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On Intersections of Generic Perturbations of Definable Sets

by

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Summary. Consider an o-minimal expansion $\mathcal{R}$ of a real closed field R and two definable sets E and M. We introduce concepts of a *locally transitive* (abbreviated to l.t.) and a *strongly locally transitive* (abbreviated to s.l.t.) action of E on M. In the former case, M is supposed to be of pure dimension m; in the latter, both M and E are supposed to be of pure dimension. We treat the elements of E as perturbations of the set M. We prove that if E acts l.t. on M, and A and B are two non-empty definable subsets of M of dimension $\dim A \leq \dim B < \dim M$, then

$$\dim(\sigma(A) \cap B) < \dim A$$

for a generic σ in E; here $\dim \emptyset = -1$. And if E acts s.l.t. on M and A and B are two definable subsets of M, then

$$\dim(\sigma(A) \cap B) \leq \max\{\dim A + \dim B - m, -1\}$$

for a generic σ in E. We give an example of a l.t. action E on M for which the latter conclusion of the intersection theorem fails. We also prove a theorem on the intersections of generic perturbations in terms of the exceptional set $T \subset M$ of points at which E is not l.t. Finally, we provide some natural conditions which imply that T is a nowhere dense subset of M.

1. Introduction and main results. Consider an o-minimal expansion $\mathcal{R}$ of a real closed field R and two definable (with parameters) subsets E and M of R^n. Let $\dim E = e$ and assume that M is of pure dimension m, i.e. the dimension of M at each point $x \in M$ is m. Examples of such sets are, for instance, definable topological manifolds (possibly with boundary). In this paper we set $\dim \emptyset = -1$. We shall investigate continuous definable

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maps

$$\alpha : E \times M \to M, \qquad \alpha(\sigma, x) = \sigma \cdot x = \sigma(x),$$

which we call *actions* of E on M. We treat the elements of E as perturbations of the set M. For subsets $X \subset E$ and $Y \subset M$, write

$$X \cdot Y := \{x \cdot y \in M : x \in X,\ y \in Y\}.$$

It will be often convenient to think of the points $x \in M$ as maps from E into M. Thus we set

$$\alpha^x : E \ni \sigma \mapsto \sigma \cdot x \in M.$$

We say that α or E is *locally transitive* (abbreviated to l.t.) at a point $x \in M$ if $\Omega \cdot x = \alpha^x(\Omega)$ is a subset of M of dimension m for every non-empty, open subset Ω of E. Of course, if E is l.t. at x, then $e \geq m$.

REMARK 1.1. It is not difficult to prove that E is l.t. at $x \in M$ iff the map α^x is generically a submersion, i.e. there exists a nowhere dense subset F of E such that the restriction of α^x to $E \setminus F$ is a submersion of class $\mathcal{C}^1$. Indeed, this follows directly from the fact that there exists a finite definable stratification of E of class $\mathcal{C}^1$ such that α^x is a $\mathcal{C}^1$ map on each stratum.

The set E is called l.t. on a subset A of M if it is l.t. at every point $x \in A$. Locally transitive actions can be characterized in terms of the rank of definable maps defined below.

Let $f : V \to W$ be a definable map between definable subsets of R^n. For any $x \in R^n$ and $r > 0$, denote by $B(x, r)$ the ball with center x and radius r. The function

$$V \times (0, \infty) \ni (x, r) \mapsto \dim f(V \cap B(x, r)) \in \mathbb{N}$$

is definable, because the dimension of fibers from a definable family depends definably on the parameters (cf. [1, Chap. 4]). Consequently, for a fixed $x \in V$, the function $\dim f(V \cap B(x, r))$ is constant for $r > 0$ small enough. Its common value $r_x f$ near zero will be called the *rank* of f at x; $r_x f$ is a definable function of the variable x. It is clear that E is l.t. at x iff the map α^x is of constant rank m.

Suppose now that the definable set E is also of pure dimension e. We say that E is *strongly locally transitive* (abbreviated to s.l.t.) at a point $x \in M$ if the fibers of the map α^x are of dimension $\leq e - m$. An easy dimension calculus, based on the following proposition (see e.g. [1, Chap. 4, Prop. 1.5]), shows that E is l.t. at x if E is s.l.t. at x.

PROPOSITION 1.2. *Let $f : V \to W$ be a definable map between non-empty definable sets. Then*

$$\dim f^{-1}f(v) \leq k \text{ for all } v \in V \ \Rightarrow\ \dim V \leq k + \dim f(V),$$
$$\dim f^{-1}f(v) \geq k \text{ for all } v \in V \ \Rightarrow\ \dim V \geq k + \dim f(V).$$

The set E is called s.l.t. on a subset A of M if it is s.l.t. at every point $x \in A$.

The main purpose of this paper is to prove the following two theorems on intersections of generic perturbations.

THEOREM 1.3. *Suppose M is a definable set of pure dimension m. Let A and B be non-empty, definable subsets of M. If E is l.t. on A and* $\dim A \leq \dim B < m$, *then there exists a definable, nowhere dense subset Z of E such that*

$$\dim(\sigma(A) \cap B) < \dim A$$

for all $\sigma \in E \setminus Z$.

THEOREM 1.4. *Suppose E and M are definable sets of pure dimension e and m, respectively. Let A and B be definable subsets of M. If E is s.l.t. on A, then there exists a definable subset Z of E such that* $\dim Z < e$ *and*

$$\dim(\sigma(A) \cap B) \leq d := \max\{\dim A + \dim B - m, -1\}$$

for all $\sigma \in E \setminus Z$.

The above results generalize the theorem on generic intersections from [6], which treated only the case where E is a definable group. Their proofs will be given in the next two sections. Section 4 gives an example of a l.t. action E for which Theorem 1.4 fails. In Section 5, we define an *exceptional* set $T \subset M$ of points at which E is not l.t. and prove a theorem on intersections of generic perturbations in terms of T (Corollary 5.1). Finally, we provide some natural conditions which imply that T is a nowhere dense subset of M (Theorem 5.2).

Some natural examples of manifolds which act l.t. (but not s.l.t.) are the following: the set of all reflections of $\mathbb{R}^m$ in affine hyperplanes, the set of all rotations in $\mathbb{R}^m$ around affine subspaces of dimension $m-2$, the set of such rotations by a fixed non-zero angle, the counterparts of these sets in the sphere $\mathbb{S}^m$ and in the hyperbolic space $\mathbb{H}^m$, as well as non-empty open subsets of the above-mentioned sets. The paper [8] provides a study of the l.t. action of the set of rotations in $\mathbb{R}^m$, $\mathbb{S}^m$ and $\mathbb{H}^m$, and its results are applied in [4], devoted to a concept of a small set which refines the concept of a Tarski nullset.

REMARK 1.5. If $\mathcal{R}$ is a polynomially bounded, o-minimal expansion of the field $\mathbb{R}$, then smooth (i.e. of class $\mathcal{C}^\infty$) definable functions constitute a quasianalytic class, i.e. the identity principle holds: two quasianalytic functions on a connected open subset $U \subset \mathbb{R}^m$ coincide if so do their germs at a point $a \in U$. Hence the following characterization of local transitivity. Suppose E and M are connected, smooth manifolds definable in $\mathcal{R}$ and $\alpha : E \times M \to M$ is a smooth, definable map. Then α is l.t. at a point $x \in M$ iff the set $E \cdot x$ is a subset of M of dimension m. For example, $\mathcal{R}$ may be an analytic structure $\mathbb{R}_{\mathrm{an}}$ (i.e. the expansion of the field $\mathbb{R}$ by restricted analytic functions) or, more generally, a quasianalytic structure (i.e. the expansion of the field $\mathbb{R}$ by restricted quasianalytic functions; see e.g. [9, 5, 7]).

REMARK 1.6. Strong local transitivity can be expressed in terms of Remmert rank. Denote by $\dim A_x$ the dimension of a definable set A at a point x. By the *Remmert rank* of a definable map $f : V \to W$ at a point $x \in V$ (cf. [3, Chap. V]) we mean the number

$$\varrho_x f := \dim V_x - \dim f^{-1}(f(x))_x.$$

Clearly, E is s.l.t. at x iff the map α^x is of Remmert rank $\geq m$ everywhere on E.

2. Proof of Theorem 1.3. We first prove that the definable set

$$Z := \{\sigma \in E : \exists a \in A\ \exists r > 0\ \ [\sigma \cdot (A \cap B(a, r)) \subset B]\}$$

is a nowhere dense subset of E. Otherwise it would contain an open definable subset U of E. By definable choice, there exist definable functions

$$a : U \to A \quad \text{and} \quad r : U \to (0, 1)$$

such that

$$\sigma \cdot (A \cap B(a(\sigma), r(\sigma))) \subset B.$$

After shrinking the open subset U, we may assume that the maps $a(\sigma)$ and $r(\sigma)$ are continuous. Take any point $\sigma_0 \in U$, $\varepsilon := r(\sigma_0)/3$ and a neighbourhood $U_0 \subset U$ of σ_0 such that

$$d(a(\sigma), a(\sigma_0)) < \varepsilon \quad \text{and} \quad r(\sigma) > 2\varepsilon \quad \text{ for all } \sigma \in U_0;$$

here d stands for the Euclidean distance in R^n. Then

$$B(a(\sigma_0), \varepsilon) \subset B(a(\sigma), 2\varepsilon) \subset B(a(\sigma), r(\sigma))$$

for all $\sigma \in U_0$, and thus

$$\sigma \cdot (A \cap B(a(\sigma_0), \varepsilon)) \subset B \quad \text{ for all } \sigma \in U_0.$$

In particular, we get

$$\dim(U_0 \cdot a(\sigma_0)) < m,$$

which contradicts the fact that E is l.t. on the set A. Therefore the definable set Z is a nowhere dense subset of E, as asserted.

Consequently, we get

$$\sigma \cdot (A \cap B(a, r)) \not\subset B$$

for all $a \in A$, $r > 0$ and $\sigma \in E \setminus Z$. Hence

$$\dim(\sigma(A) \cap B) < \dim A$$

for all $\sigma \in E \setminus Z$. Indeed, this is an immediate consequence of

Lemma 2.1. *Given definable subsets C and D, if* $\dim(C \cap D) = \dim C$, *then $C \cap B(a, r) \subset D$ for some $a \in C$ and $r > 0$.*

This, in turn, follows directly from the existence of a finite definable cell decomposition compatible with the sets C and D. Thus the proof of Theorem 1.3 is complete. ∎

3. Proof of Theorem 1.4. The proof relies on dimension calculus similar to that applied in the proof of the theorem on generic intersections in [6]. We adopt the notation from that paper with the definable group G replaced by the definable topological manifold E. Let

$$\varDelta = \varDelta_M := \{(x, x) : x \in M\} \quad \text{and} \quad \pi : \varDelta \to M$$

be the diagonal and the projection onto the first factor. Then

$$\begin{aligned} \sigma(A) \cap B &= \pi((\sigma(A) \times B) \cap \varDelta) \\ &= \pi \circ (\sigma \times \mathrm{Id}_M)((A \times B) \cap \{(x, \sigma(x)) : x \in M\}). \end{aligned}$$

Hence the sets $\sigma(A) \cap B$ and $(A \times B) \cap \{(x, \sigma(x)) : x \in M\}$ are definably homeomorphic, and thus we have to find a definable, nowhere dense subset Z of E such that

$$\dim(A \times B) \cap \{(x, \sigma(x)) : x \in M\} \le d \qquad \text{for all } \sigma \in G \setminus Z.$$

Therefore Theorem 1.4 follows immediately from the lemma below (cf. [6, p. 23]).

Lemma 3.1. *The subset Z of all $\sigma \in E$ such that*

$$\dim(A \times B) \cap \{(x, \sigma(x)) : x \in M\} > d$$

is definable and nowhere dense in E.

Its proof can be repeated verbatim because it requires only that the fibers

$$\{\sigma \in E : \sigma(x) = y\} = (\alpha^x)^{-1}(y), \qquad x \in A,\ y \in B,$$

be of dimension $\le e - m$ (in the present setting). But this is just the assumption that E is s.l.t. on A. □

4. Examples. We will show that Theorem 1.3 is asymmetric in the sense that the assumption $\dim A \le \dim B$ is essential, even in the case where E is a submanifold of a definable Lie group G which acts transitively on a definable manifold M. In particular, in this example the action of E on M will be l.t. but the action of

$$E^{-1} := \{\sigma^{-1} \in G : \sigma \in E\}$$

will not be l.t. on M.

Let $M := R^2 \setminus \{(0,0)\}$ and

$$E := \left\{ \frac{1}{1-xy} \begin{bmatrix} 1 & -x \\ -y & 1 \end{bmatrix} \in \mathrm{GL}(2,R) : x,y \in R \right\};$$

of course, the general linear group $\mathrm{GL}(2,R)$ acts transitively on M and E is an algebraic submanifold of $\mathrm{GL}(2,R)$. Then, by an easy calculation, for every $(a,b) \in M$ the set $E \cdot (a,b)$ contains all points (c,d) with $c \neq 0$ and $d \neq 0$. Hence E is l.t. on M. Clearly,

$$E^{-1} := \left\{ \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \in \mathrm{GL}(2,R) : x,y \in R \right\}.$$

Further,

$$E^{-1} \cdot (u,0) = \{(u,v) : v \in R\}$$

for all $u \neq 0$. Since the above set is of dimension 1, E^{-1} is not l.t. at $(u,0)$.

Now let us show that the assumption $\dim A \leq \dim B$ is needed. Keep the notation of the above example. Let A be the line $\{(1,t) : t \in R\}$ and $B := \{(1,v)\}$. Thus E is l.t. but $\dim A > \dim B$. For all $\sigma \in E$, we have $B \subset \sigma(A)$ (the equivalent relation $\sigma^{-1}(B) \subset A$ is obvious). Hence $\dim(\sigma(A) \cap B) = 0$, while the conclusion of Theorem 1.3 would require it to be -1, i.e. the intersection to be empty.

REMARK 4.1. Suppose that a definable group G acts transitively on a definable manifold M and that E is a definable subset of G of pure dimension. It is not difficult to check that if E is s.l.t. on M, then E^{-1} is l.t. on M.

When $\alpha : E \times M \to M$ is a definable map of class $\mathcal{C}^1$ between two definable manifolds of class $\mathcal{C}^1$, we call α a *submersive* action at a point $x \in M$ if the map α^x is a submersion. E is called *submersive* on a subset A of M if it is submersive at every point $x \in A$. Obviously, every submersive action is s.l.t. We immediately obtain the following corollary to Theorem 1.4.

COROLLARY 4.2. *Let A and B be definable subsets of M. If E is submersive on A, then there exists a definable subset Z of E such that* $\dim Z < e$ *and*

$$\dim(\sigma(A) \cap B) \leq d := \max\{\dim A + \dim B - m, -1\}$$

for all $\sigma \in E \setminus Z$. □

REMARK 4.3. This corollary follows also from an o-minimal version of the Thom transversality theorem (see e.g. [2, Chap. 3, Theorem 2.7] for the classical version) and the existence of a definable stratification.

PROPOSITION 4.4. *The weaker assumptions of Theorem* 1.1 *do not imply the inequality* $\dim(\sigma(A) \cap B) \leq d$ *of Theorem* 2.2.

Proof. Let $M := (R^2 \setminus \{(0,0)\}) \times R$ and

$$E := \left\{ \frac{1}{x_1^2 - x_2^2} \begin{bmatrix} x_1 & -x_2 & 0 \\ -x_2 & x_1 & 0 \\ x_2x_3 & -x_1x_3 & x_1^2 - x_2^2 \end{bmatrix} \in \mathrm{GL}(3, R) \right\};$$

of course, E is an algebraic submanifold of $\mathrm{GL}(3, R)$. Then, by an easy calculation, for every $a = (a_1, a_2, a_3) \in M$ the set $E \cdot a$ contains all points $b = (b_1, b_2, b_3)$ with $b_1 \neq 0$, $b_2 \neq 0$, $b_1^2 - b_2^2 \neq 0$ and $a_2b_1 - a_1b_2 \neq 0$. Hence E is l.t. on M. Clearly,

$$E^{-1} := \left\{ \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & x_3 & 1 \end{bmatrix} \in \mathrm{GL}(3, R) \right\},$$

E^{-1} is not l.t. on M and thus E is not s.l.t. on M.

Let A be the circle

$$A := \{a \in M : a_1^2 + a_2^2 = 1,\ a_3 = 1\}$$

and let B be the line

$$B := \{b \in M : b_2 = 0,\ b_3 = 1\}.$$

Then $d = \max\{\dim A + \dim B - m, -1\} = -1$.

Observe now that if $b = (b_1, 0, 1) \in B$ and

$$\sigma^{-1} := \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & x_3 & 1 \end{bmatrix} \in E^{-1},$$

then $\sigma^{-1}(b) = (b_1x_1, b_1x_2, 1)$. Hence $\sigma^{-1}(B)$ is the line through the points $(0, 0, 1)$ and $(x_1, x_2, 1)$. Therefore $A \cap \sigma^{-1}(B)$ is a two-point set. Hence $\dim(\sigma(A) \cap B) = 0 > d$ for every $\sigma \in E$, contrary to the inequality of Theorem 1.2. ∎

5. Exceptional set of the action. Consider further an action of a definable set E on a definable set M of pure dimension m. By the *exceptional subset* T of M with respect to the action E we mean the definable set of all points $x \in M$ at which E is not l.t., i.e. of those $x \in M$ for which $U \cdot x$ is of dimension $< m$ for some non-empty, open subset U of E. Thus the assumption of Theorem 1.3 says that $A \cap T = \emptyset$. But this can be replaced by the weaker assumption $\dim(A \cap T) < \dim A$:

Corollary 5.1. *If A and B are non-empty, definable subsets of M such that*

$$\dim A \leq \dim B < m \quad \text{and} \quad \dim(A \cap T) < \dim A,$$

then there exists a definable, nowhere dense subset Z of E such that

$$\dim(\sigma(A) \cap B) < \dim A \quad \textit{for all } \sigma \in E \setminus Z.$$

Proof. Subtract T from A and apply Theorem 1.3. ∎

The smaller the dimension of T, the larger the class of pairs A, B for which Theorem 1.3 and Corollary 5.1 apply. Now we are going to provide some conditions under which T is of dimension $< m$. For a subset F of E set

$$M(F) := \{x \in M : \dim(F \cdot x) < m\}.$$

Then

$$F_1 \subset F_2 \Rightarrow M(F_1) \supset M(F_2) \tag{5.1}$$

and

$$M(F_1 \cup F_2) = M(F_1) \cap M(F_2). \tag{5.2}$$

Obviously, if F is a definable set, so is $M(F)$. Further, we get

$$T = \bigcup_{U \subset E \text{ open}} M(U) = \bigcup_{\sigma \in E,\, r>0} M(E \cap B(\sigma, r)).$$

This gives rise to the following definition. We call an element $\sigma \in E$ a *singular perturbation* with respect to E if

$$\dim M(E \cap B(\sigma, r)) = m \quad \text{for some } r > 0.$$

The set of all singular perturbations with respect to E is an open definable subset of E. We call its closure E_s the *singular locus* of E. The complement $E_t := E \setminus E_s$ is an open definable subset of E, called the *tame locus* of E. It is not difficult to check that E_t has no singular perturbation with respect to E_t.

THEOREM 5.2. *If E has no singular perturbation with respect to E, then the exceptional set T is a subset of M of dimension $< m$.*

Proof. Towards a contradiction, suppose that T is of dimension m and thus contains an open definable subset Ω of M. By definable choice, there are definable functions

$$c : \Omega \to E \quad \text{and} \quad r : \Omega \to (0, \infty) \subset R$$

such that $x \in M(E \cap B(c(x), r(x))$ for all $x \in \Omega$. We may assume, after shrinking Ω, that the functions c and r are continuous. Take a point $x_0 \in \Omega$ and a neighbourhood Ω_0 of x_0 such that

$$d(c(x), c(x_0) < r_0/3 \quad \text{and} \quad r(x) > 2r_0/3 \quad \text{for all } x \in \Omega_0,$$

where $r_0 := r(x_0)$. Then

$$B(c(x), r(x)) \supset B(c(x_0), r_0/3) \quad \text{for all } x \in \Omega_0,$$

and

$$x \in M(E \cap B(c(x), r(x))) \subset M(E \cap B(c(x_0), r_0/3)) \quad \text{for all } x \in \Omega_0.$$

Hence

$$\Omega_0 \subset M(E \cap B(c(x_0), r_0/3)),$$

and thus $c(x_0) \in E$ is a singular perturbation with respect to E, contrary to the assumption. This finishes the proof. ∎

REMARK 5.3. Assume that $\mathcal{R}$ is a polynomially bounded, o-minimal expansion of the field $\mathbb{R}$, and that E and M are connected, smooth, definable manifolds. It follows from the identity principle for quasianalytic functions (cf. Remark 1.5) that $M(U) = M(E)$ for any non-empty, open subset U of E. Further, $M(E)$ is a definable quasianalytic subset of M, and hence so is the exceptional set $T = M(E)$. Thus, unlike in the general settings, T is either a closed, nowhere dense subset of M or the whole manifold M.

References

[1] L. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge Univ. Press, 1998.

[2] M. W. Hirsch, *Differential Topology*, Springer, New York, 1976.

[3] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel, 1991.

[4] J. Mycielski and G. Tomkowicz, *On small subsets in Euclidean spaces*, Bull. Polish Acad. Sci. Math., to appear.

[5] K. J. Nowak, *Quantifier elimination, valuation property and preparation theorem in quasianalytic geometry via transformation to normal crossings*, Ann. Polon. Math. 96 (2009), 247–282.

[6] K. J. Nowak, *A theorem on generic intersections in an o-minimal structure*, Fund. Math. 227 (2014), 21–25.

[7] K. J. Nowak, *Quantifier elimination in quasianalytic structures via non-standard analysis*, Ann. Polon. Math. 114 (2015), 235–267.

[8] K. J. Nowak and G. Tomkowicz, *Intersection of generic rotations in some classical spaces*, Bull. Polish Acad. Sci. Math. 64 (2016), 105–107.

[9] J.-P. Rolin, P. Speissegger and A. J. Wilkie, *Quasianalytic Denjoy–Carleman classes and o-minimality*, J. Amer. Math. Soc. 16 (2003), 751–777.

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