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Strong homology, derived limits, and set theory

by

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Abstract. We consider the question of the additivity of strong homology. This entails isolating the set-theoretic content of the higher derived limits of an inverse system indexed by the functions from $\mathbb{N}$ to $\mathbb{N}$. We show that this system governs, at a certain level, the additivity of strong homology over sums of arbitrary cardinality. We show in addition that, under the assumption of the Proper Forcing Axiom, strong homology is not additive, not even on closed subspaces of $\mathbb{R}^4$.

1. Introduction. The strong homology theory $\overline{H}_*$, defined for all topological spaces, has the following desirable properties:

1. It satisfies all the Eilenberg–Steenrod axioms on paracompact pairs (X, A).
2. It is strong shape invariant.
3. It is a Steenrod-type homology theory (and therefore Alexander dual to $\check{H}^*$); it satisfies, in other words, two of the three axioms Milnor proposed to supplement Eilenberg and Steenrod's ([Mi1], [Mi2]; see [Ma]).

It remains an open question on what class of spaces it may satisfy the third of those axioms, *additivity*, the condition that every mapping

$$i : \bigoplus_A \overline{H}_p(X_\alpha) \to \overline{H}_p\Big(\coprod_A X_\alpha\Big)$$

induced by the inclusion maps $i_\alpha : X_\alpha \hookrightarrow \coprod_A X_\alpha$ be an isomorphism.

It was in investigation of this question that Mardešić and Prasolov computed the strong homology of $Y^{(k)}$, the topological sum of countably many k-dimensional Hawaiian earrings. They showed, in particular, that $\overline{H}_p(Y^{(k)}) = \lim^{k-p} \mathbf{A}$ for $0 < p < k$, where $\mathbf{A}$ is an abelian pro-group indexed by $\mathbb{N}^\mathbb{N}$ (see below). For a single k-dimensional Hawaiian earring $X^{(k)}$, $\overline{H}_p(X^{(k)}) = 0$ for

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$0 < p < k$; thus additivity requires at least that $\lim^n \mathbf{A} = 0$ for $n > 0$. Mardešić and Prasolov then showed [MP] that the continuum hypothesis implies that $\lim^1 \mathbf{A} \neq 0$. Shortly thereafter, Dow, Simon, and Vaughan showed [Do] that the Proper Forcing Axiom (PFA) implies that $\lim^1 \mathbf{A} = 0$ and, hence, that the vanishing of $\lim^1 \mathbf{A}$ is independent of the axioms of ZFC. This vanishing, in fact, is a question of broad interest in its own right; see [To1], for instance, and the discussion therein.

It is the purpose of this note to extend those investigations. In Sections 3 and 4, we show the vanishing of $\lim^2 \mathbf{A}$ also to be independent of the axioms of ZFC and characterize, more generally, the higher $\lim^n \mathbf{A}$. In particular, we prove that, under PFA, strong homology is not additive, not even on the category of, e.g., closed subspaces of $\mathbb{R}^4$ (our witness in this case is $\overline{H}_1(Y^{(3)})$)). In Section 5, for κ infinite, we let $\mathbf{A}_\kappa$ denote a pro-group analogous to $\mathbf{A}$ but indexed by $\mathbb{N}^\kappa$; we show $\lim^1 \mathbf{A}_\kappa = 0$ if and only if $\lim^1 \mathbf{A} = 0$. Extending, as it does, the topological significance of the system $\mathbf{A}$, this is the main theorem of the paper. In Section 6, we list some open problems.

In Section 2, we define our notation, the system $\mathbf{A}$, and briefly review the derived functors $\lim^n$ of lim. This paper aims to interest readers in both homological algebra and set theory, and therefore—with a few mild exceptions in Section 4—assumes no more than a basic knowledge of either. In particular, no knowledge of forcing is presumed; the reader need only understand that the Proper Forcing Axiom, $\diamondsuit(S^2_1)$, $\mathrm{MA}_{\aleph_1}$, wKH, and $\mathfrak{d} = \aleph_1$ (or $\aleph_2$) are prominent set-theoretic hypotheses independent of the axioms of ZFC. For more on the Proper Forcing Axiom, see in particular [Mo]. For more on set theory generally, see [Je] or [Ku]. For further information on lim and its derived functors, see [Ma, §11] and [Jen].

2. Background and notation. Our inverse systems all will consist of abelian groups X_f and "bonding" homomorphisms $p_{fg} : X_g \to X_f$ for all $f \le g$. Our index set will typically be $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, ordered coordinatewise: $f \le g$ if and only if $f(i) \le g(i)$ for all $i \in \mathbb{N}$. Our particular focus is $\mathbf{A} = (A_f, p_{fg}, \mathcal{N})$, where

$$A_f = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}^{f(i)}$$

with projection mappings p_{fg}. Relatedly, $\mathbf{B} = (B_f, p_{fg}, \mathcal{N})$, where

$$B_f = \prod_{i\in\mathbb{N}} \mathbb{Z}^{f(i)}.$$

We consider only level morphisms $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$ among such systems, i.e., collections of functions $F_f : X_f \to Y_f$ which commute with all the bonding maps. Likewise, terms of the quotient $\mathbf{Y}/\mathbf{X}$ are of the form Y_f/X_f, so that

$$0 \to \mathbf{A} \xrightarrow{\mathbf{I}} \mathbf{B} \xrightarrow{\mathbf{Q}} \mathbf{B}/\mathbf{A} \to 0, \tag{2.1}$$

for example, is exact. In the language of category theory, we study the abelian category $\mathcal{A}b^{\mathcal{N}}$.

An abelian group X together with $\mathbf{p} = \{p_f : X \to X_f \mid f \in \mathcal{N}\}$ is an inverse limit of $\mathbf{X}$ if

$$p_f = p_{fg} p_g \quad \text{for all } f \leq g \in \mathcal{N}, \tag{2.2}$$

and for any $(Y, \mathbf{q})$ satisfying (2.2) there exists a unique $q : Y \to X$ such that $\mathbf{p}q = \mathbf{q}$. Such an X and $\mathbf{p}$ are unique up to isomorphism; we henceforth write $X = \lim \mathbf{X}$ for the group alone. X admits the following description:

$$\lim \mathbf{X} = \Big\{ \langle x_f \rangle \in \prod_{f \in \mathcal{N}} X_f \;\Big|\; p_{fg}(x_g) = x_f \text{ for all } f \leq g \in \mathcal{N} \Big\}. \tag{2.3}$$

Note that

$$\lim \mathbf{B} \cong \prod_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} \mathbb{Z}, \qquad \lim \mathbf{A} \cong \bigoplus_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} \mathbb{Z}.$$

Returning to (2.3), for $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$, define $\lim \mathbf{F} : \lim \mathbf{X} \to \lim \mathbf{Y}$ as the induced mapping of products. We define thereby a functor $\lim : \mathcal{A}b^{\mathcal{N}} \to \mathcal{A}b$. We are interested in the following phenomenon: lim applied to a sequence (2.1), for example, may fail to be exact. More precisely, lim is left exact: $\lim \mathbf{I}$ will be injective, but $\lim \mathbf{Q}$ may fail to be surjective, to a degree the long exact sequence

$$0 \to \lim \mathbf{A} \xrightarrow{\lim \mathbf{I}} \lim \mathbf{B} \xrightarrow{\lim \mathbf{Q}} \lim \mathbf{B}/\mathbf{A} \xrightarrow{\theta_0} \lim^1 \mathbf{A} \xrightarrow{\lim^1 \mathbf{I}} \lim^1 \mathbf{B} \to \cdots \tag{2.4}$$

in some sense measures. The nonexactness of lim induces, in other words, a sequence of derived functors $\lim^n$ connected, for any short exact sequence in $\mathcal{A}b^{\mathcal{N}}$, by a long exact sequence of abelian groups of the above form, with connecting transformations θ_n. These functors $\lim^n$, like lim, admit explicit description; see the proof of Theorem 3.4 below. From this description, the reader may verify the following:

(i) For any constant system $\mathbf{X} = (X_f, p_{fg}, \mathcal{N})$, i.e., any system with $X_f = X$ and $p_{fg} = \mathrm{id}$ for all $f \leq g \in \mathcal{N}$, $\lim^n \mathbf{X} = 0$ for $n \geq 1$.
(ii) $\lim^n \mathbf{B} = 0$ for $n \geq 1$.

Returning to (2.4): by (ii), $\lim^1 \mathbf{A} = 0$ if and only if $\lim \mathbf{Q}$ is surjective. To better articulate that equivalence, we introduce the following conventions, basic to all that follows:

For $f \in \mathcal{N}$, let $I_f = \{(i,j) \mid j \leq f(i)\}$. For $f, g \in \mathcal{N}$ write $f <^* g$ if $\{i \mid f(i) \not< g(i)\}$ is finite. Write $f \leq^* g$ if $\{i \mid f(i) \not\leq g(i)\}$ is finite or, equivalently, if $I_f \subseteq^* I_g$. Further, ϕ_f will denote a function of the form

$I_f \to \mathbb{Z}$. Write $\phi =^\star \psi$ if $\{x \in \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi) \mid \phi(x) \neq \psi(x)\}$ is finite; note that this is not, in general, an equivalence relation. A collection $\Phi = \{\phi_f \mid f \in \mathcal{N}\}$ is *coherent* if $\phi_f =^\star \phi_g$ for all $f, g \in \mathcal{N}$; and Φ is *trivial* if there exists $\phi : \mathbb{N}^2 \to \mathbb{Z}$ such that $\phi =^\star \phi_f$ for all $f \in \mathcal{N}$. We may view any ϕ_f as an element of B_f; write $[\phi_f]$ for its image in B_f/A_f. We may view Φ, likewise, as an element of $\prod_{\mathcal{N}} B_f$; writing $[\Phi]$ for $\{[\phi_f] \mid f \in \mathcal{N}\}$, we have

$$\lim \mathbf{B}/\mathbf{A} \cong \{[\Phi] \mid \Phi \text{ is coherent}\}.$$

Hence $\lim \mathbf{Q}$ is surjective if and only if every coherent $[\Phi]$ equals $\{[\phi\restriction_{I_f}] \mid f \in \mathcal{N}\}$ for some $\phi : \mathbb{N}^2 \to \mathbb{Z}$ in $\lim \mathbf{B}$. In other words:

Theorem 2.1 ([MP]). $\lim^1 \mathbf{A} = 0$ *if and only if every coherent family* $\Phi = \{\phi_f \mid f \in \mathcal{N}\}$ *of functions is trivial.*

It is this observation we generalize in Section 3.

We recall, finally, the following notions from set theory. The *cofinality* of a partial order P is

$$\mathrm{cf}(P) = \min\{|Q| \mid \text{for all } p \in P \text{ there exists a } q \in Q \text{ with } q \geq p\}.$$

The cofinality of an inverse system is the cofinality of its index set. Observe that $\mathrm{cf}(\mathcal{N}, <) = \mathrm{cf}(\mathcal{N}, <^*)$. We write $\mathfrak{d}$ for either. We set

$$\mathfrak{b} = \min\{|\mathcal{F}| \mid \text{for all } g \in \mathcal{N} \text{ there exists an } f \in \mathcal{F} \text{ with } f \not<^* g\}.$$

Symbols α, β, ξ denote ordinals; κ denotes a cardinal; and $[\kappa]^{<\kappa} = \{y \subset \kappa \mid \kappa > |y|\}$. For $A \subseteq \mathrm{dom}(f)$, we write $f''A = \{f(a) \mid a \in A\}$. A cofinal subset C of β is *club* if it is closed in β under the topology induced by the membership relation. We say $S \subseteq \beta$ is *stationary* if it intersects all club subsets of β.

Definition 2.2. $\diamondsuit(S^2_1)$ is the assertion that there exists a family $\mathcal{S} = \{S_\beta \mid \beta < \omega_2 \text{ and } \mathrm{cof}(\beta) = \aleph_1\}$ such that, for any $A \subset \omega_2$, the set $\{\beta \mid A \cap \beta = S_\beta\}$ is stationary.

The reader may verify that the intersection of two club subsets of β is a club and, hence, that the intersection of a club and a stationary subset of β is stationary; these facts and the straightforward implication $\diamondsuit(S^2_1) \Rightarrow 2^{\aleph_0} \leq \aleph_2$ play a role in the proof of Theorem 4.1.

3. Characterizing higher derived limits of A.

Let

$$\mathcal{N}^{[n]} = \{(f_0, \ldots, f_{n-1}) \in \mathcal{N}^n \mid f_i \leq f_j \text{ for all } i < j\}$$

for $n > 0$, and let $\mathcal{N}^{[0]} = \{\emptyset\}$. Let $\vec{f}^{\,i}$ denote the $(n-1)$-tuple obtained by deleting f_i from $\vec{f} \in \mathcal{N}^{[n]}$; and $\phi_{\vec{f}}$ will denote a function of the form $I_{f_0} \to \mathbb{Z}$ unless $\vec{f} = \emptyset$, in which case $\phi_{\vec{f}} : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$.

DEFINITION 3.1. A collection $\Phi = \{\phi_{\vec{f}} \mid \vec{f} \in \mathcal{N}^{[n]}\}$ is *n-coherent* if, for all $\vec{f} \in \mathcal{N}^{[n+1]}$,

$$\phi_{\vec{f}^0} \restriction_{I_{f_0}} + \sum_{i=1}^{n} (-1)^i \phi_{\vec{f}^i} =^\star 0.$$

For readability, we henceforth write such sums, simply, as $\sum_{i=0}^{n} (-1)^i \phi_{\vec{f}^i}$.

DEFINITION 3.2. We say Φ is *n-trivial* if there exists $\{\psi_{\vec{f}} \mid \vec{f} \in \mathcal{N}^{[n-1]}\}$ such that, for all $\vec{f} \in \mathcal{N}^{[n]}$,

$$\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{f}^i} =^\star \phi_{\vec{f}}.$$

Observe that every n-trivial Φ is n-coherent. The following instance will play a role in the proof of Theorem 4.1.

EXAMPLE 3.3. If $\Phi_1 = \{\phi_f \mid f \in \mathcal{N}\}$ 2-trivializes $\Phi_2 = \{\phi_{fg} \mid fg \in \mathcal{N}^{[2]}\}$, then for all $(f, g, h) \in \mathcal{N}^{[3]}$,

$$\phi_{fg} - \phi_{fh} + \phi_{gh} =^\star \phi_g - \phi_f - (\phi_h - \phi_f) + \phi_h - \phi_g =^\star 0,$$

i.e., Φ_2 is 2-coherent. (Here again for readability we have suppressed the obvious restrictions.)

The question of whether every n-coherent Φ is n-trivial is subtler.

THEOREM 3.4. *For $n > 0$, $\lim^n \mathbf{A} = 0$ if and only if every n-coherent $\Phi = \{\phi_{\vec{f}} \mid \vec{f} \in \mathcal{N}^{[n]}\}$ is n-trivial.*

Proof. Define a cochain complex $\mathcal{K}(\mathbf{B}) : 0 \to K^0(\mathbf{B}) \to K^1(\mathbf{B}) \to \cdots$ by

$$K^n(\mathbf{B}) = \prod_{\vec{f} \in \mathcal{N}^{[n+1]}} B_{f_0}$$

with differential $d^n : K^{n-1}(\mathbf{B}) \to K^n(\mathbf{B})$ defined, for $n > 0$, by

$$(d^n \Psi)(\vec{f}) = \sum_{i=0}^{n} (-1)^i \Psi(\vec{f}^i).$$

Analogously define $\mathcal{K}(\mathbf{A})$, a subcomplex of $\mathcal{K}(\mathbf{B})$. View any Φ as in the statement of the theorem as an element of $K^{n-1}(\mathbf{B})$; observe that Φ is n-coherent if and only if $d^n\Phi \in K^n(\mathbf{A})$, and is n-trivial if and only if $\Phi - d^{n-1}\Psi \in K^{n-1}(\mathbf{A})$ for some $\Psi \in K^{n-2}(\mathbf{B})$.

Nöbeling and Roos independently established that, in general, $\lim^n \mathbf{X} \cong H^n\mathcal{K}(\mathbf{X})$ (see [Ma, §11.5] for proof; the reader may more immediately verify that $H^0(\mathcal{K}(\mathbf{X})) = \lim \mathbf{X}$). In particular, $\lim^n \mathbf{A} = 0$ if and only if, in $\mathcal{K}(\mathbf{A})$, every n-cocycle is an n-coboundary. Assume the latter, and take $n \geq 2$ (the case $n = 1$ was Theorem 2.1); if Φ is n-coherent, then $d^n\Phi \in K^n(\mathbf{A})$ is an

n-cocycle and hence, by assumption, equals $d^n\Upsilon$ for some $\Upsilon \in K^{n-1}(\mathbf{A})$. Since $\lim^{n-1} \mathbf{B} = 0$, the cocycle $\Phi - \Upsilon$ equals $d^{n-1}\Psi$ for some $\Psi \in K^{n-2}(\mathbf{B})$; in other words, $\Phi - d^{n-1}\Psi \in K^{n-1}(\mathbf{A})$, i.e., Φ is n-trivial.

On the other hand, if every n-coherent Φ is n-trivial, take an n-cocycle $\Upsilon \in K^n(\mathbf{A})$. Since $\lim^{n-1} \mathbf{B} = 0$, $\Upsilon = d^n\Phi$ for some $\Phi \in K^{n-1}(\mathbf{B})$. As Φ is n-coherent, by assumption $\Phi - d^{n-1}\Psi \in K^{n-1}(\mathbf{A})$ for some $\Psi \in K^{n-2}(\mathbf{B})$; hence $\Upsilon = d^n(\Phi - d^{n-1}\Psi)$ is an n-coboundary in $\mathcal{K}(\mathbf{A})$. ■

We will sometimes consider systems indexed by orders extending or contained in $\mathcal{N}$; the appropriate modification of definitions should in such cases be obvious.

Early indications of the relevance of set-theoretic considerations to higher derived limits were the following:

THEOREM 3.5 ([Go]). *For any inverse system* $\mathbf{X}$ *of cofinality* $\aleph_k$, $\lim^n \mathbf{X} = 0$ *for all* $n \geq k+2$.

THEOREM 3.6 ([Mit]). *For every* $k \geq 0$ *there exists an inverse system* $\mathbf{X}$ *of cofinality* $\aleph_k$ *with* $\lim^{k+1} \mathbf{X} \neq 0$.

COROLLARY 3.7. *If* $\mathfrak{d} = \aleph_k$, *then* $\lim^n \mathbf{A} = 0$ *for all* $n \geq k+1$.

Proof. Immediate, by Theorem 3.5, for $n > k+1$. Let $\mathbf{D} = (D_f, p^d_{fg}, \mathcal{N})$, with $D_f = \{\phi : \mathbb{N}^2 \setminus I_f \to \mathbb{Z} \mid \operatorname{supp}(\phi) \text{ is finite}\}$ and p^d_{fg} the inclusion map; let $\mathbf{E} = (E_f, p^e_{fg}, \mathcal{N})$, with $E_f = \{\phi : \mathbb{N}^2 \to \mathbb{Z} \mid \operatorname{supp}(\phi) \text{ is finite}\}$ and p^e_{fg} the identity. Form the short exact sequence

$$0 \to \mathbf{D} \to \mathbf{E} \to \mathbf{A} \to 0$$

inducing a long exact sequence

$$\cdots \to \lim^{k+1} \mathbf{E} \to \lim^{k+1} \mathbf{A} \to \lim^{k+2} \mathbf{D} \to \cdots.$$

As noted, for $k \geq 0$, $\lim^{k+1} \mathbf{E} = 0$, so by Theorem 3.5, $0 = \lim^{k+2} \mathbf{D} = \lim^{k+1} \mathbf{A}$. ■

By the corollary, together with the following theorem, $\mathfrak{d} = \aleph_1$ fully determines when $\lim^n \mathbf{A} = 0$.

THEOREM 3.8 ([Do]). *If* $\mathfrak{d} = \aleph_1$ *then* $\lim^1 \mathbf{A} \neq 0$.

4. PFA and $\lim^2 \mathbf{A}$**.** By the following theorem, PFA fully determines when $\lim^n \mathbf{A} = 0$ as well—but in a different direction.

THEOREM 4.1. *If* $\mathfrak{b} = \mathfrak{d} = \aleph_2$ *and* $\diamondsuit(S^2_1)$ *then* $\lim^2 \mathbf{A} \neq 0$.

COROLLARY 4.2. *Assuming the Proper Forcing Axiom,* $\lim^n \mathbf{A} \neq 0$ *if and only if* $n = 2$.

Proof. Among the consequences of PFA:

1. $\mathfrak{d} = \aleph_2$ ([Ve], [Be]). So by Corollary 3.7, $\lim^n \mathbf{A} = 0$ for $n > 2$.
2. $\lim^1 \mathbf{A} = 0$ ([Do]). This and $\mathfrak{b} = \aleph_2$ follow in fact from a strictly weaker assumption, the Open Coloring Axiom, a consequence of PFA ([To2]).
3. $\diamondsuit(S^2_1)$ ([Ba], [Ve]).

Theorem 4.1 then completes the proof. ∎

The condition $\mathfrak{b} = \mathfrak{d} = \aleph_2$ is equivalent to the existence of an ω_2-scale.

DEFINITION 4.3. A γ-*chain* in $\mathcal{N}$ is a collection $\{f_\alpha \mid \alpha < \gamma\} \subset \mathcal{N}$ such that $\alpha < \beta$ implies $f_\alpha <^* f_\beta$. A γ-*scale* is a γ-chain which is $<^*$-cofinal in $\mathcal{N}$.

Theorem 3.8 is perhaps better understood as a ZFC phenomenon:

THEOREM 4.4. *For any ω_1-chain $\mathcal{F}$ in $\mathcal{N}$, there exists a nontrivial coherent $\Phi^{\mathcal{F}} = \{\phi_f \mid f \in \mathcal{F}\}$.*

In other words, $\lim^1 \mathbf{A}^{\mathcal{F}} \neq 0$, where $\mathbf{A}^{\mathcal{F}} = (A_f, p_{fg}, \mathcal{F})$. Theorem 4.4 is simply a recasting of [Be, pp. 96–98], which inscribes a gap in any $\subset^*$-increasing ω_1-chain of subsets of $\mathbb{N}$. Let $\mathcal{F}^* = \{g \in \mathcal{N} \mid g \leq^* f$ for some $f \in \mathcal{F}\}$; write $\Phi^{\mathcal{F}}$ for $\{\phi_f \mid f \in \mathcal{F}\}$, as above. Any coherent $\Phi^{\mathcal{F}}$ extends to a coherent $\Phi^{\mathcal{F}^*}$, so the theorem gives a nontrivial coherent $\Phi^{\mathcal{G}}$ for any $\mathcal{G} \subseteq \mathcal{N}$ of cofinality $\aleph_1$ in the $<^*$-ordering. Such $\Phi^{\mathcal{G}}$ admit no "upward" extensions:

OBSERVATION 1. *For any h with $g \leq^* h$ for all $g \in \mathcal{G}$, no nontrivial coherent $\Phi^{\mathcal{G}}$ extends to a coherent $\Phi^{\mathcal{G} \cup \{h\}}$.*

For if it did, then any $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ extending ϕ_h would trivialize $\Phi^{\mathcal{G}}$, a contradiction. This is one key to the proof below. The other is the following:

OBSERVATION 2. *If $\Phi_1 = \{\phi_f \mid f \in \mathcal{N}\}$ and $\Upsilon_1 = \{\upsilon_f \mid f \in \mathcal{N}\}$ 2-trivialize the same $\Phi_2 = \{\phi_{fg} \mid (f, g) \in \mathcal{N}^{[2]}\}$, then they differ by a 1-coherent $\Psi_1 = \{\psi_f \mid f \in \mathcal{N}\}$.*

For, letting $\psi_f = \phi_f - \upsilon_f$, we have

$$\psi_g - \psi_f = (\phi_g - \upsilon_g) - (\phi_f - \upsilon_f) = (\phi_g - \phi_f) - (\upsilon_g - \upsilon_f) =^\star \phi_{fg} - \phi_{fg} = 0$$

for all $f \leq g$. Hence Ψ_1 is 1-coherent.

Proof of Theorem 4.1. Fix an ω_2-scale $\mathcal{F} = \{f_\alpha \mid \alpha < \omega_2\}$ and an $\mathcal{S}$ witnessing $\diamondsuit(S^2_1)$. Let $\mathcal{F}_\beta = \{f \in \mathcal{N} \mid f \leq^* f_\alpha$ for some $\alpha < \beta\}$. Let $Y_\beta = \bigcup_{f \in \mathcal{F}_\beta} \mathbb{Z}^{I_f}$ and $Y = \bigcup_{\beta < \omega_2} Y_\beta$, and fix a bijection $\rho : \omega_2 \to Y$. For any $\Phi_1 = \{\phi_f \mid f \in \mathcal{N}\}$ those β for which $\Phi_1 \cap Y_\beta \subseteq \rho''\beta$ form a club subset of ω_2, so the set $S(\Phi_1) = \{\beta \in S^2_1 \mid \rho'' S_\beta = \Phi_1 \cap Y_\beta\}$ is stationary.

We show $\lim^2 \mathbf{A} \neq 0$ by constructing, in stages Φ_2^β ($\beta < \omega_2$), a non-2-trivial 2-coherent Φ_2: each Φ_2^β will be of the form

$$\{\phi_{fg} \mid f \leq g \leq^* f_\alpha \text{ for some } \alpha < \beta\}$$

and Φ_2 will be their union. By the argument of Corollary 3.7, $\lim^2 \mathbf{A}^{\mathcal{F}_\beta} = 0$ for every $\beta < \omega_2$, so every 2-coherent Φ_2^β is 2-trivial, and therefore extends to some 2-trivial (hence 2-coherent) $\Phi_2^{\beta+1}$. For limit β, let $\Phi_2^\beta = \bigcup_{\gamma<\beta} \Phi_2^\gamma$. At stage β, if $\rho'' S_\beta$ is of the form $\{\phi_f \mid f \in \mathcal{F}_\beta\}$ and 2-trivializes Φ_2^β, we extend with greater care. Since $\operatorname{cf}(\beta) = \aleph_1$, by Theorem 4.4 there exists a nontrivial coherent family $\Psi_1^{\mathcal{F}_\beta}$. Take any extension $\Upsilon_1^{\beta+1} = \{\upsilon_f : f \in \mathcal{F}_{\beta+1}\}$ of $\Upsilon_\beta = \rho'' S_\beta + \Psi_1^{\mathcal{F}_\beta}$. Letting $\phi_{fg} = \upsilon_g\restriction_{I_f} - \upsilon_f$ for any $g \in \mathcal{F}_{\beta+1} \backslash \mathcal{F}_\beta$ defines a 2-coherent extension $\Phi_2^{\beta+1}$ of Φ_2^β which is 2-trivialized by $\Upsilon_1^{\beta+1}$.

Clearly Φ_2 is 2-coherent. Suppose for contradiction that Φ_1 2-trivializes Φ_2. Then for $\beta \in S(\Phi_1)$, $\Phi_1 \cap Y_\beta$ and $\Phi_1 \cap Y_{\beta+1}$ 2-trivialize Φ_2^β and $\Phi_2^{\beta+1}$, respectively. By construction, $\Upsilon_1^{\beta+1}$ also 2-trivializes $\Phi_2^{\beta+1}$. By Observation 2, then, $\Upsilon_1^{\beta+1} - (\Phi_1 \cap Y_{\beta+1})$ is a coherent family extending the nontrivial coherent family $\Upsilon_1^\beta - (\Phi_1 \cap Y_\beta) = \Psi_1^{\mathcal{F}_\beta}$, contradicting Observation 1. ■

Beginning from a model of PFA, Todorcevic forced $\lim^1 \mathbf{A} \neq 0$ while preserving $\mathrm{MA}_{\aleph_1}$. The forcing in question is ω_2-distributive; in consequence, $\neg$wKH, and hence $\diamondsuit(S_1^2)$, are preserved (see [To1], [Ba]). Therefore:

Theorem 4.5. *Under the assumption of the Proper Forcing Axiom, $MA_{\aleph_1}$ is consistent with "$\lim^n \mathbf{A} \neq 0$ if and only if $n \leq 2$".*

Remark. The large cardinal consistency strength of PFA is not really needed here. More precisely, it is unneeded in Todorcevic's construction. One may then follow his forcing (over a model of $\mathfrak{b} = \mathfrak{d} = \aleph_2$) with the usual forcing for $\diamondsuit(S_1^2)$: the latter, being ω_2-closed, adds no reals, hence preserves $\lim^1 \mathbf{A} \neq 0$, and clearly $\mathrm{MA}_{\aleph_1}$ as well.

5. Relating $\lim^1 \mathbf{A}$ to $\lim^1 \mathbf{A}_\kappa$. Let $\mathbf{A}_\kappa = (A_f, p_{fg}, \omega^\kappa)$, where $A_f = \bigoplus_{\alpha<\kappa} \mathbb{Z}^{f(\alpha)} = \{\phi_f : I_f \to \mathbb{Z} \mid \operatorname{supp}(f) \text{ is finite}\}$ and $p_{fg} : \phi_f \mapsto \phi_f\restriction_{I_g}$, as before. Then $\mathbf{A}_\kappa$ generalizes $\mathbf{A}$ both in form ($\mathbf{A}_\omega = \mathbf{A}$) and in content: $\bar{H}_p(Y^{(k)}) = \lim^{k-p} \mathbf{A}_\kappa$ for $Y^{(k)}$ the disjoint union of κ many k-dimensional Hawaiian earrings when $0 < p < k$. We show the following relation:

Theorem 5.1. $\lim^1 \mathbf{A} = 0$ *if and only if* $\lim^1 \mathbf{A}_\kappa = 0$ *for all infinite* κ.

Proof. If we replace $\mathcal{N}$ with ω^κ in Definitions 3.1 and 3.2, the arguments of Theorem 3.4 apply equally to $\lim^n \mathbf{A}_\kappa$, so one direction of the theorem is clear: if every n-coherent family $\{\phi_{\vec{f}} \mid \vec{f} \in (\omega^\kappa)^{[n]}\}$ is n-trivial, so too must be every n-coherent family $\{\phi_{\vec{f}} \mid \vec{f} \in (\omega^\omega)^{[n]}\}$. In other words:

OBSERVATION 3. *For* $n > 0$, $\lim^n \mathbf{A}_\kappa = 0$ *implies* $\lim^n \mathbf{A} = 0$.

For the other direction of Theorem 5.1, we assume $\lim^1 \mathbf{A} = 0$, fix a coherent family $\Phi = \{\phi_f \mid f \in \omega^\kappa\}$ and show it trivial. This we will argue by transfinite induction on κ. The argument separates into the two cases $\mathrm{cof}(\kappa) = \omega$ and $\mathrm{cof}(\kappa) > \omega$. The hypothesis in all cases is that $\lim^1 \mathbf{A}_\lambda = 0$ for $\lambda < \kappa$; hence $\Phi\restriction_x = \{\phi_f\restriction_{I_f \cap (x\times\omega)} \mid f \in \omega^\kappa\}$ is trivial for any $x \in [\kappa]^{<\kappa}$. We want to measure the failure of various ϕ to trivialize Φ; for this the notation $e(\phi,\psi) = \{\alpha \mid \phi(\alpha,i) \neq \psi(\alpha,i) \text{ for some } i\}$ will be useful.

CASE 1: $\mathrm{cof}(\kappa) = \omega$. Fix $\{\beta_j \mid j < \omega\}$ cofinal in κ, with $\beta_0 = 0$. Let $L_j = [\beta_j, \beta_{j+1})$ and fix, for all $j < \omega$, some $\phi_j : L_j \times \omega \to \mathbb{Z}$ trivializing $\Phi\restriction_{L_j}$. For all $\alpha < \kappa$, there is a unique $j(\alpha)$ such that $\alpha \in L_{j(\alpha)}$. Define $\phi : \kappa \times \omega \to \mathbb{Z}$ by $\phi(\alpha, i) = \phi_{j(\alpha)}(\alpha, i)$. Let $\mathrm{err}(\phi) = \{f \in \omega^\kappa \mid \phi_f \neq^\star \phi\}$, i.e., $\mathrm{err}(\phi)$ collects those f such that $e(\phi_f, \phi)$ is infinite. Note that $e(\phi_f, \phi)$ is countable for every f, and that $\mathrm{err}(\phi) = \emptyset$ if and only if ϕ trivializes Φ.

Say x *bounds* a collection $\mathcal{C} \subset P(\kappa)$ if $c \subset^* x$ for all $c \in \mathcal{C}$. For any $x \in [\kappa]^{<\kappa}$ bounding $\{e(\phi_f, \phi) : f \in \mathrm{err}(\phi)\}$, it is our induction hypothesis that some $\psi : x \times \omega \to \mathbb{Z}$ trivializes $\Phi\restriction_x$. Define $\phi' : \kappa \times \omega \to \mathbb{Z}$:

$$\phi'(\alpha, i) = \begin{cases} \psi(\alpha, i) & \text{if } \alpha \in x, \\ \phi(\alpha, i) & \text{otherwise.} \end{cases}$$

Observe that ϕ' trivializes Φ.

We show that such an x must always exist. If not, then there exist $f_\xi \in \mathrm{err}(\phi)$ $(\xi < \omega_1)$ such that $u(\xi) = e(\phi_{f_\xi}, \phi) \setminus \bigcup_{\eta<\xi} e(\phi_{f_\eta}, \phi)$ is infinite for every ξ. Define $g : \kappa \to \omega$ by $g(\alpha) = f_\xi(\alpha)$ if $\alpha \in u(\xi)$, and $g(\alpha) = 0$ otherwise. For some $j < \omega$, $A = \{\xi < \omega_1 \mid e(\phi_{f_\xi}, \phi_g) < \beta_j\}$ is uncountable. For some $k \geq j$, $\{\xi \in A \mid u(\xi) \cap L_k \neq \emptyset\}$ is uncountable as well. But this gives uncountably many $\alpha_\xi \in L_k$ such that, for some i, $\phi_g(\alpha_\xi, i) = \phi_{f_\xi}(\alpha_\xi, i) \neq \phi(\alpha_\xi, i) = \phi_k(\alpha_\xi, i)$. Hence ϕ_k does not trivialize $\Phi\restriction_{L_k}$, a contradiction.

CASE 2: $\mathrm{cof}(\kappa) > \omega$. *Stacked functions* are natural attempts to trivialize Φ:

DEFINITION 5.2. A collection of functions $f_j \in \omega^\kappa$ such that $\bigcup_{j\in\omega} I_{f_j} = \kappa \times \omega$ is a *stack*. Further, $\phi : \kappa \times \omega \to \mathbb{Z}$ is *stacked* if $\phi : (\alpha, i) \mapsto \phi_{f_k}(\alpha, i)$ for some stack $\mathcal{F} = \langle f_j \rangle$, where $k = \min\{j : (\alpha, i) \in I_{f_j}\}$.

If $\mathcal{F}$ so determines ϕ, write $\phi = \phi^{\mathcal{F}}$.

Lemma 5.3. *For any stacked functions ϕ, ψ, there exists $\delta < \kappa$ such that $\phi(\alpha, i) = \psi(\alpha, i)$ whenever $\alpha > \delta$.*

Proof. Let $\mathcal{F} = \langle f_j \rangle$ and $\mathcal{G} = \langle g_k \rangle$ determine ϕ and ψ, respectively. Were $e(\phi, \psi) = \{\alpha : \phi(\alpha, i) \neq \psi(\alpha, i) \text{ for some } i\}$ uncountable, so too would be $e(\phi_{f_j}, \phi_{g_k})$ for some $j, k \in \omega$. This cannot be; hence $e(\phi, \psi)$ is bounded below κ. ∎

Applying the induction hypothesis, for $\beta < \kappa$ fix $\phi_\beta : \beta \times \omega \to \mathbb{Z}$ trivializing $\Phi \restriction_\beta$. Note that these ϕ_β "cohere": $e(\phi_\beta, \phi_\gamma)$ is finite for every $\beta < \gamma < \kappa$. Now fix a stack $\mathcal{F} = \langle f_j \mid 0 < j < \omega \rangle$. Note the index shift: though $\phi = \phi^{\mathcal{F}}$ is defined, we have left room at index 0 for one more function f_0 (room, in other words, to revise $\phi^{\mathcal{F}} \restriction_{I_{f_0}}$ to ϕ_{f_0}). Now assume, towards contradiction, that Φ is nontrivial.

For all $\alpha < \kappa$ there exists a least $\alpha^+ < \kappa$ such that $e(\phi_{\alpha^+}, \phi) \cap [\alpha, \alpha^+)$ is infinite; if for some $\beta < \kappa$ this were not so, then

$$\phi'(\alpha, i) = \begin{cases} \phi_\beta(\alpha, i) & \text{if } \alpha < \beta, \\ \phi(\alpha, i) & \text{otherwise} \end{cases}$$

would trivialize Φ. Let $A = \{\alpha < \kappa \mid \alpha \in e(\phi_{\alpha^+}, \phi)\}$. If $\alpha \in A$, let $f_0(\alpha) = \min\{i \mid \phi_{\alpha^+}(\alpha, i) \neq \phi(\alpha, i)\}$. For $\alpha \in \kappa \setminus A$ let $f_0(\alpha) = 0$.

Let $\psi = \psi^{\mathcal{F} \cup \{f_0\}}$; by Lemma 5.3, take $\delta < \kappa$ such that $\psi(\alpha, i) = \phi(\alpha, i)$ for all $\alpha > \delta$. By the coherence of $\{\phi_\beta \mid \beta < \kappa\}$, $\alpha^+ = \delta^+$ for $\alpha \in A \cap [\delta, \delta^+)$. So $\phi_{\delta^+}(\alpha, i) \neq \phi(\alpha, i)$ for infinitely many $(\alpha, i) \in I_{f_0} \cap ([\delta, \delta^+) \times \omega)$, by the definition of f_0. But $\psi(\alpha, i) = \phi_{f_0}(\alpha, i)$ for such (α, i), and $\phi_{f_0}(\alpha, i) = \phi_{\delta^+}(\alpha, i)$ for all but finitely many (α, i), hence $\psi(\alpha, i) \neq \phi(\alpha, i)$ for some $\alpha > \delta$—a contradiction completing the proof of Theorem 5.1. ∎

6. Open problems. The foregoing suggests further questions:

1. *For $n > 1$, does* $\lim^n \mathbf{A} = 0$ *imply* $\lim^n \mathbf{A}_\kappa = 0$*?*
2. *Does* $\lim^n \mathbf{A}_\kappa = 0$ *for all $n > 0$, $\kappa \geq \omega$ imply strong homology additive on, e.g., locally compact metric spaces?*

Here there are two questions, really, in play. Andrei Prasolov has exhibited a paracompact, nonmetrizable ZFC counterexample to the additivity of strong homology (see [Pr]). So a first question is: *On what class of spaces can strong homology be additive?* Prasolov's example is a kind of upper bound. Secondly: *On that class of spaces, are nonzero* $\lim^n \mathbf{A}_\kappa$ *the only obstructions to additivity?*

3. *Is it consistent that* $\lim^n \mathbf{A}_\kappa = 0$ *for all $n > 0$ and $\kappa \geq \omega$?*

This extends a question of Moore's (see [Mo]): *Is it consistent that* $\lim^1 \mathbf{A} = \lim^2 \mathbf{A} = 0$*?*

4. *Is it consistent that* $\lim^3 \mathbf{A} \neq 0$*?*

Arguments like ours for Theorem 4.1 would require higher analogues of Theorem 4.4. An affirmative answer to 4, in other words, would follow from an affirmative answer to 5 below, in the case $n = 2$.

5. *Given an ω_n-chain $\mathcal{F} \subset \mathcal{N}$, does* $\lim^n \mathbf{A}^{\mathcal{F}} \neq 0$*?*
6. *Can a witness to* $\lim^n \mathbf{A} \neq 0$ *be analytic?*

Todorcevic [To1] has given a negative answer to 6 in the case $n = 1$.

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