u(z) = \int_{\mathbb{C}} K(z/\zeta)\, d\mu(\zeta), \tag{4.3}$$

where $d\mu$ is a non-negative locally finite measure on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} \min\left(\frac{1}{|\zeta|}, \frac{1}{|\zeta|^2}\right) d\mu(\zeta) < \infty. \tag{4.4}$$

Subharmonic functions represented in this form are called *canonical integrals of genus one.*

Let $M(r, u) = \max_{|z|\le r} u(z)$. A standard estimate of the kernel

$$K(z) \lesssim \frac{|z|^2}{1+|z|}, \qquad z \in \mathbb{C},$$

yields Borel's estimate (cf. [6, Chapter II])

$$M(r, u) \lesssim r \int_0^r \frac{\mu(t)}{t^2}\, dt + r^2 \int_r^\infty \frac{\mu(t)}{t^3}\, dt.$$

In particular,

$$M(r, u) = \begin{cases} o(r), & r \to 0 \\ o(r^2), & r \to \infty. \end{cases}$$

THEOREM 4.5. *Let $u(z) \geq 0$ be a canonical integral (4.3) of genus one, then*

$$M(r,u) \lesssim r \int_0^r \frac{\mathfrak{n}(t)}{t^2}\,dt + r^2 \int_r^\infty \frac{\mathfrak{n}(t)}{t^3}\,dt. \tag{4.6}$$

The RHS of (4.6) does not depend on the bound for the integral (4.4), this makes the result not so obvious. By Jensen's formula, $\mu(r) \leq M(er,u)$, so that $\mu(r) \lesssim$ the RHS of (4.6). As a corollary we immediately obtain (4.2) and the Marcinkiewicz estimate (4.1).

5. Here we sketch the proof of Theorem 4.5.

We shall need two auxiliary lemmas. The first one is a version of the Levin integral formula without remainder term (cf. [10, Section IV.2], [6, Chapter 1]). The proof can be found in [13].

LEMMA 5.1. *Let v be a subharmonic function in $\mathbb{C}$ such that*

$$\int_0^{2\pi} |v(re^{i\theta})|\,|\sin\theta|\,d\theta = o(r), \qquad r \to 0, \tag{5.2}$$

and

$$\int_0 \frac{\mathfrak{n}(t) + v^-(t) + v^-(-t)}{t^2}\,dt < \infty. \tag{5.3}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} v(Re^{i\theta}|\sin\theta|)\frac{d\theta}{R\sin^2\theta} = \int_0^R \frac{\mathfrak{n}(t)}{t^2}\,dt, \qquad 0 < R < \infty, \tag{5.4}$$

where $\mathfrak{n}(t)$ is the Levin-Tsuji counting function, and the integral on the LHS is absolutely convergent.

The next lemma was proved in a slightly different setting in [11, §2], see also [6, Lemma 5.2, Chapter 6]

LEMMA 5.5. *Let $v(z)$ be a subharmonic function in $\mathbb{C}$ satisfying conditions (5.2) and (5.3) of the previous lemma, let*

$$T(r,v) = \frac{1}{2\pi}\int_0^{2\pi} v^+(re^{i\theta})\,d\theta$$

be its Nevanlinna characteristic function, and let

$$\mathfrak{T}(r,v) = \frac{1}{2\pi}\int_0^{2\pi} v^+(re^{i\theta}|\sin\theta)|)\frac{d\theta}{r\sin^2\theta}$$

be its Tsuji characteristic function. Then

$$\int_R^\infty \frac{T(r,v)}{r^3}\,dr \leq \int_R^\infty \frac{\mathfrak{T}(r,v)}{r^2}\,dr, \qquad 0 < R < \infty. \tag{5.6}$$

For the reader's convenience, we recall the proof. Consider the integral

$$I(R) = \frac{1}{2\pi}\iint_{\Omega_R} \frac{v^+(re^{i\theta})}{r^3}\,dr\,d\theta,$$

where $\Omega_R = \{z = re^{i\theta} : r > R|\sin\theta|\} = \{z : |z \pm iR/2| > R/2\}$. Introducing a new variable $\rho = r/|\sin\theta|$ instead of r, we get

$$I(R) = \int_R^\infty \frac{d\rho}{\rho^2}\left\{\frac{1}{2\pi}\int_0^{2\pi} v^+(\rho|\sin\theta|e^{i\theta})\frac{d\theta}{\rho\sin^2\theta}\right\} = \int_R^\infty \frac{\mathfrak{T}(\rho, v)}{\rho^2}\,d\rho.$$

Now, consider another integral

$$J(R) = \frac{1}{2\pi}\iint_{K_R} \frac{v^+(re^{i\theta})}{r^3}\,dr\,d\theta,$$

where $K_R = \{z : |z| > R\}$. Since $K_R \subset \Omega_R$, we have $J(R) \le I(R)$. Taking into account that

$$J(R) = \int_R^\infty \frac{dr}{r^3}\left\{\frac{1}{2\pi}\int_0^{2\pi} v^+(re^{i\theta})\,d\theta\right\} = \int_R^\infty \frac{T(r, v)}{r^3}\,dr$$

we obtain (5.6). ∎

Proof of Theorem 4.5. Due to Borel's estimate condition (5.2) is fulfilled. Due to non-negativity of u and (4.4), condition (5.3) holds as well. Using monotonicity of $T(r, u)$, Lemma 5.5, and then Lemma 5.1, we obtain

$$\begin{aligned}\frac{T(R,u)}{R^2} &\le 2\int_R^\infty \frac{T(r,u)}{r^3}\,dr \overset{(5.6)}{\le} 2\int_R^\infty \frac{\mathfrak{T}(r,u)}{r^2}\,dr \\ &\overset{(5.4)}{=} 2\int_R^\infty \frac{dr}{r^2}\int_0^r \frac{\mathfrak{n}(t)}{t^2}\,dt = \frac{2}{R}\int_0^R \frac{\mathfrak{n}(t)}{t^2}\,dt + 2\int_R^\infty \frac{\mathfrak{n}(t)}{t^3}\,dt.\end{aligned}$$

The inequality $M(r, u) \le 3T(2r, u)$ completes the job. ∎

6. Non-negativity of $u(z)$ in $\mathbb{C}$ seems to be a too strong assumption, a more natural one is non-negativity of $u(x)$ on $\mathbb{R}$.

THEOREM 6.1. *Let $u(z)$ be a canonical integral (4.3) of genus one, and let $u(x) \ge 0$, $x \in \mathbb{R}$. Then*

$$M(r, u) \lesssim r^2\left[\int_r^\infty \frac{\sqrt{\mathfrak{n}^*(t)}}{t^2}\,dt\right]^2, \tag{6.2}$$

where

$$\mathfrak{n}^*(r) = r\int_0^r \frac{\mathfrak{n}(t)}{t^2}\,dt + r^2\int_r^\infty \frac{\mathfrak{n}(t)}{t^3}\left(1 + \log\frac{t}{r}\right)dt. \tag{6.3}$$

The proof of Theorem 6.1 is given in [13]. The method of proof differs from that of Theorem 4.5, and is more technical than one would wish.

Fix an arbitrary $\epsilon > 0$. Then by the Cauchy inequality

$$\begin{aligned}\left[\int_r^\infty \frac{\sqrt{\mathfrak{n}^*(t)}}{t^2}dt\right]^2 &= \left[\int_r^\infty \frac{\sqrt{\left(1+\log^{1+\epsilon}\frac{t}{r}\right)\mathfrak{n}^*(t)}}{t^{3/2}}\,\frac{dt}{t^{1/2}\sqrt{1+\log^{1+\epsilon}\frac{t}{r}}}\right]^2 \\ &\lesssim_\epsilon \int_r^\infty \frac{\mathfrak{n}^*(t)}{t^3}\left(1+\log^{1+\epsilon}\frac{t}{r}\right)dt \\ &\lesssim_\epsilon \frac{1}{r}\int_0^r \frac{\mathfrak{n}(s)}{s^2}\,ds + \int_r^\infty \frac{\mathfrak{n}(s)}{s^3}\left(1+\log^{3+\epsilon}\frac{s}{r}\right)ds.\end{aligned}$$

Thus we get

Corollary 6.4. *For each $\epsilon > 0$,*

$$M(r,u) \lesssim_\epsilon r\int_0^r \frac{\mathfrak{n}(t)}{t^2}\,dt + r^2\int_r^\infty \frac{\mathfrak{n}(t)}{t^3}\left(1+\log^{3+\epsilon}\frac{t}{r}\right)dt. \tag{6.5}$$

Estimate (6.5) is slightly weaker than (4.6); however, it suffices for deriving inequalities of M. Riesz and Kolmogorov. Using Jensen's estimate $\mu(r) \le M(er,u)$, we arrive at

Corollary 6.6. *The following inequalities hold for canonical integrals of genus one which are non-negative on the real axis:*

- *M. Riesz-type estimate:*

$$\int_0^\infty \frac{\mu(r)}{r^{p+1}}\,dr \lesssim_p \int_0^\infty \frac{\mathfrak{n}(r)}{r^{p+1}}\,dr, \qquad 1<p<2, \tag{6.7}$$

- *weak (p,∞)-type estimate:*

$$\sup_{r\in(0,\infty)} \frac{\mu(r)}{r^p} \lesssim_p \sup_{r\in(0,\infty)} \frac{\mathfrak{n}(r)}{r^p}, \qquad 1<p<2, \tag{6.8}$$

- *Kolmogorov-type estimate:*

$$\sup_{r\in(0,\infty)} \frac{\mu(r)}{r} \lesssim \int_0^\infty \frac{\mathfrak{n}(r)}{r^2}\,dr. \tag{6.9}$$

Remark 6.10. If the integral on the RHS of (6.9) is finite, then $u(z)$ has positive harmonic majorants in the upper and lower half-planes which can be efficiently estimated near the origin and infinity, see [13, Theorem 3].

7. Here we mention several questions related to our results.

7.1. How to distinguish the logarithmic determinants (2.7) of $f = g + i\mathcal{H}g$ from other canonical integrals (4.3) which are non-negative in $\mathbb{C}$? In other words, let dm_f be a distribution measure of f; i.e. a locally-finite non-negative measure in $\mathbb{C}$ defined by $m_f(E) = \text{meas}\{t\in\mathbb{R} : f(t)\in E\}$ for an arbitrary borelian subset $E\subset\mathbb{C}$. It should be interesting to find properties of dm_f which do not follow only from non-negativity of the subharmonic function $u_f(z)$. A similar question can be addressed for analytic functions $f(z)$ of Smirnov's class in the unit disk.

7.2. Let X be a rearrangement invariant Banach space of measurable functions on $\mathbb{R}$. That is, the norm in X is the same for all rearrangements of $|g|$, and $||g_1||_X \le ||g_2||_X$ provided that $|g_1|\le|g_2|$ everywhere. For which spaces does the inequality

$$||\mathcal{H}g_d||_X \le C_X||\mathcal{H}g||_X$$

hold? This question is interesting only for spaces X where the Hilbert transform is unbounded; i.e. for spaces which are close in a certain sense either to L^1 or to L^∞. Some natural restrictions on X can be assumed: the linear span of the characteristic functions χ_E of bounded measurable subsets E is dense in X, and $||\chi_E||_X \to 0$, when meas(E) $\to 0$, see [2, Chapter 3].

7.3. We do not know how to extend estimate (1.2) (as well as (1.8)) to more general operators like the maximal Hilbert transform, the non-tangential maximal conjugate harmonic function, or Calderón-Zygmund operators. A similar question can be naturally posed for the Riesz transform [17].

7.4. Does Marcinkiewicz-type inequality (4.6) hold under the assumption that a canonical integral u of genus one is non-negative on $\mathbb{R}$? According to a personal communication from A. Ph. Grishin, the exponent $3+\epsilon$ can be improved in (6.5). However, his technique also does not allow to get rid at all of the logarithmic factor.

7.5. Let $u(z)$ be a non-negative subharmonic function in $\mathbb{C}$, $u(0)=0$. As before, by $\mu(r)$ and $\mathfrak{n}(r)$ we denote the conventional and the Levin-Tsuji counting functions of the Riesz measure $d\mu$ of u. Assume that $\mu(r)=o(r)$, $r\to 0$. This condition is needed to exclude from consideration the function $u(z)=|\mathrm{Im}(z)|$ which is non-negative in $\mathbb{C}$ and harmonic outside of $\mathbb{R}$. Let $\mathcal{M}$, $\mathcal{M}(0)=0$, $\mathcal{M}(\infty)=\infty$, be a (regularly growing) majorant for $\mathfrak{n}(r)$. What can be said about the majorant for $\mu(r)$? If $\mathcal{M}(r)=r^p$, $1<p<\infty$, then we know the answer:

$$\sup_{r\in(0,\infty)}\frac{\mu(r)}{r^p}\le C_p\sup_{r\in(0,\infty)}\frac{\mathfrak{n}(r)}{r^p},$$

and

$$\int_0^\infty\frac{\mu(r)}{r^{p+1}}\,dr\le C_p\int_0^\infty\frac{\mathfrak{n}(r)}{r^{p+1}}\,dr.$$

It is more difficult and interesting to study majorants $\mathcal{M}(r)$ which grow faster than any power of r when $r\to\infty$, and decay to zero faster than any power of r when $r\to 0$. The question might be related to the classical Carleman-Levinson-Sjoberg "log log-theorem", and the progress may lead to new results about the Hilbert transform.

References

[1] A. Baernstein, *Some sharp inequalities for conjugate functions*, Indiana Univ. Math. J. 27 (1978), 833–852.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Appl. Math. 129, Academic Press, 1988.

[3] B. Davis, *Hardy spaces and rearrangements*, Trans. Amer. Math. Soc. 261 (1980), 211–233.

[4] M. Essén, *Some best constants inequalities for conjugate functions*, Internat. Ser. Numer. Math. 103, Birkhäuser, Basel, 1992.

[5] I. Ts. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space*, Transl. Math. Monographs 18, Amer. Math. Soc., Providence, RI, 1969.

[6] A. A. Goldberg and I. V. Ostrovskii, *Value Distribution of Meromorphic Functions*, Nauka, Moscow, 1970 (in Russian).

[7] P. Koosis, *Introduction to H_p Spaces*, 2nd ed., Cambridge Univ. Press, 1998.

[8] I. Klemes, *Finite Toeplitz matrices and sharp Littlewood conjectures*, Algebra i Analiz 13 (2001), 39–59.

[9] B. Ya. Levin, *On functions holomorphic in a half-plane*, Travaux de l'Université d'Odessa (Math.) 3 (1941), 5–14 (in Russian).
[10] B. Ya. Levin, *Distribution of zeros of entire functions*, Transl. Math. Monographs 5, AMS, Providence, RI, 1980.
[11] B. Ya. Levin and I. V. Ostrovskii, *The dependence of the growth of an entire function on the distribution of the zeros of its derivatives*, Sibirsk. Mat. Zh. 1 (1960), 427–455 (in Russian); English transl.: Amer. Math. Soc. Transl. (2) 32 (1963), 323–357.
[12] J. Marcinkiewicz, *Sur l'interpolation d'opérations*, C. R. Acad. Sci. Paris 208 (1939), 1272–1273.
[13] V. Matsaev, I. Ostrovskii and M. Sodin, *Variations on the theme of Marcinkiewicz' inequality*, J. Analyse Math. 86 (2002), 289–317.
[14] V. Matsaev and M. Sodin, *Variations on the theme of M. Riesz and Kolmogorov*, Int. Math. Res. Notices 6 (1999), 287–297.
[15] V. Matsaev and M. Sodin, *Distribution of Hilbert transforms of measures*, Geom. Funct. Anal. 10 (2000), 160–184.
[16] V. Matsaev and M. Sodin, *Compact operators with S_p-imaginary component and entire functions*, in: Entire Functions in Modern Analysis (Tel Aviv, 1997), Proc. Israel Math. Conf. 15, Bar Ilan Univ., Ramat Gan, 2001, 243–260.
[17] E. Stein, *Harmonic Analysis*, Princeton Univ. Press, Princeton, NJ, 1993.
[18] E. C. Titchmarsh, *On conjugate functions*, Proc. London Math. Soc. (2) 29 (1929), 49–80.
[19] O. Tsereteli, *A metric characterization of the set of functions whose conjugate functions are integrable*, Bull. Acad. Sci. Georgian SSR 81 (1976), 281–283 (in Russian).
[20] M. Tsuji, *On Borel's directions of meromorphic functions of finite order*, Tôhoku Math. J. 2 (1950), 97–112.
[21] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operators*, J. Math. Pures Appl. 35 (1956), 223–248.