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The number of unimodular zeros of self-reciprocal polynomials with coefficients in a finite set

by

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1. Introduction and notation. Research on the distribution of the zeros of algebraic polynomials has a long and rich history. In fact all the items [A-02]–[T-07] in our list of references are just some of the publications devoted to this topic. The study of the number of real zeros of trigonometric polynomials and the number of unimodular zeros (that is, zeros lying on the unit circle of the complex plane) of algebraic polynomials with various constraints on their coefficients is the subject of quite a few of these. We do not try to survey these in our introduction.

Let $S \subset \mathbb{C}$. Let $\mathcal{P}_n(S)$ be the set of all algebraic polynomials of degree at most n with each of their coefficients in S. A polynomial

$$P_n(z) = \sum_{j=0}^{n} p_{j,n} z^j, \qquad p_{j,n} \in \mathbb{C}, \tag{1.1}$$

is called *conjugate-reciprocal* if

$$\bar{p}_{j,n} = p_{n-j,n}, \qquad j = 0, 1, \ldots, n. \tag{1.2}$$

A polynomial P_n of the form (1.1) is called *plain-reciprocal* or *self-reciprocal* if

$$p_{j,n} = p_{n-j,n}, \qquad j = 0, 1, \ldots, n. \tag{1.3}$$

If a conjugate-reciprocal polynomial P_n has only real coefficients, then it is

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obviously plain-reciprocal. We note also that if

$$P_{2n}(z) = \sum_{j=0}^{2n} p_{j,2n} z^j, \qquad p_{j,2n} \in \mathbb{C},$$

is conjugate-reciprocal, then there are $\theta_j \in \mathbb{R}$, $j = 1, \ldots, n$, such that

$$T_n(t) := P_{2n}(e^{it})e^{-int} = p_{n,2n} + \sum_{j=1}^{n} 2|p_{j,2n}| \cos(jt + \theta_j).$$

If the polynomial P_{2n} above is plain-reciprocal, then

$$T_n(t) := P_{2n}(e^{it})e^{-int} = p_{n,2n} + \sum_{j=1}^{n} 2p_{j,2n} \cos(jt).$$

In this paper, whenever we write "$P_n \in \mathcal{P}_n(S)$ is conjugate-reciprocal" we mean that P_n is of the form (1.1) with each $p_{j,n} \in S$ satisfying (1.2). Similarly, whenever we write "$P_n \in \mathcal{P}_n(S)$ is self-reciprocal" we mean that P_n is of the form (1.1) with each $p_{j,n} \in S$ satisfying (1.3). This is going to be our understanding even if the degree of $P_n \in \mathcal{P}_n(S)$ is less than n.

Associated with an algebraic polynomial P_n of the form (1.1) we introduce the numbers

$$\mathrm{NC}(P_n) := |\{j \in \{0, 1, \ldots, n\} : p_{j,n} \neq 0\}|.$$

Here and in what follows, $|A|$ denotes the number of elements of a finite set A. Let $\mathrm{NZ}(P_n)$ denote the number of real zeros (counting multiplicities) of an algebraic polynomial P_n that lie on the unit circle. Associated with a trigonometric polynomial

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt)$$

we introduce the numbers

$$\mathrm{NC}(T_n) := |\{j \in \{0, 1, \ldots, n\} : a_{j,n} \neq 0\}|.$$

Let $\mathrm{NZ}(T_n)$ denote the number of real zeros (by counting multiplicities) of a trigonometric polynomial T_n in a period $[a, a + 2\pi)$, $a \in \mathbb{R}$. The (slightly modified) quotation below is from [B-07].

> Let $0 \leq n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T_N(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period $[a, a + 2\pi)$, $a \in \mathbb{R}$. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the Littlewood Conjecture proved by S. Konyagin [K-81] and independently by McGehee, Pigno, and Smith [M-81] in 1981. See also [D-93, pp. 285–288] for a book proof. It is not difficult to prove the statement in

general even in the case $n_1 = 0$ without using Littlewood's Conjecture. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T_N((2j-1)\pi/n_N) = 0.$$

See [K-04a], for example. Another way is to use Theorem 2 of [M-06a]. So there is certainly no shortage of possible approaches to prove the starting observation of this paper even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication B. Conrey asked how fast the number of real zeros of the above sums in a period tends to ∞ as a function N. In [C-00] the authors observed that for an odd prime p the Fekete polynomial $f_p(z) = \sum_{k=0}^{p-1} \binom{k}{p} z^k$ (the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Conrey's question in general does not appear to be easy.

Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [L-68] (problem 22) poses the following research problem, which appears to still be open: "If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m\theta)$? Possibly $N-1$, or not much less." Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half-century. In a recent paper [B-08a] we showed that this is false. There exists a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ with the n_m integral and all different so that the number of its real zeros in the period is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ always has many zeros in the period.

One of the highlights of the present paper, Corollary 2.7, shows that the number of real zeros of the sums $T_N(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ in a period $[a, a+2\pi)$, $a \in \mathbb{R}$, tends to ∞ whenever $0 \le n_1 < \cdots < n_N$ are integers and N tends to ∞, even though the part "how fast" in Conrey's question remains open. In fact, we will prove more general results of this variety. Let

$$\mathcal{L}_n := \Big\{ P : P(z) = \sum_{j=0}^{n} p_{j,n} z^j,\ p_{j,n} \in \{-1, 1\} \Big\}.$$

Elements of $\mathcal{L}_n$ are often called *Littlewood polynomials of degree* n. Let

$$\mathcal{K}_n := \Big\{ P : P(z) = \sum_{j=0}^{n} p_{j,n} z^j,\ p_{j,n} \in \mathbb{C},\ |p_{0,n}| = |p_{n,n}| = 1,\ |p_{j,n}| \le 1 \Big\}.$$

Observe that $\mathcal{L}_n \subset \mathcal{K}_n$. In [B-08b] we proved that any polynomial $P \in \mathcal{K}_n$ has at least $8n^{1/2}\log n$ zeros in any open disk centered at a point on the unit circle with radius $33n^{-1/2}\log n$. Thus polynomials in $\mathcal{K}_n$ have a few zeros near the unit circle. One may naturally ask how many unimodular roots a polynomial in $\mathcal{K}_n$ can have. Mercer [M-06a] proved that if a Littlewood polynomial $P \in \mathcal{L}_n$ of the form (1.1) is *skew-reciprocal*, that is, $p_{j,n} = (-1)^j p_{n-j,n}$ for each $j = 0, 1, \ldots, n$, then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [E-01] and [M-06a] that if a Littlewood polynomial P of the form (1.1) is self-reciprocal, that is, $p_{j,n} = p_{n-j,n}$ for each $j = 0, 1, \ldots, n$, $n \geq 1$, then it has at least one zero on the unit circle. Mukunda [M-06b] improved this result by showing that every self-reciprocal Littlewood polynomial of odd degree at least 3 has at least three zeros on the unit circle. Drungilas [D-08] proved that every self-reciprocal Littlewood polynomial of odd degree $n \geq 7$ has at least five zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree $n \geq 14$ has at least four zeros on the unit circle.

In [B-15] two types of Littlewood polynomials are considered: Littlewood polynomials with one sign change in the sequence of coefficients and Littlewood polynomials with one negative coefficient, and the numbers of the zeros such Littlewood polynomials have on the unit circle and inside the unit disk, respectively, are investigated. Note that the Littlewood polynomials studied in [B-15] are very special. In [B-08a] we proved that the average number of unimodular zeros of self-reciprocal Littlewood polynomials of degree n is at least $n/4$.

However, it is much harder to give decent lower bounds for the quantities

$$\mathrm{NZ}_n := \min_P \mathrm{NZ}(P),$$

where $\mathrm{NZ}(P)$ denotes the number of zeros of a polynomial P lying on the unit circle and the minimum is taken for all self-reciprocal Littlewood polynomials $P \in \mathcal{L}_n$. It has been conjectured for a long time that $\lim_{n\to\infty} \mathrm{NZ}_n = \infty$. In this paper we show that $\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty$ whenever $P_n \in \mathcal{L}_n$ is self-reciprocal and $\lim_{n\to\infty} |P_n(1)| = \infty$. This is a consequence of a more general result (see Corollary 2.3) in which the coefficients of the self-reciprocal polynomials P_n of degree at most n belong to a fixed finite set of real numbers. In [B-07] we proved the following.

THEOREM 1.1. *If the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite, the sequence (a_j) is not eventually periodic, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt),$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

In [B-07] this is stated without the assumption that the sequence (a_j) is not eventually periodic. However, as the following example shows, Lemma 3.4 in [B-07], dealing with the case of eventually periodic sequences (a_j), is incorrect. Let

$$\begin{aligned} T_n(t) :=\ & \cos t + \cos((4n+1)t) + \sum_{k=0}^{n-1} \big(\cos((4k+1)t) - \cos((4k+3)t)\big) \\ =\ & \frac{1+\cos((4n+2)t)}{2\cos t} + \cos t. \end{aligned}$$

It is easy to see that $T_n(t) \neq 0$ on $[-\pi, \pi] \setminus \{-\pi/2, \pi/2\}$ and the zeros of T_n at $-\pi/2$ and $\pi/2$ are simple. Hence T_n has only two (simple) zeros in the period. So the conclusion of Theorem 1.1 above is false for the sequence (a_j) with $a_0 := 0$, $a_1 := 2$, $a_3 := -1$, $a_{2k} := 0$, $a_{4k+1} := 1$, $a_{4k+3} := -1$ for every $k = 1, 2, \dots$. Nevertheless, Theorem 1.1 can be saved even in the case of eventually periodic sequences (a_j) if we assume that $a_j \neq 0$ for all sufficiently large j (see Lemma 3.11). So Theorem 1 in [B-07] can be corrected as

THEOREM 1.2. *If the set* $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ *is finite,* $a_j \neq 0$ *for all sufficiently large* j*, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt),$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

It was expected that the conclusion of the above theorem remains true even if the coefficients of T_n do not come from the same sequence, that is,

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt),$$

where the set $S := \{a_{j,n} : j \in \{0, 1, \dots, n\},\ n \in \mathbb{N}\} \subset \mathbb{R}$ is finite and $\lim_{n\to\infty} |\{j \in \{0, 1, \dots, n\},\ a_{j,n} \neq 0\}| = \infty$. Our purpose is to prove such an extension of Theorem 1.1. This extension is formulated as Theorem 2.1, which is the main result of this paper.

The already mentioned Littlewood Conjecture, proved by Konyagin [K-81] and independently by McGehee, Pigno, and Smith [M-81], plays a key role in the proof of main results. It is stated as follows:

THEOREM 1.3. *There is an absolute constant* $c > 0$ *such that*

$$\int_0^{2\pi} \Big| \sum_{j=1}^{m} a_j e^{i\lambda_j t} \Big| \, dt \geq c\gamma \log m$$

whenever $\lambda_1, \dots, \lambda_m$ *are distinct integers and* $a_1, \dots, a_m$ *are complex numbers of modulus at least* $\gamma > 0$.

This is an obvious consequence of the following result a book proof of which has been worked out by DeVore and Lorentz [D-93, pp. 285–288].

THEOREM 1.4. *If $\lambda_1 < \dots < \lambda_m$ are integers and $a_1, \dots, a_m$ are complex numbers, then*

$$\int_0^{2\pi} \Big| \sum_{j=1}^m a_j e^{i\lambda_j t} \Big| \, dt \geq \frac{1}{30} \sum_{j=1}^m \frac{|a_j|}{j}.$$

2. New results. Associated with an algebraic polynomial

$$P_n(z) = \sum_{j=0}^n p_{j,n} z^j, \qquad p_{j,n} \in \mathbb{C},$$

let

$$\mathrm{NC}_k(P_n) := \Big| \Big\{ u : 0 \leq u \leq n-k+1, \ \sum_{j=u}^{u+k-1} p_{j,n} \neq 0 \Big\} \Big|.$$

Recall that if

$$P_{2n}(z) = \sum_{j=0}^{2n} p_{j,2n} z^j, \qquad p_{j,2n} \in \mathbb{R},$$

is self-reciprocal, then

$$T_n(t) := P_{2n}(e^{it}) e^{-int} = p_{n,2n} + \sum_{j=1}^n 2p_{j,2n} \cos(jt).$$

It is also clear that $\mathrm{NZ}(P_{2n}) = \mathrm{NZ}(T_n)$.

THEOREM 2.1. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal polynomials, $T_n(t) := P_{2n}(e^{it})e^{-int}$, and*

$$\lim_{n\to\infty} \mathrm{NC}_k(P_{2n}) = \infty \tag{2.1}$$

for every $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} \mathrm{NZ}(P_{2n}) = \lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty. \tag{2.2}$$

COROLLARY 2.2. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal polynomials, $T_n(t) := P_{2n}(e^{it})e^{-int}$, and*

$$\lim_{n\to\infty} |P_{2n}(1)| = \infty, \tag{2.3}$$

then (2.2) *holds.*

Our next result is slightly more general than Corollary 2.2, yet it is a simple consequence of it.

COROLLARY 2.3. *If $S \subset \mathbb{R}$ is a finite set, $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials, and*

$$\lim_{n\to\infty} |P_n(1)| = \infty, \tag{2.4}$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty. \tag{2.5}$$

Consider the following property of a set $S \subset \mathbb{R}$: for every $k \in \mathbb{N}$,

$$s_1 + \cdots + s_k = 0,\ s_1, \ldots, s_k \in S, \quad \text{implies} \quad s_1 = \cdots = s_k = 0, \tag{2.6}$$

that is, any sum of nonzero elements of S is different from 0.

Corollary 2.4. *If the finite set $S \subset \mathbb{R}$ has property* (2.6)*, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal polynomials, $T_n(t) := P_{2n}(e^{it})e^{-int}$, and*

$$\lim_{n\to\infty} \mathrm{NC}(P_{2n}) = \infty, \tag{2.7}$$

then (2.2) *holds.*

Our next result, slightly more general than Corollary 2.4, follows from Corollary 2.4 simply.

Corollary 2.5. *If the finite set $S \subset \mathbb{R}$ has property* (2.6)*, $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials, and*

$$\lim_{n\to\infty} \mathrm{NC}(P_n) = \infty, \tag{2.8}$$

then (2.5) *holds.*

Our next result is an obvious consequence of Corollary 2.2.

Corollary 2.6. *If*

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt),$$

where the set $S := \{a_{j,n} : j \in \{0, 1, \ldots, n\},\ n \in \mathbb{N}\} \subset \mathbb{R}$ is finite and

$$\lim_{n\to\infty} \Big| \sum_{j=0}^{n} a_{j,n} \Big| = \infty,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

Our next result is an obvious consequence of Corollary 2.6.

Corollary 2.7. *If*

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt),$$

where the set $S := \{a_{j,n} : j \in \{0, 1, \ldots, n\},\ n \in \mathbb{N}\} \subset [0, \infty)$ is finite, and $\lim_{n\to\infty} \mathrm{NC}(T_n) = \infty$, *then* $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$.

3. Lemmas. Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with complex coefficients.

LEMMA 3.1. *If $S \subset \mathbb{C}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$, and $H \in \mathcal{P}_m$ is a polynomial of minimal degree m such that*

$$\sup_{n \in \mathbb{N}} \mathrm{NC}(P_{2n}H) < \infty, \tag{3.1}$$

then each zero of H is a root of unity, and each zero of H is simple.

Proof. Suppose to the contrary that $H(\alpha) = 0$, where $0 \neq \alpha \in \mathbb{C}$ is not a root of unity. Let $G \in \mathcal{P}_{m-1}$ be defined by

$$G(z) := \frac{H(z)}{z - \alpha}. \tag{3.2}$$

Let S_n^* be the set of the coefficients of $P_{2n}G$, and let

$$S^* := \bigcup_{n \in \mathbb{N}} S_n^*.$$

As $P_{2n} \in \mathcal{P}_{2n}(S)$ and the set S is finite, the set S^* is also finite. Let

$$(P_{2n}H)(z) = \sum_{j=0}^{2n+m} a_{j,n} z^j \quad \text{and} \quad (P_{2n}G)(z) = \sum_{j=0}^{2n+m} b_{j,n} z^j. \tag{3.3}$$

Note that $b_{2n+m,n} = 0$. Due to the minimality of H we have

$$\sup_{n \in \mathbb{N}} \mathrm{NC}(P_{2n}G) = \infty. \tag{3.4}$$

Observe that (3.2) implies

$$a_{j,n} = b_{j-1,n} - \alpha b_{j,n}, \qquad j = 1, \dots, 2n+m. \tag{3.5}$$

Let

$$A_n := \{j : 1 \leq j \leq 2n+m,\ b_{j-1,n} \neq \alpha b_{j,n}\}.$$

Combining (3.1), (3.3), and (3.5), we can deduce that

$$\mu := \sup_{n \in \mathbb{N}} |A_n| < \infty.$$

Hence $A_n = \{j_{1,n} < j_{2,n} < \cdots < j_{u_n,n}\}$, where $u_n \leq \mu$ for each $n \in \mathbb{N}$. We introduce the numbers $j_{0,n} := 1$ and $j_{u_n+1,n} := 2n+m$. As $\alpha \in \mathbb{C}$ is not a root of unity, the inequality

$$j_{l+1,n} - j_{l,n} \geq |S^*|$$

for some $l = 0, 1, \dots, u_n$ implies

$$b_{j,n} = 0, \qquad j = j_{l,n},\ j_{l,n}+1,\ j_{l,n}+2, \dots,\ j_{l+1,n} - 1.$$

But then $b_{j,n} \neq 0$ is possible only for $(\mu+1)|S^*|$ values of $j = 1, \dots, 2n+m$, which contradicts (3.4). Thus each zero of H is a root of unity.

Now we prove that each zero of H is simple. Without loss of generality it is sufficient to prove that $H(1)=0$ implies that $H'(1)\neq 0$; the general case can easily be reduced to this. Assume to the contrary that $H(1)=0$ and $H'(1)=0$. Let $G_1\in\mathcal{P}_{m-1}$ and $G_2\in\mathcal{P}_{m-2}$ be defined by

$$G_1(z):=\frac{H(z)}{z-1}\quad\text{and}\quad G_2(z):=\frac{H(z)}{(z-1)^2}=\frac{G_1(z)}{z-1}, \tag{3.6}$$

respectively. Let

$$\begin{aligned}(P_{2n}H)(z)&=\sum_{j=0}^{2n+m}a_{j,n}z^j,\\ (P_{2n}G_1)(z)&=\sum_{j=0}^{2n+m}b_{j,n}z^j,\qquad (P_{2n}G_2)(z)=\sum_{j=0}^{2n+m}c_{j,n}z^j.\end{aligned} \tag{3.7}$$

Due to the minimality of the degree of H we have

$$\sup_{n\in\mathbb{N}}\mathrm{NC}(P_{2n}G_1)=\infty. \tag{3.8}$$

Observe that (3.6) implies

$$a_{j,n}=b_{j-1,n}-b_{j,n},\qquad j=1,\ldots,2n+m, \tag{3.9}$$

$$b_{j,n}=c_{j-1,n}-c_{j,n},\qquad j=1,\ldots,2n+m. \tag{3.10}$$

Combining (3.1), (3.7), and (3.9), we can deduce that

$$\mu:=\sup_{n\in\mathbb{N}}|\{j:1\le j\le 2n+m,\ b_{j-1,n}\neq b_{j,n}\}|<\infty. \tag{3.11}$$

By (3.8) and (3.11), for every $N\in\mathbb{N}$ there are $n,L\in\mathbb{N}$ such that

$$0\neq b:=b_{L,n}=b_{L+1,n}=\cdots=b_{L+N,n}.$$

Combining this with (3.10), we get

$$c_{j-1,n}=c_{j,n}+b,\qquad j=L,L+1,\ldots,L+N.$$

Hence

$$\sup_{n\in\mathbb{N}}\max_{j=0,1,\ldots,2n+m}|c_{j,n}|=\infty. \tag{3.12}$$

On the other hand, since $P_{2n}\in\mathcal{P}_{2n}(S)$ and S is finite, the set

$$\{|c_{j,n}|:j\in\{0,1,\ldots,2n+m\},\ n\in\mathbb{N}\}$$

is also finite. This contradicts (3.12), and so each zero of H is simple. ■

Lemma 3.2. *If $S\subset\mathbb{C}$ is a finite set, $P_{2n}\in\mathcal{P}_{2n}(S)$, $H(z):=z^k-1$, and*

$$\mu:=\sup_{n\in\mathbb{N}}\mathrm{NC}(P_{2n}H)<\infty, \tag{3.13}$$

then there are constants $c_1, c_2 > 0$ depending only on μ, k, and S and independent of n and δ such that

$$\int_{-\delta}^{\delta} |P_{2n}(e^{it})|\,dt > c_1 \log(\mathrm{NC}_k(P_{2n})) - c_2\delta^{-1}$$

for every $\delta \in (0,\pi)$, and hence assumption (2.1) *implies*

$$\lim_{n\to\infty} \int_{-\delta}^{\delta} |P_{2n}(e^{it})|\,dt = \infty \qquad \textit{for every } \delta \in (0,\pi).$$

Proof. We define

$$G(z) := \sum_{j=0}^{k-1} z^j$$

so that $H(z) = G(z)(z-1)$. Let S_n^* be the set of the coefficients of $P_{2n}G$. We define

$$S^* := \bigcup_{n=1}^{\infty} S_n^*.$$

As $P_{2n} \in \mathcal{P}_{2n}(S)$ and S is finite, the set S^* is also finite. So by Theorem 1.3 there is an absolute constant $c > 0$ such that

$$\int_{0}^{2\pi} |(P_{2n}G)(e^{it})|\,dt \ge c\gamma \log(\mathrm{NC}(P_{2n}G)) \ge c\gamma \log(\mathrm{NC}_k(P_{2n})) \tag{3.14}$$

for every $n \in \mathbb{N}$ with $\gamma := \min_{z\in S^*\setminus\{0\}} |z|$. Observe that

$$\begin{aligned} |(P_{2n}G)(e^{it})| &= \frac{1}{|e^{it}-1|}\,|(P_{2n}H)(e^{it})| \le \frac{\mu M}{|e^{it}-1|} = \frac{\mu M}{\sin(t/2)} \\ &\le \frac{\pi\mu M}{t}, \qquad t \in (-\pi,\pi), \end{aligned}$$

where μ is defined by (3.13) and $M := \max\{|z| : z \in S^*\}$ depends only on S^*, and hence $M > 0$ depends only on k and S. It follows that

$$\int_{[-\pi,\pi]\setminus[-\delta,\delta]} |(P_{2n}G)(e^{it})|\,dt \le 2\pi\frac{\pi\mu M}{2\delta} = \frac{\pi^2\mu M}{\delta}. \tag{3.15}$$

Now (3.14) and (3.15) give

$$\begin{aligned} \int_{-\delta}^{\delta} |P_{2n}(e^{it})|\,dt &\ge \frac{1}{k}\int_{-\delta}^{\delta} |(P_{2n}G)(e^{it})|\,dt \\ &= \frac{1}{k}\Big(\int_{0}^{2\pi} |(P_{2n}G)(e^{it})|\,dt - \int_{[-\pi,\pi]\setminus[-\delta,\delta]} |(P_{2n}G)(e^{it})|\,dt\Big) \\ &\ge \frac{1}{k}\, c\gamma \log(\mathrm{NC}_k(P_{2n})) - \frac{\pi^2\mu M}{k\delta}. \quad \blacksquare \end{aligned}$$

LEMMA 3.3. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal, $H(z) := z^k - 1$, (3.13) holds,*

$$T_n(t) := P_{2n}(e^{it})e^{-int}, \qquad R_n(x) := \int_0^x T_n(t)\,dt,$$

and $0 < \delta \le (2k)^{-1}$, then

$$\sup_{n\in\mathbb{N}} \max_{x\in[-\delta,\delta]} |R_n(x)| < \infty.$$

Proof. Let

$$T_n(t) = a_{0,n} + \sum_{j=1}^{n} 2a_{j,n}\cos(jt), \qquad a_{j,n} \in S.$$

Observe that (3.13) implies that

$$\sup_{n\in\mathbb{N}} |\{j : k \le j \le n,\ a_{j-k,n} \ne a_{j,n}\}| \le \mu := \sup_{n\in\mathbb{N}} \mathrm{NC}(P_{2n}H) < \infty. \tag{3.16}$$

We have

$$R_n(x) = a_{0,n}x + \sum_{j=1}^{n} \frac{2a_{j,n}}{j}\,\sin(jx).$$

Now (3.16) implies that

$$R_n(x) = a_{0,n}x + \sum_{m=1}^{u_n} F_{m,n}(x),$$

where

$$F_{m,k,n}(x) := \sum_{j=0}^{n_m} \frac{2A_{m,k,n}\sin((j_m + jk)x)}{j_m + jk}$$

with some $A_{m,k,n} \in S$, $m = 1, \dots, u_n$, $j_m \in \mathbb{N}$, and $n_m \in \mathbb{N}$, where

$$\sup_{n\in\mathbb{N}} u_n \le k\mu < \infty$$

(we do not know much about j_m and n_m). Since $S \subset \mathbb{R}$ is finite, and hence bounded, it is sufficient to prove that

$$\max_{x\in[-\delta,\delta]} |F_{m,k,n}(x)| \le M,$$

where M is a uniform bound valid for all $n \in \mathbb{N}$, $j_m \in \mathbb{N}$, $n_m \in \mathbb{N}$, $m = 1, \dots, u_n$, that is, it is sufficient to prove that if

$$F(x) := \sum_{j=0}^{\nu} \frac{\sin((j_0 + jk)x)}{j_0 + jk}$$

then

$$\max_{x\in[-\delta,\delta]} |F(x)| = \max_{x\in[0,\delta]} |F(x)| \le M, \tag{3.17}$$

where M is a uniform bound valid for all $\nu \in \mathbb{N}$ and $j_0 \in \mathbb{N}$. Note that the equality in (3.17) holds as F is odd.

To prove the inequality in (3.17) let $x \in (0, \delta]$, where $0 < \delta \le (2k)^{-1}$. We break the sum as

$$F = R + S, \tag{3.18}$$

where

$$R(x) := \sum_{\substack{j=0 \\ j_0+jk \le x^{-1}}}^{\nu} \frac{\sin((j_0+jk)x)}{j_0+jk}, \qquad S(x) := \sum_{\substack{j=0 \\ x^{-1} < j_0+jk}}^{\nu} \frac{\sin((j_0+jk)x)}{j_0+jk}.$$

Here

$$\begin{aligned} |R(x)| &\le \sum_{\substack{j=0 \\ j_0+jk \le x^{-1}}}^{\nu} \left| \frac{\sin((j_0+jk)x)}{j_0+jk} \right| \le (x^{-1}+1)|x| \\ &\le 1 + |x| \le 1 + \delta = 1 + (2k)^{-1} \le 3/2, \end{aligned} \tag{3.19}$$

where each term in the sum is estimated by

$$\left| \frac{\sin((j_0+jk)x)}{j_0+jk} \right| \le \left| \frac{(j_0+jk)x)}{j_0+jk} \right| = |x|,$$

and the number of terms is clearly at most $x^{-1}+1$. Further, using Abel rearrangement, we have

$$S(x) = -\frac{B_v(x)}{j_0+vk} + \frac{B_u(x)}{j_0+uk} + \sum_{\substack{j=0 \\ x^{-1} < j_0+jk}}^{\nu} B_j(x) \left(\frac{1}{j_0+jk} - \frac{1}{j_0+(j+1)k} \right)$$

with

$$B_j(x) := B_{j,k}(x) := \sum_{h=0}^{j} \sin((j_0+hk)x)$$

and with some $u, v \in \mathbb{N}_0$ for which $x^{-1} < j_0+(u+1)k$ and $x^{-1} < j_0+(v+1)k$. Hence,

$$\begin{aligned} |S(x)| \le & \left| \frac{B_v(x)}{j_0+vk} \right| + \left| \frac{B_u(x)}{j_0+uk} \right| \\ & + \sum_{\substack{j=0 \\ x^{-1} < j_0+jk}}^{\nu} |B_j(x)| \left(\frac{1}{j_0+jk} - \frac{1}{j_0+(j+1)k} \right). \end{aligned} \tag{3.20}$$

Observe that $x \in (0, \delta]$, $0 < \delta \le (2k)^{-1}$, $x^{-1} < j_0 + (w+1)k$, and $w \in \mathbb{N}_0$ imply

$$x^{-1} < j_0 + (w+1)k < 2(j_0 + wk) \quad \text{if } w \ge 1,$$

and

$$2k \le \delta^{-1} \le x^{-1} < j_0 + k \quad \text{if } w = 0,$$

and hence

$$\frac{1}{j_0 + wk} \le 2x, \quad w \in \mathbb{N}_0. \tag{3.21}$$

Observe also that $x \in (0, \delta]$ and $0 < \delta \le (2k)^{-1}$ imply that $0 < x < \pi k^{-1}$. Hence, with $z = e^{ix}$ we have

$$\begin{aligned} |B_j(x)| &= \left| \frac{1}{2} \operatorname{Im} \Big(\sum_{h=0}^{j} z^{j_0 + hk} \Big) \right| \le \left| \frac{1}{2} \sum_{h=0}^{j} z^{j_0+hk} \right| = \left| \frac{1}{2} \sum_{h=0}^{j} z^{hk} \right| \\ &= \left| \frac{1}{2} \, \frac{1 - z^{(j+1)k}}{1 - z^k} \right| \le \frac{1}{2} \, |1 - z^{(j+1)k}| \, \frac{1}{|1 - z^k|} \le \frac{1}{|1 - z^k|} \\ &\le \frac{1}{\sin(kx/2)} \le \frac{\pi}{kx}. \end{aligned} \tag{3.22}$$

Combining (3.20)–(3.22), we conclude that

$$|S(x)| \le \frac{\pi}{kx}\, 2x + \frac{\pi}{kx}\, 2x + \frac{\pi}{kx}\, 2x \le \frac{6\pi}{k}. \tag{3.23}$$

Now (3.18), (3.19), and (3.23) give the inequality in (3.17) with $M := 6\pi/k \le 6\pi$. ∎

Our next lemma is well known and may be proved simply by contradiction.

Lemma 3.4. *If R is a continuously differentiable function on the interval $[-\delta, \delta]$, $\delta > 0$, and*

$$\int_{-\delta}^{\delta} |R'(x)| \, dx = L \quad \textit{and} \quad \max_{x \in [-\delta, \delta]} |R(x)| = M,$$

then there is an $\eta \in [-M, M]$ such that $R - \eta$ has at least $L(2M)^{-1}$ zeros in $[-\delta, \delta]$.

Lemma 3.5. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal,*

$$T_n(t) := P_{2n}(e^{it}) e^{-int},$$

$H(z) := z^k - 1$, and (2.1) and (3.13) hold, then (2.2) holds, that is, $\lim_{n \to \infty} \mathrm{NZ}(T_n) = \infty$.

Proof. Let $0 < \delta \le (2k)^{-1}$. Let

$$R_n(x) := \int_0^x T_n(t) \, dt.$$

Observe that $|T_n(x)| = |P_{2n}(e^{ix})|$ for all $x \in \mathbb{R}$. By Lemmas 3.2 and 3.3,

$$\lim_{n\to\infty} \int_{-\delta}^{\delta} |R_n'(x)|\,dx = \lim_{n\to\infty} \int_{-\delta}^{\delta} |T_n(x)|\,dx = \lim_{n\to\infty} \int_{-\delta}^{\delta} |P_{2n}(e^{ix})|\,dx = \infty \tag{3.24}$$

and

$$\sup_{n\in\mathbb{N}} \max_{x\in[-\delta,\delta]} |R_n(x)| < \infty. \tag{3.25}$$

(Note that to obtain (3.24) from Lemma 3.2 we use the second statement of that lemma, which is valid under the assumption (2.1)—that is why (2.1) is also assumed in this lemma.) Therefore, by Lemma 3.4 there are $c_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathrm{NZ}(R_n - c_n) = \infty.$$

However, $T_n(x) = (R_n - c_n)'(x)$ for all $x \in \mathbb{R}$, and hence $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$ by Rolle's Theorem. ∎

Our next lemma follows immediately from Lemmas 3.1 and 3.5.

Lemma 3.6. *If $S \subset \mathbb{R}$ is a finite set, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal, $T_n(t) := P_{2n}(e^{it})e^{-int}$,* (2.1) *holds, and there is a polynomial $H \in \mathcal{P}_m$ such that* (3.13) *holds, then* (2.2) *holds.*

Moreover, we have the following observation.

Lemma 3.7. *Let (n_ν) be a strictly increasing sequence of positive integers. If $S \subset \mathbb{R}$ is a finite set, $P_{2n_\nu} \in \mathcal{P}_{2n_\nu}(S)$ are self-reciprocal,*

$$T_{n_\nu}(t) := P_{2n_\nu}(e^{it})e^{-in_\nu t}, \qquad \lim_{\mu\to\infty} \mathrm{NC}_k(P_{2n_\mu}) = \infty \quad \text{for every } k \in \mathbb{N},$$

and there is a polynomial $H \in \mathcal{P}_m$ such that $\sup_{\nu\in\mathbb{N}} \mathrm{NC}(P_{2n_\nu}H) < \infty$, then

$$\lim_{\nu\to\infty} \mathrm{NZ}(T_{n_\nu}) = \infty.$$

Proof. Without loss of generality we may assume that $0 \in S$. We define the self-reciprocal polynomials $P_{2n} \in \mathcal{P}_{2n}(S)$ by

$$P_{2n}(z) := z^{n-n_\nu} P_{2n_\nu}(z), \qquad n_\nu \le n < n_{\nu+1},$$

and apply Lemma 3.6. ∎

The next lemma is a straightforward consequence of Theorem 1.4.

Lemma 3.8. *Let $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ be nonnegative integers and let*

$$Q_m(t) = \sum_{j=0}^{m} A_j \cos(\lambda_j t), \qquad A_j \in \mathbb{R},\ j = 0, 1, \ldots, m.$$

Then

$$\int_{-\pi}^{\pi} |Q_m(t)|\,dt \ge \frac{1}{60} \sum_{j=0}^{m} \frac{|A_{m-j}|}{j+1}.$$

We will also need the lemma below in the proof of Theorem 2.1.

LEMMA 3.9. *Let* $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ *be nonnegative integers and let* $Q_m(t)$ *be as in Lemma* 3.8. *Let* $A := \max_{j=0,1,\dots,m} |A_j|$. *Suppose* Q_m *has at most* $K-1$ *zeros in* $[-\pi,\pi)$. *Then*

$$\int_{-\pi}^{\pi} |Q_m(t)|\,dt \le 2KA\biggl(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j}\biggr) \le 2KA(5+\log m).$$

Proof. We may assume that $\lambda_0 = 0$; the case $\lambda_0 > 0$ can be handled similarly. Associated with Q_m in the lemma let

$$R_m(t) := A_0 t + \sum_{j=0}^{m} \frac{A_j}{\lambda_j} \sin(\lambda_j t).$$

Clearly

$$\max_{t\in[-\pi,\pi]} |R_m(t)| \le A\biggl(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j}\biggr).$$

Also, for every $c \in \mathbb{R}$ the function $R_m - c$ has at most K zeros in $[-\pi,\pi)$, otherwise Rolle's Theorem implies that $Q_m = (R_m - c)'$ has at least K zeros there. Hence

$$\begin{aligned}\int_{-\pi}^{\pi} |Q_m(t)|\,dt = \int_{-\pi}^{\pi} |R_m'(t)|\,dt = V_{-\pi}^{\pi}(R_m) \le 2K \max_{t\in[-\pi,\pi]} |R_m(t)| \\ \le 2KA\biggl(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j}\biggr) \le 2KA(5+\log m),\end{aligned}$$

where $V_{-\pi}^{\pi}(R_m)$ is the total variation of R_m on $[-\pi,\pi]$. ■

The lemma below is needed only in the proof of Lemma 3.11.

LEMMA 3.10. *Suppose* $k \in \mathbb{N}$. *Let*

$$z_j := \exp(2\pi j i/k), \qquad j = 0,1,\dots,k-1,$$

be the k*th roots of unity. Suppose*

$$\{b_0,b_1,\dots,b_{k-1}\} \subset \mathbb{R}, \quad b_0 \ne 0, \quad \textit{and} \quad Q(z) := \sum_{j=0}^{k-1} b_j z^j.$$

Then there is a value of $j \in \{0,1,\dots,k-1\}$ *for which* $\mathrm{Re}(Q(z_j)) \ne 0$.

Proof. If the statement were false, then

$$z^{k-1}(Q(z) + Q(1/z)) = (z^k - 1) \sum_{\nu=0}^{k-2} \alpha_\nu z^\nu$$

with some $\alpha_\nu \in \mathbb{R}$, $\nu = 0, 1, \dots, k-2$. Observe that the coefficient of z^{k-1} on the right hand side is 0, while the coefficient of z^{k-1} on the left hand side is $2b_0 \neq 0$, a contradiction. ∎

Our final lemma has already been used in Section 1, where Theorem 1 of [B-07] has been corrected to Theorem 1.2.

LEMMA 3.11. *If* $0 \notin \{b_0, b_1, \dots, b_{k-1}\} \subset \mathbb{R}$, $\{a_0, a_1, \dots, a_{m-1}\} \subset \mathbb{R}$, *where* $m = uk$ *with some integer* $u \geq 0$,

$$a_{m+lk+j} = b_j, \qquad l = 0, 1, \dots, \; j = 0, 1, \dots, k-1,$$

and $n = m + lk + r$ *with integers* $m \geq 0$, $l \geq 0$, $k \geq 1$, *and* $0 \leq r \leq k-1$, *then there is a constant* $c_3 > 0$ *depending only on the sequence* (a_j) *but independent of* n *such that*

$$T_n(t) := \operatorname{Re}\Big(\sum_{j=0}^{n} a_j e^{ijt}\Big)$$

has at least $c_3 n$ *zeros in* $[-\pi, \pi)$.

Proof. Note that

$$\begin{aligned}\sum_{j=0}^{n} a_j z^j &= \sum_{j=0}^{m-1} a_j z^j + z^m\Big(\sum_{j=0}^{k-1} b_j z^j\Big)\frac{z^{(l+1)k}-1}{z^k-1} + z^{m+lk}\sum_{j=0}^{r} b_j z^j \\ &= P_1(z) + P_2(z),\end{aligned}$$

where

$$P_1(z) := \sum_{j=0}^{m-1} a_j z^j + z^{m+lk}\sum_{j=0}^{r} b_j z^j m,$$

$$P_2(z) := z^{uk}\sum_{j=0}^{k-1} b_j z^j \frac{z^{(l+1)k}-1}{z^k-1} = Q(z) z^{uk}\frac{z^{(l+1)k}-1}{z^k-1}$$

with

$$Q(z) := \sum_{j=0}^{k-1} b_j z^j.$$

By Lemma 3.10 there is a kth root of unity $\xi = e^{i\tau}$ such that $\operatorname{Re}(Q(\xi)) \neq 0$. Then, for every $K > 0$ there is a $\delta \in (0, 2\pi/k)$ such that $\operatorname{Re}(P_2(e^{it}))$ oscillates between $-K$ and K at least $c_4(l+1)k\delta$ times on the interval $[\tau - \delta, \tau + \delta]$, where $c_4 > 0$ is a constant independent of n. Now we choose $\delta \in (0, 2\pi/k)$ for

$$K := 1 + \sum_{j=0}^{m-1} |a_j| + \sum_{j=0}^{k-1} |b_j|.$$

Then

$$T_n(t) := \mathrm{Re}\Big(\sum_{j=0}^{n} a_j e^{ijt}\Big) = \mathrm{Re}(P_1(e^{it})) + \mathrm{Re}(P_2(e^{it}))$$

has at least one zero on each interval on which $\mathrm{Re}(P_2(e^{it}))$ oscillates between $-K$ and K, and hence it has at least $c_{10}(l+1)k\delta > c_3 n$ zeros on $[-\pi,\pi)$, where $c_3 > 0$ is a constant independent of n. ■

4. Proofs of the theorems. We denote by $\mathcal{T}_k$ the set of all real trigonometric polynomials of degree at most k.

Proof of Theorem 2.1. Suppose the theorem is false. Then there are $k \in \mathbb{N}$, a strictly increasing sequence $(n_\nu)_{\nu=1}^{\infty}$ of positive integers, and even trigonometric polynomials $Q_{n_\nu} \in \mathcal{T}_k$ with maximum norm 1 such that T_{n_ν} has a sign change on the period $[-\pi,\pi)$ exactly at

$$t_{1,n_\nu} < t_{2,n_\nu} < \cdots < t_{m_\nu,n_\nu},$$

where m_ν are nonnegative even integers and $m_\nu \le k$ for each ν, and hence the even trigonometric polynomials $Q_{n_\nu} \in \mathcal{T}_k$ defined by

$$Q_{n_\nu}(t) := h_{n_\nu} \prod_{j=1}^{m_\nu} \sin \frac{t - t_{j,n_\nu}}{2}$$

with an appropriate choice of $h_{n_\nu} \in \mathbb{R}$ have maximum norm 1 on $[-\pi,\pi)$ and

$$T_{n_\nu}(t) Q_{n_\nu}(t) \ge 0, \qquad t \in \mathbb{R}. \tag{4.1}$$

Picking a subsequence of $(n_\nu)_{\nu=1}^{\infty}$ if necessary, we may assume that Q_{n_ν} converges to a $Q \in \mathcal{T}_k$ uniformly on $[-\pi,\pi)$. That is,

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0 \quad \text{with} \quad \varepsilon_\nu := \max_{t\in[-\pi,\pi]} |Q(t) - Q_{n_\nu}(t)|. \tag{4.2}$$

We introduce the notation

$$T_{n_\nu}(t) = \sum_{j=0}^{n_\nu} a_{j,\nu} \cos(jt),$$

and

$$T_{n_\nu}(t) Q(t)^3 = \Big(\sum_{j=0}^{n_\nu} a_{j,\nu} \cos(jt)\Big) Q(t)^3 = \sum_{j=0}^{K_\nu} b_{j,\nu} \cos(\beta_{j,\nu} t), \tag{4.3}$$

$$b_{j,\nu} \ne 0,\ j = 0,1,\dots,K_\nu,$$

and

$$(4.4)\qquad T_{n_\nu}(t)Q(t)^4 = \Big(\sum_{j=0}^{n_\nu} a_{j,\nu}\cos(jt)\Big)Q(t)^4 = \sum_{j=0}^{L_\nu} d_{j,\nu}\cos(\delta_{j,\nu}t),$$

$$d_{j,\nu} \neq 0,\ j = 0,1,\dots,L_\nu,$$

where $\beta_{0,\nu} < \beta_{1,\nu} < \cdots < \beta_{K_\nu,\nu}$ and $\delta_{0,\nu} < \delta_{1,\nu} < \cdots < \delta_{L_\nu,\nu}$ are nonnegative integers. Since the set $S^* := \{a_{j,\nu} : j \in \{0,1,\dots,n_\nu\},\ \nu \in \mathbb{N}\} \subset \mathbb{R}$ is finite, so are the sets

$$\big\{b_{j,\nu} : j \in \{0,1,\dots,K_\nu\},\ \nu \in \mathbb{N}\big\}, \qquad \big\{d_{j,\nu} : j \in \{0,1,\dots,L_\nu\},\ \nu \in \mathbb{N}\big\}.$$

Hence there are $\rho, M \in (0,\infty)$ such that

$$(4.5)\qquad |a_{j,\nu}| \le M, \quad j = 0,1,\dots,n_\nu,\ \nu \in \mathbb{N},$$

$$(4.6)\qquad \rho \le |b_{j,\nu}| \le M, \quad j = 0,1,\dots,K_\nu,\ \nu \in \mathbb{N},$$

$$(4.7)\qquad \rho \le |d_{j,\nu}| \le M, \quad j = 0,1,\dots,L_\nu,\ \nu \in \mathbb{N}.$$

As

$$T_{n_\nu}(t) = \sum_{j=0}^{n_\nu} a_{j,\nu}\cos(jt),$$

and S^* is finite, orthogonality and $Q_{n_\nu} \in \mathcal{T}_k$ imply that

$$(4.8)\qquad \int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt \le 2\pi M(k+1) \max_{t\in[-\pi,\pi]} |Q_{n,\nu}(t)| = 2\pi M(k+1),$$

where $M := \max\{|z| : z \in S^*\} \le 2\max\{|z| : z \in S\}$ depends only on S.

Observe that our indirect assumption together with Lemma 3.7 implies that

$$(4.9)\qquad \lim_{\nu\to\infty} K_\nu = \infty \quad\text{and}\quad \lim_{\nu\to\infty} L_\nu = \infty.$$

Indeed, if

$$\lim_{\nu\to\infty} K_\nu < \infty,$$

then Lemma 3.7 with $H \in \mathcal{P}_m$ defined by $H(e^{it}) := e^{imt/2}Q(t)^3$, $m := 6\deg(Q)$, while if

$$\lim_{\nu\to\infty} L_\nu < \infty,$$

then Lemma 3.7 with $H \in \mathcal{P}_m$ defined by $H(e^{it}) := e^{imt/2}Q(t)^4$, $m := 8\deg(Q)$, gives (2.2), that is, $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty$, which is the conclusion of the theorem, contradicting our indirect assumption that the theorem is false.

We claim that

$$(4.10)\qquad K_\nu \le c_5 L_\nu$$

with some $c_5 > 0$ independent of $\nu \in \mathbb{N}$. Indeed, using Parseval's formula, (4.2), (4.3), and (4.7) we deduce

$$(4.11)\qquad \frac{1}{\pi}\int_{-\pi}^{\pi} T_{n_\nu}(t)^2 Q(t)^4 Q_{n_\nu}(t)^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} (T_{n_\nu}(t)Q(t)^2 Q_{n_\nu}(t))^2\,dt \geq \frac{1}{2}\rho^2 K_\nu$$

for every sufficiently large $\nu \in \mathbb{N}$. Also, (4.1)–(4.8) imply

$$\begin{aligned}(4.12)\qquad & \frac{1}{\pi}\int_{-\pi}^{\pi} T_{n_\nu}(t)^2 Q(t)^4 Q_{n_\nu}(t)^2\,dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (T_{n_\nu}(t)Q_{n_\nu}(t))(T_{n_\nu}(t)Q(t)^4)Q_{n_\nu}(t)\,dt \\ &\leq \frac{1}{\pi}\Big(\int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt\Big)\Big(\max_{t\in[-\pi,\pi]} |T_{n_\nu}(t)Q(t)^4|\Big)\Big(\max_{t\in[-\pi,\pi]} |Q_{n_\nu}(t)|\Big) \\ &\leq \frac{1}{\pi}\Big(\int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt\Big) L_\nu M \Big(\max_{t\in[-\pi,\pi]} |Q_{n_\nu}(t)|\Big) \leq c_6 L_\nu\end{aligned}$$

with a constant $c_6 > 0$ independent of ν for every $\nu \in \mathbb{N}$. Now (4.10) follows from (4.11) and (4.12).

From Lemma 3.8 we deduce

$$(4.13)\qquad \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^4|\,dt \geq c_7 \rho \log L_\nu$$

with some constant $c_7 > 0$ independent of $\nu \in \mathbb{N}$. On the other hand, using (4.1), Lemma 3.9, (4.2), (4.4), (4.8), and (4.10), we obtain

$$\begin{aligned}(4.14)\qquad & \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^4|\,dt \\ &\leq \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,|Q_{n_\nu}(t)|\,dt + \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,|Q(t)-Q_{n_\nu}(t)|\,dt \\ &= \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q_{n_\nu}(t)|\,|Q(t)^3|\,dt + \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,|Q(t)-Q_{n_\nu}(t)|\,dt \\ &\leq \Big(\int_{-\pi}^{\pi} |T_{n_\nu}(t)Q_{n_\nu}(t)|\,dt\Big)\Big(\max_{t\in[-\pi,\pi]} |Q(t)|^3\Big) \\ &\quad + \Big(\int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,dt\Big)\Big(\max_{t\in[-\pi,\pi]} |Q(t)-Q_{n_\nu}(t)|\Big)\end{aligned}$$

$$= \Big(\int_{-\pi}^{\pi} T_{n_\nu}(t) Q_{n_\nu}(t)\, dt \Big) \Big(\max_{t \in [-\pi,\pi]} |Q(t)|^3 \Big) + \Big(\int_{-\pi}^{\pi} |T_{n_\nu}(t) Q(t)^3|\, dt \Big) \varepsilon_\nu$$

$$\le c_8 + c_9 (\log K_\nu) \varepsilon_\nu \le c_8 + c_9 (\log(c_5 L_\nu)) \varepsilon_\nu \le c_{10} + c_9 (\log L_\nu) \varepsilon_\nu,$$

where c_8, c_9, and c_{10} are constants independent of $\nu \in \mathbb{N}$, and $\varepsilon_\nu \to 0$ as $\nu \to \infty$. Combining (4.13) and (4.14), we obtain

$$c_7 \rho \log L_\nu \le c_{10} + c_9 (\log L_\nu) \varepsilon_\nu,$$

which contradicts (4.9). Hence our indirect assumption is false, and the theorem is true. ■

Proof of Corollary 2.2. Observe that (2.3) implies (2.1), and hence the corollary follows from Theorem 2.1. ■

Proof of Corollary 2.3. We will show that

$$\lim_{k \to \infty} \mathrm{NZ}(P_{2k}) = \infty, \tag{4.15}$$

$$\lim_{k \to \infty} \mathrm{NZ}(P_{2k+1}) = \infty. \tag{4.16}$$

Note that (4.15) is an obvious consequence of Theorem 2.1. To see (4.16) observe that if $P_{2k+1} \in \mathcal{P}_{2k+1}(S)$ are self-reciprocal then $\widetilde{P}_{2k+2}$ defined by

$$\widetilde{P}_{2k+2}(z) := (z+1) P_{2k+1}(z) \in \mathcal{P}_{2k+2} c(\widetilde{S}) \tag{4.17}$$

are also self-reciprocal, where the fact that S is finite implies that the set

$$\widetilde{S} := \{ s_1 + s_2 : \ s_1, s_2 \in S \cup \{0\} \} \subset \mathbb{R} \tag{4.18}$$

is also finite. Furthermore, (2.4) implies

$$\lim_{k \to \infty} |\widetilde{P}_{2k+2}(1)| = \lim_{k \to \infty} 2 |P_{2k+1}(1)| = \infty.$$

Hence the polynomials $\widetilde{P}_{2k+2} \in \mathcal{P}_{2k+2}(\widetilde{S})$ satisfy the assumptions of Corollary 2.2, and it follows that

$$\lim_{k \to \infty} \mathrm{NZ}(\widetilde{P}_{2k+2}) = \infty. \tag{4.19}$$

Combining this with

$$\mathrm{NZ}(P_{2k+1}) = \mathrm{NZ}(\widetilde{P}_{2k+2}) - 1 \tag{4.20}$$

we obtain (4.16). Finally, (4.15) and (4.16) give (2.5). ■

Proof of Corollary 2.4. If (2.7) holds and the finite set $S \subset \mathbb{R}$ has property (2.6), then assumption (2.1) is satisfied, and (2.2) follows from Theorem 2.1. ■

Proof of Corollary 2.5. Corollary 2.4 implies (4.15) and (4.16). For (4.15) this is obvious. To see (4.16) observe that if $P_{2k+1} \in \mathcal{P}_{2k+1}(S)$ are self-reciprocal, then $\widetilde{P}_{2k+2}$ defined by (4.17) are also self-reciprocal, where the fact that S is finite implies that the set $\widetilde{S} \subset \mathbb{R}$ defined by (4.18) is also finite.

It is easy to see that the fact that S satisfies (2.6) implies that so does $\widetilde{S}$. Similarly, (2.6) implies that $\mathrm{NC}(\widetilde{P}_{2k+2}) \geq \mathrm{NC}(P_{2k+1})$. Combining this with (2.8) leads to

$$\lim_{k\to\infty} \mathrm{NC}(\widetilde{P}_{2k+2}) = \infty.$$

Hence the polynomials $\widetilde{P}_{2k+2} \in \mathcal{P}_{2k+2}(\widetilde{S})$ defined by (4.17) satisfy the assumptions of Corollary 2.4, and (4.19) follows. Combining this with (4.20) we obtain (4.16). From (4.15) and (4.16) we deduce (2.5), the conclusion of the corollary. ■

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