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# PL($M$) admits no Polish group topology

by

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**Abstract.** We show that the group of piecewise linear homeomorphisms of any compact PL manifold does not admit a Polish group topology, using both general results on topologies on groups of homeomorphisms, and results on the algebraic structure of PL homeomorphism groups. The proof also shows that the group of piecewise projective homeomorphisms of $S^1$ has no Polish topology.

**1. Introduction.** Let $M$ be a compact topological manifold, and let Homeo($M$) denote its group of self-homeomorphisms. There is a remarkable interplay between the algebraic structure of Homeo($M$), the possible group topologies on Homeo($M$), and the topology of $M$ itself. For example, in [Kal86] Kallman uses this to show that many "large" subgroups of Homeo($M$) admit a unique Polish group topology. Other instances of this algebraic-topological relationship appear in the main results of [Fil82], [Man15], [Hur15], and [Man16], all of which derive topological statements (e.g. continuity of group homomorphisms) from purely algebraic constraints.

In this note, we develop this relationship further, giving additional information on possible topologies on subgroups of Homeo($M$) (Proposition 2.3 and Theorem 1.2 below). Though these results are general, our primary motivation is to understand the group PL($M$) of piecewise linear homeomorphisms of PL manifolds. Recall that an orientation-preserving homeomorphism $f$ of the $n$-cube $I^n$ is *piecewise linear* if there exists a subdivision of $I^n$ into finitely many linear simplices such that the restriction of $f$ to each simplex is an affine linear homeomorphism onto its image. A manifold $M$ has a *PL structure* if it is locally modeled on $(I^n, \mathrm{PL}(I^n))$, in which case PL($M$) is the automorphism group of this structure. This group is interesting for many reasons, including its rich algebraic structure (see e.g. [BS85], [CR15])

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and its relationship with the Thompson groups. There has also been significant historical interest in how to best topologize $\mathrm{PL}(M)$. For instance, in [Las65, Problems 39–40], three different topologies are proposed by Milnor, Stasheff, and Wall, and the choice of most appropriate topology appears to be unresolved.

We ask here if $\mathrm{PL}(M)$ admits a Polish group topology. This question is inherently interesting from the perspective of descriptive set theory (see Rosendal's work on non-Polishable groups in [Ros05] and references therein) but also interesting from the perspective of transformation groups. Much like the group $\mathrm{Diff}(M)$ of diffeomorphisms of a manifold, $\mathrm{PL}(M)$ is not complete in—and arguably not best described by—the compact-open topology inherited from $\mathrm{Homeo}(M)$. However $\mathrm{Diff}(M)$ does have a different topology, the standard $C^\infty$ topology, that makes it a Polish group. It is natural to ask whether $\mathrm{PL}(M)$ might as well.

In [CK16], Cohen and Kallman showed that $\mathrm{PL}(I)$ and $\mathrm{PL}(S^1)$ have no Polish group structure. However, their proof uses 1-dimensionality in an essential way. Here we follow a different strategy, giving an elementary and independent proof of the following stronger result.

THEOREM 1.1. *Let $M$ be a PL manifold. Then* $\mathrm{PL}(M)$ *does not admit a Polish group topology.*

In fact, we will show that $\mathrm{PL}(M)$ is *very far* from being a Polish group. In the case of $M = I$, this distinction is easy to summarize: while $\mathrm{PL}(I)$ is known to contain no free subgroups, our structure results imply that, for many groups of homeomorphisms including $\mathrm{PL}(I)$, the generic pair of elements with respect to *any* Polish group topology generate a free subgroup. Precisely, we prove the following.

THEOREM 1.2. *Let $G \subset$* $\mathrm{Homeo}(M)$ *be a group satisfying a "local perturbations property"* (*cf. Definition* 2.2 *below*). *If $\tau$ is any Polish group topology on $G$, then the generic pair $(f, g)$ in $G \times G$ with the product topology generates a free subgroup.*

Loosely speaking, the *local perturbations property* is a statement that there exist many homeomorphisms supported on small sets and close to the identity in $G$. While $\mathrm{PL}(M)$ contains many homeomorphisms supported on small sets, the application of Theorem 1.2 is not completely straightforward, as these need not a priori be close to the identity in any Polish topology on $\mathrm{PL}(M)$. However, our results on structure theory of homeomorphism groups, following observations of Kallman, show that this is indeed the case.

The situation is more complicated for higher-dimensional manifolds, since there are many examples of free subgroups in $\mathrm{PL}(M)$ as soon as $M$ has dimension at least 2. (See Section 3.) However, we are able to produce a proof much in the same spirit as the PL($I$) case: we describe a natural subgroup of PL($M$), show that this subgroup necessarily inherits any Polish group structure from PL($M$), and then show that the subgroup is both large enough to have the local perturbations property and small enough to contain no free subgroup.

Our strategy also applies to other transformation groups, such as the group of *piecewise projective* homeomorphisms of $S^1$ discussed in [Mon13]. In Section 4.1 we show the following.

COROLLARY 1.3. *The group of piecewise projective homeomorphisms of $S^1$ admits no Polish topology.*

One might interpret these results as explaining *why* we have yet to settle on a choice of "best" topology for PL($M$). It is suspected that several other transformation groups, such as the group of bi-Lipschitz homeomorphisms of a manifold, or diffeomorphisms of intermediate regularity on a smooth manifold, also fail to admit a Polish group topology. Progress is made in [CK16] for the one-dimensional case in some categories; it would be interesting to extend these results to higher-dimensional manifolds as well.

**2. Structure theory of transformation groups.** As before, let $M$ denote a compact topological manifold. In this section we discuss general constraints on topologies on subgroups $G \subset \mathrm{Homeo}(M)$.

Surprisingly, it is often the case that any choice of reasonable topology on $G$ (even without requiring that the point evaluation maps $G \to M$ given by $g \mapsto g(x)$ be continuous) is forced to reflect the topology of $M$. The most basic result along these lines is the following lemma of Kallman.

Say that $G \subset \mathrm{Homeo}(M)$ has *property* $(*)$ if the following holds:

$(*)$ for each nonempty open set $U \subset M$, there exists a nonidentity map $g_U \in G$ that fixes $M \setminus U$ pointwise.

LEMMA 2.1 (Kallman [Kal86]). *Let $G \subset \mathrm{Homeo}(M)$ have property $(*)$. If $\tau$ is any Hausdorff group topology on $G$, then each set of the form*

$$C(U,V) := \{f \in G \mid f(\overline{U}) \subseteq \overline{V}\}$$

*is closed in $(G,\tau)$.*

A proof of Lemma 2.1 for the case of $M = I$ is given in [CK16, Lemma 2.2], but it applies equally well in the general case. The outline is as follows: first, one uses property $(*)$ to show that $C(U,V)$ is the intersection of all sets of the form

$$F(U',W') := \{f \in \mathrm{Homeo}(M) \mid f g_{U'} f^{-1} \text{ commutes with } g_{W'}\}$$

where $U'$ is open in $U$ and $W'$ is open in $M \setminus V$. Now each $F(U',W')$ is closed, since it is the pre-image of the identity under the (continuous) commutator map $\mathrm{Homeo}(M) \to \mathrm{Homeo}(M)$ given by $f \mapsto [f g_{U'} f^{-1}, g_{W'}]$.

We will work with the following strengthening of condition $(*)$. Note that this strengthening is satisfied by $\mathrm{PL}(M)$, as well as many other familiar transformation groups such as $\mathrm{Diff}(M)$, the group of volume-preserving diffeomorphisms of a Riemannian manifold, etc.

DEFINITION 2.2. Say that $G \subset \mathrm{Homeo}(M)$ satisfies the *local perturbations property* if, for each open set $U \subset M$ and $y \in U$, the set $\{f(y) \mid f|_{M\setminus U} = \mathrm{id}\}$ is uncountable.

We think of such $f$ as a "perturbation" of $y$. The word *local* refers to the fact that we can find perturbations supported on (i.e. pointwise fixing the complement of) any open neighborhood $U$ of $y$. In other words, $f$ has local support with respect to the topology of $M$.

The next proposition says that these perturbations are also inherently local in a group-theoretic sense, namely, they can be found close to the identity in $G$.

PROPOSITION 2.3 (Local perturbations are local). *Suppose $G \subset \mathrm{Homeo}(M)$ has the local perturbations property and $\tau$ is any separable, metrizable topology on $G$. Then for any open set $U \subset M$, point $y \in U$, and open neighborhood $N$ of the identity in $(G, \tau)$, there exists $f \in N$ such that $f(y) \neq y$ and $f|_{M\setminus U} = \mathrm{id}$.*

*Proof.* Given an open $U \subset M$ and $y \in U$, let $H = \{f \mid f|_{M\setminus U} = \mathrm{id}\}$. Since $(G, \tau)$ is separable and metrizable, the subset topology on $H$ is also separable.

We now claim that, for any neighborhood $N$ of the identity in $G$, the neighborhood $N \cap H$ of the identity in $H$ contains some $f$ such that $f(y) \neq y$. To see this, let $\{h_i \mid i \in \mathbb{N}\}$ be a countable dense subset of $H$. Then

$$H = \bigcup_{i=1}^{\infty} h_i(N \cap H).$$

If we had $h(y) = y$ for all $h \in N \cap H$, then the set of images $\{f(y) \mid f \in H\} = \{h_i(y)\}$ would be countable, and this contradicts the local perturbations property. ■

Proposition 2.3 readily generalizes to groups of homeomorphisms fixing a submanifold or, in the case where $\partial M \neq 0$, those fixing the boundary of $M$. For simplicity, we state only the boundary case. Let $\mathrm{Homeo}(M, \partial)$ denote the group of homeomorphisms of $M$ that fix $\partial M$ pointwise.

PROPOSITION 2.4. *Suppose that $G \subset \mathrm{Homeo}(M, \partial)$ has a separable, metrizable topology $\tau$, and suppose that the condition in the local perturbations property is satisfied for every point $y$ in the interior of $M$. Then for* *any closed set* $X \subset M$*, interior point* $y \notin X$*, and open neighborhood* $N$ *of the identity in* $(G, \tau)$*, there exists* $f \in N$ *such that* $f(x) = x$ *for all* $x \in X$*, but* $f(y) \neq y$*.*

**2.1. Generic free subgroups.** Using Proposition 2.3, we prove Theorem 1.2 on generic free subgroups. Recall that this is the statement that, for any group $G \subset \mathrm{Homeo}(M)$ with the local perturbations property, and any Polish group topology $\tau$ on $G$, the $\tau$-generic pair $(f, g) \in G \times G$ generates a free group. In fact, our technique here can be modified to show something even stronger: for any point $y \in M$, the generic pair $(f, g)$ generates a free group acting on $M$ such that $y$ has trivial stabilizer ([1]).

*Proof of Theorem 1.2.* Assume that $G$ has a Polish topology, so in particular $G \times G$ is a Baire space. For each nontrivial, reduced word $w \in F_2$, define a set $X_w := \{(f, g) \in G \times G \mid w(f, g) = \mathrm{id}\}$. This is the pre-image of the singleton $\{\mathrm{id}\}$ under the continuous map $G \times G \to G$ given by $(f, g) \mapsto w(f, g)$, so it is closed. We will show that $X_w$ has empty interior. Given this, the generic pair $(f, g)$ does not lie in any $X_w$, and hence generates a free group.

Assume for contradiction that some $X_w$ has nonempty interior and choose $(f, g) \in \mathrm{int}(X_w)$. Write $w(f, g) = t_k \dots t_1$ as a reduced word with each $t_i$ in $\{f^{\pm 1}, g^{\pm 1}\}$. Now choose any point $y_0 \in M$, and let $y_i = t_i \dots t_1(y_0)$. Let $m$ be the minimum integer such that the points $y_0, y_1, \dots, y_m$ are *not* all distinct. Since $w(f, g)$ is the identity, we have $y_k = y_0$ and so $1 \le m \le k$. Let $U$ be a small neighborhood of $y_{m-1}$ disjoint from $\{y_0, y_1, \dots, y_{m-2}\} \setminus \{y_m\}$, and such that $t_m(U)$ is also disjoint from $\{y_0, y_1, \dots, y_{m-1}\} \setminus \{y_m\}$. By Proposition 2.3, for any neighborhood $N$ of the identity in $G$, there exists $h \in N$ such that $h(y_{m-1}) \neq y_{m-1}$ and $h$ restricts to the identity on the complement of $U$. Modify $t_m$ (which is either $f$, $g$, $f^{-1}$, or $g^{-1}$) by replacing it with $t_m \circ h$, and leaving the other free generator unchanged. This gives a new pair $(f', g')$ that still lies in the interior of $X_w$, provided that $N$ was chosen small enough.

We claim that, after this modification, the images of $y_0$ under the first $m$ initial strings of $w(f', g')$—adapting the previous notation, these are the points $t'_i \dots t'_1(y)$ for $0 \le i \le m$, where $t'_i \in \{(f')^{\pm 1}, (g')^{\pm 1}\}$—are now all distinct. In fact, we will have $t'_i \dots t'_1(y) = t_i \dots t_1(y)$ for each $i < m - 1$. To see this, note that for each generator $t_i$, we have $t_i(y_{i-1}) = y_i$, except in the (intended) case $i = m$, or possibly if $t_{m-1} = t_m^{-1}$, in which case $t'_{m-1} = h^{-1} t_{m-1}$ and we would have $t'_{m-1}(y_{m-2}) = h^{-1}(y_{m-1})$. But the latter case is excluded by requiring that $w$ be a reduced word. As $t'_m(y_{m-1}) \neq y_m$ and $t'_m(y_{m-1}) \in t_m(U)$, this shows that the points $t'_i \dots t'_1(y)$ are all distinct.

([1]) Similar results for $G = \mathrm{Homeo}(\mathbb{R})$ and $G = \mathrm{Homeo}(S^1)$, *assuming that the homeomorphism groups are given the compact-open topology*, appear in [BK86] and [Ghy01]. Here more care is needed since we know much less about the topology of $G$.

If $w(f', g') \neq \mathrm{id}$, we are already done. Otherwise, we may repeat the procedure, again perturbing a generator in the neighborhood of the first repeated point in the sequence of images of $y_0$ under initial subwords of $w(f', g')$. The process terminates when we arrive at some pair $(f^{(k)}, g^{(k)})$ in the interior of $X_w$ satisfying either $w(f^{(k)}, g^{(k)}) \neq \mathrm{id}$ or the more specific condition $w(f^{(k)}, g^{(k)})(y_0) \neq y_0$. This contradicts the definition of $X_w$. ■

REMARK 2.5 (Relative case of Theorem 1.2). If we use Proposition 2.4 in place of Proposition 2.3, the proof above shows that whenever $G \subset \mathrm{Homeo}(M, \partial)$ has a Polish group topology and satisfies the local perturbations property, then the generic pair of elements of $G$ generates a free group.

**3. Free groups in** $\mathrm{PL}(I^n)$**.** Having found many free groups in subgroups of $\mathrm{Homeo}(M)$, our next goal is to show that there are relatively few in $\mathrm{PL}(M)$. Throughout this section, we assume $M$ is oriented, and work with the index two subgroup $\mathrm{PL}_+(M) \subset \mathrm{PL}(M)$ of orientation-preserving PL homeomorphisms of $M$. We use the following result of Brin and Squier.

LEMMA 3.1 ([BS85]). $\mathrm{PL}_+(I)$ *contains no free subgroup. More specifically, the subgroup generated by any* $f, g \in \mathrm{PL}_+(I)$ *is either abelian or contains a copy of* $\mathbb{Z}^\infty$.

The proof is not hard, for completeness we give a sketch here. Recall the standard notation $\mathrm{supp}(f)$ for the *support* of $f$, which is the closure of the set $\{x \in M \mid f(x) \neq x\}$.

*Proof of Lemma 3.1.* Let $f, g \in \mathrm{PL}_+(I)$. Suppose $x \in I$ is fixed by both $f$ and $g$. Then the left and right derivatives (note that one-sided derivatives are always defined for PL homeomorphisms) of the commutator $[f, g]$ at $x$ are both equal to 1, and it follows that $[f, g]$ is the identity on a neighborhood of $x$. This shows that $\mathrm{supp}([f, g])$ is contained in $I \setminus (\mathrm{fix}(f) \cap \mathrm{fix}(g))$. Assuming that the subgroup generated by $f$ and $g$ is not abelian, let $W$ then be the (nonempty) subgroup of $\mathrm{PL}_+(I)$ consisting of all homeomorphisms $w$ such that $\mathrm{supp}(w)$ is nonempty and contained in $I \setminus (\mathrm{fix}(f) \cap \mathrm{fix}(g))$.

Choose some $w \in W$ such that $\mathrm{supp}(w)$ meets a *minimum* number of connected components of the set $I \setminus (\mathrm{fix}(f) \cap \mathrm{fix}(g))$. Let $A$ be a connected component of $I \setminus (\mathrm{fix}(f) \cap \mathrm{fix}(g))$ that meets $\mathrm{supp}(w)$. Let $a$ and $b$ denote $\min\{\mathrm{supp}(w) \cap A\}$ and $\max\{\mathrm{supp}(w) \cap A\}$ respectively.

As $\sup\{h(a) \mid h \in \langle f, g\rangle\}$ is fixed by $f$ and $g$, there exists some $h \in \langle f, g\rangle$ with $h(a) > b$. It follows that $hwh^{-1}$ and $w$ have disjoint supports on $A$, so $[hwh^{-1}, w]$ restricts to the identity on $A$. Since $\mathrm{supp}(w)$ was assumed to meet a minimum number of connected components of $I \setminus (\mathrm{fix}(f) \cap \mathrm{fix}(g))$, we must have $\mathrm{supp}([hwh^{-1}, w]) = \emptyset$, i.e. $hwh^{-1}$ and $w$ commute. This process can be iterated, leading to $h_n$ that displaces the support of $w$ off of $\bigcup_{i<n} \mathrm{supp}(h_i w h_i^{-1}) \cap A$, giving a copy of $\mathbb{Z}^\infty$ in $\langle f, g\rangle$. ∎

By contrast, as soon as $\dim(M) = n \geq 2$, the groups $\mathrm{PL}_+(M)$ and $\mathrm{PL}_+(M, \partial)$ contain many free subgroups. A number of examples are given in [CR15]; the easiest ones are the following.

EXAMPLE 3.2. Let $n \geq 2$, and consider a free subgroup of $\mathrm{GL}_n(\mathbb{R})$ freely generated by $\alpha$ and $\beta$. For any $p \in M$, there exist orientation-preserving PL homeomorphisms $f$ and $g$, fixing $p$, supported on a neighborhood of $p$, and with derivatives $Df(p) = \alpha$ and $Dg(p) = \beta$. Taking the derivative at $p$ defines an injective homomorphism from the group generated by $f$ and $g$ to a free subgroup of $\mathrm{GL}_n(\mathbb{R})$, hence $f$ and $g$ satisfy no relation.

We now show how to exclude these free subgroups by restricting attention to the subgroup of PL homeomorphisms that preserve a 1-dimensional foliation.

DEFINITION 3.3. Let $\mathrm{PL}_+(I^n, \mathcal{F})$ denote the subgroup of $\mathrm{PL}_+(I^n)$ consisting of homeomorphisms that preserve each leaf of the foliation of $I^n$ by vertical lines $\{x\} \times I$, where $x \in I^{n-1}$.

PROPOSITION 3.4. $\mathrm{PL}_+(I^n, \mathcal{F})$ *does not contain a free subgroup.*

*Proof.* Let $f, g \in \mathrm{PL}_+(I^n, \mathcal{F})$. The restrictions of $f$ and $g$ to any vertical line $L = \{x\} \times I$ are orientation-preserving piecewise linear homeomorphisms of $L$. Fixing $L$, restriction gives a homomorphism $\langle f, g\rangle \to \mathrm{PL}_+(L) \cong \mathrm{PL}_+(I)$ whose image, by Lemma 3.1, is either abelian or contains a copy of $\mathbb{Z}^\infty$. In particular, the kernel of this homomorphism contains two nonconjugate (i.e. noncommuting) elements in $F_2$. In other words, there exist nontrivial words $u$ and $v$ in the letters $f$ and $g$ such that $u$ and $v$ restrict to the identity on $L$, and the commutator $[u, v]$ is not the trivial word.

Notice that if $u \in \mathrm{PL}_+(I^n, \mathcal{F})$ restricts to the identity on $L$, then after identifying $L$ with the $n$th coordinate axis, $u$ is locally linear of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & a_{n-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

and the collection of all such linear maps forms an abelian group. In particular, the commutator $[u, v]$ agrees with the identity on a *neighborhood* of $L$. Shrinking this neighborhood if needed, we may take it to be of the form $U \times I$, where $U$ is a neighborhood of $x$ in $I^{n-1}$.

Now consider the collection of all open sets of the form $U' \times I$ such that

- $U'$ is open in $I^{n-1}$, and
- there exists a nontrivial reduced word $w$ in $f$ and $g$ with $w|_{U' \times I} = \mathrm{id}$.

The argument given above shows that this collection of sets forms an open cover of $I^n$. Let $\{U_1 \times I, \dots, U_m \times I\}$ be a finite subcover of *minimal* cardinality, and for each $1 \leq i \leq m$ let $w_i$ be a nontrivial word that restricts to the identity on $U_i \times I$. We claim that $m = 1$, and therefore $f$ and $g$ satisfy a nontrivial relation.

To see that $m = 1$, assume for contradiction that we have more than one set in the cover and choose $i$ and $j$ such that $U_i \cap U_j \neq \emptyset$. If $[w_i, w_j]$ reduces to the trivial word, then $w_i$ and $w_j$ would both be powers of some word $w'$. Since $\mathrm{PL}_+(I)$ is torsion-free, this implies that $w'$ restricts to the identity on both $U_i$ and $U_j$, so we could replace our cover with a smaller one, using the single set $(U_i \cup U_j) \times I$ on which $w'$ is the identity. Thus, the assumption of minimal cardinality of the cover implies that $[w_i, w_j]$ is a nontrivial word. However, since $w_i$ pointwise fixes $U_i \times I$ and $w_j$ pointwise fixes $U_j \times I$, the commutator $[w_i, w_j]$ restricts to the identity on $(U_i \cup U_j) \times I$, and this again contradicts our choice of a minimal cover. ∎

**4. Completing the proof.** Combining the results from the previous two sections, we now prove Theorem 1.1, starting with the special case $M = I^n$. Suppose for contradiction that $\mathrm{PL}(I^n)$ admits a Polish topology; its restriction to $\mathrm{PL}_+(I^n)$ then gives a Polish topology on $\mathrm{PL}_+(I^n)$. Let $\mathrm{PL}_+(I^n, \mathcal{F})$ be the subgroup of vertical-line-preserving homeomorphisms defined in the previous section. We claim that $\mathrm{PL}_+(I^n, \mathcal{F})$ is a closed subgroup, and hence Polish. This is a consequence of the following general lemma.

LEMMA 4.1. *Let $M = A \times B$ be a product manifold. Let $G \subset \mathrm{Homeo}_+(M)$ be a subgroup satisfying condition $(*)$, and $\tau$ a Hausdorff group topology on $G$. Then*

$$G(B) := \{f \in G \mid f(\{a\} \times B) = \{a\} \times B \text{ for all } a \in A\}$$

*is a closed subgroup.*

*Proof.* We show that $G(B)$ is an intersection of sets of the form $C(U, V)$, hence is closed by Lemma 2.1. Let $\Lambda$ be the collection of all open sets in $M$ of the form $\{U'\} \times B$, where $U'$ is open in $A$. Then

$$G(B) = \bigcap_{U \in \Lambda} C(U, U).$$

For one inclusion, if $f \in \mathrm{Homeo}_+(M)$ satisfies $f(\{U'\} \times B) \subset \{U'\} \times B$ for each $U'$ in a neighborhood basis of $a \in A$, then $f(\{a\} \times B) = \{a\} \times B$. This shows that $\bigcap_{U \in \Lambda} C(U, U) \subset G(B)$. The reverse inclusion is immediate. ∎

To continue the proof of the main theorem, note that $\mathrm{PL}_+(I^n, \mathcal{F})$ also satisfies the local perturbations property—for example, given $y \in M$ and any neighborhood $U$ of $y$, one can define for each $t \in (0, \epsilon)$ an element of $\mathrm{PL}_+(I^n, \mathcal{F})$ supported on $U$, and agreeing with the map $(x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n + t)$ on a small linear simplex containing $y$. Thus, by Theorem 1.2, the generic $f, g \in \mathrm{PL}_+(I^n, \mathcal{F})$ generate a free subgroup. But this contradicts Proposition 3.4. We conclude that $\mathrm{PL}(I^n)$ has no Polish topology.

This strategy also works to show that the group $\mathrm{PL}(I^n, \partial)$ of piecewise linear homeomorphisms of $I^n$ that pointwise fix the boundary admits no Polish group topology. In detail, the proof of Lemma 4.1 shows that the subgroup of homeomorphisms in $\mathrm{PL}(I^n, \partial) = \mathrm{PL}_+(I^n, \partial)$ that preserve each vertical line is closed, hence Polish. It also satisfies the (relative) local perturbations property. Now Remark 2.5 implies that the generic pair of elements generates a free group, which is again a contradiction.

For the general case, let $M$ be an $n$-dimensional PL manifold, and assume again for contradiction that $\mathrm{PL}(M)$ has a Polish group topology. Let $A \subset M$ be a linearly embedded copy of $I^n$, and let $G \subset \mathrm{PL}(M)$ be the subgroup of homeomorphisms that restrict to the identity on $M \setminus A$. We claim that

$$G = \bigcap_{U' \subset M \setminus \bar{A} \text{ open}} C(U', U'),$$

and hence $G$ is a closed subgroup. That $G \subset C(U', U')$ for any $U' \subset M \setminus \bar{A}$ is immediate. To see the reverse inclusion, take any point $x \in M \setminus \bar{A}$. If $f(U') \subset U'$ for all sets $U'$ in a neighborhood basis of $x$, then $f(x) = x$.

Since $G$ is closed, it is also a Polish group. As $G \cong \mathrm{PL}(I^n, \partial)$, this contradicts the case proved above, and completes the proof of Theorem 1.1. ∎

**4.1. Piecewise projective homeomorphisms.** A homeomorphism $f$ of $S^1$ is *piecewise projective* if there is a partition of $S^1$ into finitely many intervals such that the restriction of $f$ to each interval agrees with the standard action of $\mathrm{PSL}(2, \mathbb{R})$ by projective transformations on $\mathbb{R}P^1 = S^1$. Much like $\mathrm{PL}(M)$, this group has a rich algebraic structure: among its subgroups are counterexamples to the von Neumann conjecture (see [Mon13]), and the full group is closely related to a proposed "Lie algebra" for the group $\mathrm{Homeo}(S^1)$ given in [MP98].

We now prove Corollary 1.3, the analog of Theorem 1.1 for piecewise projective homeomorphisms. Let $G$ denote the group of all piecewise projective homeomorphisms of $S^1$, let $I \subset S^1$ be a small interval, and let $H \subset G$ be the subgroup of homeomorphisms pointwise fixing $I$. Suppose that $G$ is given a Polish group topology. Since

$$H = \bigcap_{U \text{ open}, U \subset I} C(U, U),$$

$H$ is a closed subgroup, hence Polish.

Note also that $H$ has the local perturbations property (in the modified sense for groups of homeomorphisms fixing a submanifold), so it follows from Theorem 1.2 that the generic pair of elements of $H$ generate a free group. However, the same argument as in Lemma 3.1 shows that the subgroup generated by any two elements of $H$ is either metabelian or contains a copy of $\mathbb{Z}^\infty$; in particular, it is not free (this is also proved in [Mon13, Theorem 14]). This gives a contradiction, showing that $G$ cannot have a Polish group topology. ∎

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## References

[BK86] A. Blass and J. Kister, *Free subgroups of the homeomorphism group of the reals*, Topology Appl. 24 (1986), 243–252.

[BS85] M. Brin and C. Squier, *Groups of piecewise linear homeomorphisms of the real line*, Invent. Math. 79 (1985), 485–498.

[CR15] D. Calegari and D. Rolfsen, *Groups of PL homeomorphisms of cubes*, Ann. Fac. Sci. Toulouse Math. (6) 24 (2015), 1261–1292.

[CK16] M. Cohen and R. Kallman, $\mathrm{PL}_+(I)$ *is not a Polish group*, Ergodic Theory Dynam. Systems 36 (2016), 2121–2137.

[Fil82] R. Filipkiewicz, *Isomorphisms between diffeomorphism groups*, Ergodic Theory Dynam. Systems 2 (1982), 159–171.

[Ghy01] É. Ghys, *Groups acting on the circle*, Enseign. Math. 47 (2001), 329–407.

[Hur15] S. Hurtado, *Continuity of discrete homomorphisms of diffeomorphism groups*, Geom. Topol. 19 (2015), 2117–2154.

[Kal86] R. Kallman, *Uniqueness results for homeomorphism groups*, Trans. Amer. Math. Soc. 295 (1986), 389–396.

[Las65] R. Lashof, *Problems in differential and algebraic topology. Seattle conference, 1963*, Ann. of Math. (2) 81 (1965), 565–591.

[MP98] F. Malikov and R. Penner, *The Lie algebra of homeomorphisms of the circle*, Adv. Math. 140 (1998), 282–322.

[Man15] K. Mann, *Homomorphisms between diffeomorphism groups*, Ergodic Theory Dynam. Systems 35 (2015), 192–214.

[Man16] K. Mann, *Automatic continuity for homeomorphism groups and applications* (with an appendix by F. Le Roux and K. Mann), Geom. Topol. 20 (2016), 3033–3056.

[Mon13] N. Monod, *Groups of piecewise projective homeomorphisms*, Proc. Nat. Acad. Sci. USA 110 (2013), 4524–4527.

[Ros05] C. Rosendal, *On the non-existence of certain group topologies*, Fund. Math. 187 (2005), 213–228.

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