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SPECIAL ODD HAMILTONIAN MODULAR LIE ALGEBRAS

BY

WEI BAI and WENDE LIU (Harbin)

Abstract. Derivations from a finite-dimensional special odd Hamiltonian Lie algebra $\mathfrak{G}$ into the odd part $\mathcal{W}$ of a generalized Witt superalgebra are described completely by means of a weight space decomposition with respect to a suitable torus. As an application, the low-dimensional cohomology spaces $H^0(\mathfrak{G};\mathcal{W})$, $H^1(\mathfrak{G};\mathcal{W})$ and the dimensional formulas for them are determined.

1. Introduction and basic notation. Eight families of finite-dimensional $\mathbb{Z}$-graded Lie superalgebras of Cartan type over fields of positive characteristic have been constructed, and the superderivations for those Lie superalgebras were determined (for example, see [BL]). Derivations from the even part into the odd part have been studied for Lie superalgebras of generalized Witt superalgebras and special superalgebras [LG]. For contact Lie superalgebras, nonnegative even $\mathbb{Z}$-homogeneous derivations from the even part into the odd part have been determined in [GL]. For odd Hamiltonian superalgebras, negative $\mathbb{Z}$-homogeneous derivations from the even part into the odd part of generalized Witt superalgebras have been studied in [HL].

The present paper aims to determine the derivation space from a special odd Hamiltonian Lie algebra $\mathfrak{G}$ into the odd part $\mathcal{W}$ of a generalized Witt superalgebra viewed as a $\mathfrak{G}$-module by means of the adjoint representation (Theorems 3.5 and 3.6). Motivated by the work on Lie algebras [F, Theorem 1.1], we do not compute directly the derivations from $\mathfrak{G}$ to $\mathcal{W}$ as in [LG], [GL] and [HL], but adopt a general reduction method: by means of the weight space decomposition with respect to a suitable torus, we can focus on $\mathbb{Z}$-homogeneous derivations with zero weight. Because a derivation is determined by its action on a generating set, some special weight vectors in $\mathcal{W}$ with the same weights for suitable generators of $\mathfrak{G}$ are given in Proposition 2.3.

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Throughout, $\mathbb{F}$ is an algebraically closed field of characteristic $p > 3$, $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the field of two elements, $\mathbb{Z}$ and $\mathbb{N}$ are the sets of integers and nonnegative integers, respectively. Fix an integer $m > 3$ and let $\mathbb{N}^m$ be the additive monoid of m-tuples of nonnegative integers. For a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, denote by $|x| = \theta$ the *parity* of a homogeneous element $x \in V_\theta$, $\theta \in \mathbb{Z}_2$. For a detailed description of superalgebras, the reader is referred to [K, ZCZ]. All vector spaces considered are assumed to be finite-dimensional.

Let us recall the notion of special odd Hamiltonian superalgebras. Fix two m-tuples of positive integers $\underline{t} = (t_1, \dots, t_m)$ and $\pi = (\pi_1, \dots, \pi_m)$, where $\pi_i = p^{t_i} - 1$. Let $\mathcal{O}(m; \underline{t})$ be the finite-dimensional divided power algebra over $\mathbb{F}$ with basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{N}^m, \alpha \le \pi\}$. For $\varepsilon_i = (\delta_{i1}, \dots, \delta_{im})$, we abbreviate $x^{(\varepsilon_i)}$ to x_i, $i = 1, \dots, m$. Let $\Lambda(m)$ be the exterior superalgebra over $\mathbb{F}$ with m variables $x_{m+1}, \dots, x_{2m}$. The tensor product $\mathcal{O}(m, m; \underline{t}) = \mathcal{O}(m; \underline{t}) \otimes_{\mathbb{F}} \Lambda(m)$ is an associative super-commutative superalgebra in the usual way. For $g \in \mathcal{O}(m, \underline{t})$ and $f \in \Lambda(m)$, write gf for $g \otimes f$. Let $u = \langle i_1, \dots, i_k \rangle$ be a strictly increasing sequence of k integers between $m+1$ and $2m$. Write $x^u = x_{i_1} \cdots x_{i_k}$. Set $\delta_{i \in u} = 1$ if $i \in u$, and 0 if $i \notin u$. Write $\omega = \langle m+1, \dots, 2m \rangle$.

Let ∂_r be the special superderivation of $\mathcal{O}(m, m; \underline{t})$ such that $\partial_r(x^{(\alpha)}) = x^{(\alpha - \varepsilon_r)}$ for $r \in \overline{1, m}$ and $\partial_r(x_s) = \delta_{rs}$ for $r, s \in \overline{1, 2m}$. The *generalized Witt superalgebra* $W(m, m; \underline{t})$ is spanned by all $f_r \partial_r$, where $f_r \in \mathcal{O}(m, m; \underline{t})$. When confusion is not likely to occur, we shall drop $(m, m; \underline{t})$ from the notation. Define a linear operator $\mathrm{T_H} : \mathcal{O} \to W$ such that $\mathrm{T_H}(a) = \sum_{i=1}^{2m} (-1)^{|\partial_i|\,|a|} \partial_i(a) \partial_{i'}$ for $a \in \mathcal{O}$, where $'$ is the involution of the index set $\{1, \dots, 2m\}$ sending i to $i + m$ if $i \le m$ and to $i - m$ otherwise. Then $\mathrm{T_H}$ is odd and $\mathbb{Z}$-homogeneous of degree -2. Note that

$$[\mathrm{T_H}(a), \mathrm{T_H}(b)] = \mathrm{T_H}(\mathrm{T_H}(a)(b)) \quad \text{for all } a, b \in \mathcal{O}.$$

Recall the odd Laplacian operator $\Delta = \sum_{i=1}^{m} \Delta_i$, $\Delta_i = \partial_i \partial_{i'}$, $i = 1, \dots, m$. Note that the kernel of $\mathrm{T_H}$ is $\mathbb{F} \cdot 1$. Writing $\bar{\mathcal{O}}$ for the quotient superspace $\mathcal{O}/\mathbb{F} \cdot 1$ and viewing $\mathrm{T_H}$ and Δ as linear operators on $\bar{\mathcal{O}}$ in the natural way, one sees that the kernel of Δ,

$$\overline{SHO} = \{a \in \bar{\mathcal{O}} \mid \Delta(a) = 0\},$$

is a finite-dimensional Lie superalgebra with bracket

$$[a, b] = \mathrm{T_H}(a)(b) \quad \text{for } a, b \in \overline{SHO}.$$

Its second derived superalgebra, denoted by SHO [if necessary, $SHO(m, m; \underline{t})$], is simple [LH], called a *special odd Hamiltonian superalgebra*, as in [K].

Hereafter, we call the even part of a special odd Hamiltonian superalgebra a *special odd Hamiltonian Lie algebra*, denoted by $\mathfrak{G}$, and write $\mathcal{W}$

for the odd part of W. The adjoint action of $\mathfrak{G}$ on $\mathcal{W}$ is

$$f \cdot D = [\mathrm{T_H}(f), D] \quad \text{for } f \in \mathfrak{G},\ D \in \mathcal{W}.$$

Some detailed information on $\mathfrak{G}$, such as its $\mathbb{Z}$-grading structure and linear generators, can be found in [LH].

2. Reduction. Let L be a Lie algebra and V an L-module. Denote by $\mathrm{Der}(L, V)$ the *derivation space*, and by $\mathrm{Inder}(L, V)$ the *inner derivation space* consisting of the derivations determined by any fixed $v \in V$, $x \mapsto x \cdot v$ for all $x \in L$. In the usual way, $\mathrm{Der}(L, V)$ and $\mathrm{Inder}(L, V)$ are L-submodules of $\mathrm{Hom}_{\mathbb{F}}(L, V)$, and

$$H^1(L; V) = \mathrm{Der}(L, V)/\mathrm{Inder}(L, V)$$

is the *first cohomology space* of L with coefficients in V.

Assume in addition that $L = \bigoplus_{r\in\mathbb{Z}} L_r$ is $\mathbb{Z}$-graded and $V = \bigoplus_{r\in\mathbb{Z}} V_r$ is a $\mathbb{Z}$-graded L-module. Then the derivation space inherits a $\mathbb{Z}$-graded module structure

$$\mathrm{Der}(L, V) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Der}_r(L, V).$$

Let $T \subset L_0$ be a torus of L with weight space decompositions

$$L = \bigoplus_{\alpha\in\Theta} L_{(\alpha)}, \qquad V = \bigoplus_{\beta\in\Omega} V_{(\beta)}.$$

Then $\mathrm{Der}(L, V)$ inherits a $\mathbb{Z} \times T^*$-grading structure from those of L and V.

It is difficult to compute directly the derivations from $\mathfrak{G}$ to $\mathcal{W}$, since the structure of $\mathfrak{G}$ is complicated. In this section, we shall give a method to simplify our task considerably by using the weight space decomposition of $\mathfrak{G}$ and $\mathcal{W}$ with respect to a suitable torus. For convenience, write $\mathbf{Y}_0 = \overline{1, m}$, $\mathbf{Y}_1 = \overline{m+1, 2m}$ and $\mathbf{Y} = \overline{1, 2m}$.

Let $\mathcal{G} = \mathrm{span}_{\mathbb{F}}\{x^u\partial_r \mid r \in \mathbf{Y},\ x^u \in \Lambda(m),\ |x^u\partial_r| = \bar{1}\}$. One can easily verify that $\mathrm{ann}_{\mathcal{W}}(\mathfrak{G}_{-1})$, the annihilator of $\mathfrak{G}_{-1}$ in $\mathcal{W}$, is $\mathcal{G}$. Note that $\mathcal{G}$ is a $\mathbb{Z}$-graded subalgebra of $\mathcal{W}$.

Let $\mathfrak{t} = \mathrm{span}_{\mathbb{F}}\{T_i = x_ix_{i'} - x_1x_{1'} \mid i \in \mathbf{Y}_0 \setminus \{1\}\}$. Obviously, $\mathfrak{t}$ is a torus in $\mathfrak{G}$ with dimension $m-1$. Set $T_{ij} = x_ix_{i'} - x_jx_{j'}$, $i, j \in \mathbf{Y}_0$, $i \neq j$. For any $\pi \geq \alpha \in \mathbb{Z}^m$ and any strictly increasing sequence $u = \langle i_1, \ldots, i_k\rangle$ of integers between $m+1$ and $2m$, we define a linear form on $\mathfrak{t}$ by

$$(\alpha + u)(T_{ij}) = \delta_{i'\in u} - \delta_{j'\in u} - \alpha_i + \alpha_j. \tag{2.1}$$

Obviously, X has a weight space decomposition with respect to $\mathfrak{t}$,

$$X = \bigoplus_{(\alpha+u)} X_{(\alpha+u)},$$

where $X = \mathfrak{G}$ or $\mathcal{G}$.

We will substantially use the following lemma to determine the derivations.

LEMMA 2.1. *Assume that* $\phi \in \mathrm{Der}(\mathfrak{G}, \mathcal{W})$.

(i) *If* ϕ *vanishes on* $\mathfrak{G}_{-1}$ *then* ϕ *vanishes on* $[D, \mathfrak{G}_{-1}]$ *if and only if* $\phi(D) \in \mathcal{G}$ *for each* $D \in \mathfrak{G}$.
(ii) ϕ *vanishes on* $\mathfrak{G}_{-1}$ *modulo a suitable inner derivation.*
(iii) ϕ *is inner modulo a suitable derivation with zero weight.*

Proof. See [LG, Lemma 2.1], [LG, Lemma 2.3] and [F, Theorem 1.1], respectively. ∎

Hereafter, we write θ *for the zero weight.*

REMARK 2.2. (i) By Lemma 2.1, it is sufficient to consider the homogeneous derivations in $\mathrm{Der}_{(t,\theta)}(\mathfrak{G}, \mathcal{W})$, where $(t, \theta) \in \mathbb{Z} \times \mathfrak{t}^*$.

(ii) A derivation from $\mathfrak{G}$ to $\mathcal{W}$ is determined by its action on the generators of $\mathfrak{G}$. In this paper, we choose the generating set $\mathfrak{L} \cup \mathfrak{M} \cup \mathfrak{N}$ of $\mathfrak{G}$ mentioned in [BLY, Proposition 2.5], where

$$\mathfrak{L} = \{x^{(b\varepsilon_i)}x_{k'} \mid i, k \in \mathbf{Y}_0,\ i \neq k,\ b \equiv 0 \pmod p\};$$
$$\mathfrak{M} = \{x_{i'}x_{j'}x_{k'} \mid i, j, k \in \mathbf{Y}_0 \text{ are distinct}\};$$
$$\mathfrak{N} = \{x^{(a\varepsilon_i)}x_{k'} \mid i, k \in \mathbf{Y}_0,\ i \neq k,\ 0 \leq a \leq 2\}.$$

From (2.1), we know that the generators are all weight vectors and are $\mathbb{Z}$-homogeneous:

$$x^{(a\varepsilon_i)}x_{k'} \in \mathfrak{G}_{(a-1, a\varepsilon_i + \langle k' \rangle)}, \quad 0 \leq a < p^{t_i}, \tag{2.2}$$

$$x_{i'}x_{j'}x_{k'} \in \mathfrak{G}_{(1, \langle i', j', k' \rangle)}. \tag{2.3}$$

In view of Lemma 2.1, we are more interested in the weight vectors of the same weights with the generators of $\mathfrak{G}$, in $\mathcal{G}$ than in $\mathcal{W}$.

PROPOSITION 2.3. *For distinct* $i, j, k \in \mathbf{Y}_0$, $b \equiv 0 \pmod p$, *we have*

(i) $\mathcal{G}_{(2\varepsilon_i + \langle k' \rangle)} = 0$;
(ii) $\mathcal{G}_{(b\varepsilon_i + \langle k' \rangle)} = \mathcal{G}_{(\langle k' \rangle)} = \sum_{t \in \mathbf{Y}_0} \mathbb{F} x_{k'}x_{t'}\partial_{t'} \oplus \mathbb{F}\frac{1-(-1)^m}{2}x^{\omega}\partial_k$;
(iii) $\mathcal{G}_{(\varepsilon_i + \langle k' \rangle)} = \mathbb{F}\frac{1+(-1)^m}{2}x^{\omega - \langle i' \rangle}\partial_k$;
(iv) $\mathcal{G}_{(\langle i', j', k' \rangle)} = \delta_{m4}\mathbb{F}\partial_{t'} \oplus \sum_{t \in \mathbf{Y}_0 \setminus \{i,j,k\}} \mathbb{F} x_{i'}x_{j'}x_{k'}x_{t'}\partial_{t'}$, *when* $m = 4$, $t \in \mathbf{Y}_0$ *and* $t \neq i, j, k$;
(v) $\mathcal{G}_{(\theta)} = \sum_{i \in \mathbf{Y}_0} \mathbb{F}\frac{1+(-1)^m}{2}x^{\omega - \langle i' \rangle}\partial_i$;
(vi) $\mathcal{G}_{(\varepsilon_i + \langle t', j', k' \rangle)} = 0$, $t \in \mathbf{Y}_0$ *and* $t \neq i, j, k$.

Proof. The " $\supset$ " inclusions are clear. To show the converses, note that for $s \neq l, t \in \mathbf{Y}_0$,

$$[T_{sl}, x^u\partial_t] = (\delta_{s'\in u} - \delta_{l'\in u} + \delta_{ts} - \delta_{tl})x^u\partial_t, \tag{2.4}$$

$$[T_{sl}, x^u\partial_{t'}] = (\delta_{s'\in u} - \delta_{l'\in u} - \delta_{ts} + \delta_{tl})x^u\partial_{t'}. \tag{2.5}$$

(i) If there exists $x^u\partial_t \in \mathcal{G}_{(2\varepsilon_i+\langle k'\rangle)}$, from (2.1) and (2.4), choosing $l = k$, $s = i$, we have

$$-3 = \delta_{s'\in u} - \delta_{l'\in u} + \delta_{ts} - \delta_{tl},$$

which is impossible. Similarly, no $x^u\partial_{t'}$ is in $\mathcal{G}_{(2\varepsilon_i+\langle k'\rangle)}$. Thus, $\mathcal{G}_{(2\varepsilon_i+\langle k'\rangle)} = 0$.

(ii) If there exists $x^u\partial_t \in \mathcal{G}_{(b\varepsilon_i+\langle k'\rangle)}$, from (2.1) and (2.4), when $t = k$, choosing $s = k$, $l \neq k$, we have

$$0 = \delta_{k'\in u} - \delta_{l'\in u},$$

which implies that $x^u\partial_t = x^\omega\partial_k$ if m is odd. When $t = i$, choosing $s = k$, $l = i$, we have

$$2 = \delta_{k'\in u} - \delta_{l'\in u},$$

which is impossible. When $t \neq i, k$, choosing $s = k$, $l = t$, we have

$$2 = \delta_{k'\in u} - \delta_{t'\in u},$$

which is impossible.

If there exists $x^u\partial_{t'} \in \mathcal{G}_{(2\varepsilon_i+\langle k'\rangle)}$, from (2.1) and (2.5) we have $t \neq i, k$. Choosing $s = i$, $l = k$ and $s = t$, $l = k$, we have

$$i' \notin u, \quad k' \in u, \quad t' \in u.$$

Choosing $s \neq i, k, t$, $l = k$, we have

$$-1 = \delta_{s'\in u} - \delta_{k'\in u},$$

which implies that $s' \notin u$. Thus $x^u\partial_{t'} = x_{k'}x_{t'}\partial_{t'}$. Then

$$\mathcal{G}_{(\langle k'\rangle)} = \sum_{t\in\mathbf{Y}_0} \mathbb{F}x_{k'}x_{t'}\partial_{t'} \oplus \mathbb{F}\frac{1-(-1)^m}{2}x^\omega\partial_k.$$

(iii) If there exists $x^u\partial_t \in \mathcal{G}_{(\varepsilon_i+\langle k'\rangle)}$, from (2.1) and (2.4), choosing $l = k$, $s = i$, we have

$$-2 = \delta_{i'\in u} - \delta_{k'\in u} + \delta_{ti} - \delta_{tk},$$

which implies that $t = k$, $k' \in u$, $i' \notin u$. Choosing $s = i$, $l \neq k$ we have

$$1 = \delta_{l'\in u}.$$

Thus $x^u\partial_t = x^{\omega-\langle i'\rangle}\partial_k$ if m is even.

If there exists $x^u\partial_{t'} \in \mathcal{G}_{(\varepsilon_i+\langle k'\rangle)}$, from (2.1) and (2.5), choosing $l = k$, $s = i$, we have

$$-2 = \delta_{i'\in u} - \delta_{k'\in u} - \delta_{ti} + \delta_{tk},$$

which implies that $t = i$, $k' \in u$, $i' \notin u$. Choosing $s = i$, $l \neq k$ we have

$$0 = \delta_{l' \in u}.$$

Thus $x^u \partial_{t'} = x_{k'} \partial_{i'}$, which is even, contradicting $|x^u \partial_{t'}| = \bar{1}$. Hence $\mathcal{G}_{(\varepsilon_i + \langle k' \rangle)} = \mathbb{F} \frac{1+(-1)^m}{2} x^{\omega - \langle i' \rangle} \partial_k$.

(iv) If there exists $x^u \partial_t \in \mathcal{G}_{(\langle i', j', k' \rangle)}$, from (2.1) and (2.4), choosing $l = j$, $s = i$ and $l = k$, $s = i$, we have

$$0 = \delta_{s' \in u} - \delta_{l' \in u} + \delta_{ts} - \delta_{tl},$$

which implies that $j', k' \in u$, $i' \notin u$ when $t = i$. In this case, choosing $s = i$, $l \neq i, j, k$ we have

$$0 = \delta_{l' \in u}.$$

Thus, we have $j', k' \in u$, $l' \notin u$ for all $l \in \mathbf{Y}_0 \setminus \{j, k\}$. Similarly, we have $i', k' \in u$, $l' \notin u$ for all $l \in \mathbf{Y}_0 \setminus \{i, k\}$ when $t = j$; and $i', j' \in u$, $l' \notin u$ for all $l \in \mathbf{Y}_0 \setminus \{j, i\}$ when $t = k$. When $t \neq i, j, k$, choosing $s = i$, $l = t$ we have

$$2 = \delta_{i' \in u} - \delta_{t' \in u},$$

which is impossible. Thus $x^u \partial_t = x_{i'} x_{j'} \partial_k$, $x_{j'} x_{k'} \partial_i$ or $x_{k'} x_{i'} \partial_j$, which is even, contrary to $|x^u \partial_{t'}| = \bar{1}$.

If there exists $x^u \partial_{t'} \in \mathcal{G}_{(\langle i', j', k' \rangle)}$, from (2.1) and (2.5), when $t = i$, choosing $l \neq j, k$ and $s = i$ we have

$$2 = \delta_{i' \in u} - \delta_{l' \in u},$$

which is impossible. Similarly, we have $t \neq j, k$. Now suppose that $t \neq i, j, k$. Choosing s, l properly, we have the following conclusions:

(1) $i, j, k \notin u$. Choosing $s = i$, $l = t$, we have $t \notin u$. Thus $x^u \partial_{t'} = \partial_{t'}$ when $m = 4$. When $m > 4$, choosing $l \neq i, j, k, t$, $s = i$, we have $1 = \delta_{i' \in u} - \delta_{l' \in u}$, contradicting $i' \notin u$.

(2) $i, j, k \in u$. Choosing $s = t$, $l = t$, we have $t \in u$. Choosing $s \neq i, j, k, t$, $l = i$, we have $s' \notin u$, Therefore $x^u \partial_{t'} = x_{i'} x_{j'} x_{k'} x_{l'} \partial_{l'}$, $i, j, k \neq l \in \mathbf{Y}_0$.

Thus, $\mathcal{G}_{(\langle i', j', k' \rangle)} = \delta_{m4} \mathbb{F} \partial_{t'} \oplus \sum_{t \in \mathbf{Y}_0 \setminus \{i,j,k\}} \mathbb{F} x_{i'} x_{j'} x_{k'} x_{t'} \partial_{t'}$ when $m = 4$, $t \in \mathbf{Y}_0$ and $t \neq i, j, k$.

(v) If there exists $x^u \partial_t \in \mathcal{G}_{(\theta)}$, from (2.1) and (2.4) we have

$$\delta_{s' \in u} - \delta_{l' \in u} = -\delta_{ts} + \delta_{tl} = \begin{cases} 1, & t = l, \\ -1, & t = s, \\ 0, & \text{otherwise}, \end{cases}$$

which means that if $t = l$ then $s' \in u$ for all $l \neq s \in \mathbf{Y}_0$. Thus $x^u \partial_t = x^{\omega - \langle i' \rangle} \partial_i$, $i \in \mathbf{Y}_0$, when m is even.

If there exists $x^u\partial_{t'} \in \mathcal{G}_{(\theta)}$, from (2.1) and (2.4) we have

$$\delta_{s'\in u} - \delta_{l'\in u} = \delta_{ts} - \delta_{tl} = \begin{cases} 1, & t = s, \\ -1, & t = l, \\ 0, & \text{otherwise}, \end{cases}$$

which means that if $t = l$ then $l' \in u$, but $s \notin u$ for all $l \neq s \in \mathbf{Y}_0$. Thus $x^u\partial_t = x_{l'}\partial_{l'}$, $l \in \mathbf{Y}_0$, which is even, contrary to $|x^u\partial_{t'}| = \bar{1}$. Thus, $\mathcal{G}_{(\theta)} = \sum_{i\in\mathbf{Y}_0} \mathbb{F}\frac{1+(-1)^m}{2}x^{\omega-\langle i'\rangle}\partial_i$.

(vi) If there exists $x^u\partial_h \in \mathcal{G}_{(\varepsilon_i+\langle t',j',k'\rangle)}$, from (2.1) and (2.4), when $h = i$, choosing $s = i$, $l = j$, we have

$$-2 = \delta_{i'\in u} - \delta_{i'\in u} + 1,$$

which is impossible. Similarly, for all $h \in \mathbf{Y}_0$, there is no $x^u\partial_h \in \mathcal{G}_{(\varepsilon_i+\langle t',j',k'\rangle)}$.

If there exists $x^u\partial_{h'} \in \mathcal{G}_{(\varepsilon_i+\langle t',j',k'\rangle)}$, from (2.1) and (2.5), when $h = i$, choosing $s = t$, $l = i$, we have

$$1 = \delta_{t'\in u} - \delta_{i'\in u},$$

which implies that $i' \notin u$, $t' \in u$. Similarly, we have $j', k' \in u$. Choosing $s \neq i, j, k, t$, $l = j$, we have $s' \notin u$. Then $x^u\partial_{h'} = x_{t'}x_{k'}x_{j'}\partial_{i'}$, which is even, contradicting $|x^u\partial_{h'}| = \bar{1}$. Choosing s, l properly, we find that, for all $h \in \mathbf{Y}_0$, no $x^u\partial_{h'}$ is in $\mathcal{G}_{(\varepsilon_i+\langle t',j',k'\rangle)}$. Thus, $\mathcal{G}_{(\varepsilon_i+\langle t',j',k'\rangle)} = 0$. ∎

3. Derivation and low-dimensional cohomology spaces. In this section we will determine the derivation space $\mathrm{Der}(\mathfrak{G}, \mathcal{W})$, and describe the low-dimensional cohomology spaces of special odd Hamiltonian algebras with coefficients in odd parts of generalized Witt superalgebras:

$$H^0(\mathfrak{G}; \mathcal{W}) = \mathrm{ann}_{\mathcal{W}}(\mathfrak{G}) = \{D \in \mathcal{W} \mid g \cdot D = 0 \text{ for all } g \in \mathfrak{G}\},$$
$$H^1(\mathfrak{G}; \mathcal{W}) = \mathrm{Der}(\mathfrak{G}, \mathcal{W})/\mathrm{Inder}(\mathfrak{G}, \mathcal{W}).$$

First, we introduce a family of "strange" derivations from $\mathfrak{G}$ to $\mathcal{W}$, the so-called *exceptional derivations*. For $q \in \mathbb{N}$ and $i \in \mathbf{Y}_0$ define a linear mapping $\Phi_i^{(q)} : \mathfrak{G} \to W$ satisfying

$$\Phi_i^{(q)}(f) = x^{\omega}(\partial_i^{p^q}(f)),$$

where $x^\omega \in \mathcal{O}$, $f \in \mathfrak{G}$. Obviously, $|\Phi_i^{(q)}|$ depends on the parity of m. From [BLY, Proposition 3.1], we know that $\Phi_i^{(q)}$ is a derivation from $\mathfrak{G}$ to W with $\mathbb{Z}$-degree $m - p^q$ and zero weight. Moreover, when m is odd, we obtain

$$\Phi_i^{(q)} \in \mathrm{Der}_{(m-p^q,\theta)}(\mathfrak{G}, \mathcal{W}),$$

and by a direct computation we have $\Phi_i^{(q)}(\mathfrak{G}_{-1}) = 0$.

From Lemma 2.1(ii), for any $\phi \in \mathrm{Der}(\mathfrak{G}, \mathcal{W})$, there exists $D \in \mathcal{W}$ such that

$$(\phi - \operatorname{ad} D)(\mathfrak{G}_{-1}) = 0.$$

From Lemma 2.1(iii), there exists $E \in \mathcal{W}$ such that

$$\phi - \operatorname{ad} E \in \mathrm{Der}_{(\theta)}(\mathfrak{G}, \mathcal{W}).$$

Moreover, $(\phi - \operatorname{ad} E)(\mathfrak{G}_{-1}) = 0$ if $\phi(\mathfrak{G}_{-1}) = 0$. Thus, *hereafter, we only consider the $\mathbb{Z}$-homogeneous derivations $\phi \in \mathrm{Der}_{(\theta)}(\mathfrak{G}, \mathcal{W})$ which vanish on $\mathfrak{G}_{-1}$.*

Now we discuss the action of ϕ on $\mathfrak{G}_0$.

PROPOSITION 3.1. *ϕ vanishes on $\mathfrak{G}_0$ modulo a suitable inner derivation.*

Proof. Note that $\mathfrak{G}_0$ is spanned by $\mathfrak{t} \cup \{x_i x_{k'} \mid i, k \in \mathbf{Y}_0, i \neq k\}$ and $\mathfrak{t} \subset \mathfrak{G}_{(0,\theta)}$, $x_i x_{k'} \in \mathfrak{G}_{(0,\varepsilon_i + \langle k' \rangle)}$. By Lemma 2.1(i) and Proposition 2.3(iii)&(v), one may assume that m is even,

$$\phi(T_i) = \sum_{r \in \mathbf{Y}_0} c_r^i x^{\omega - \langle r' \rangle} \partial_r \quad \text{and} \quad \phi(x_s x_{l'}) = d^{(sl)} x^{\omega - \langle s' \rangle} \partial_l,$$

where $c_r^i, d^{(sl)} \in \mathbb{F}$, $s, l \in \mathbf{Y}_0$, $s \neq l$. For distinct $1, i, j \in \mathbf{Y}_0$, applying ϕ to

$$[x_1 x_{i'}, x_i x_{1'}] = T_i, \quad [T_i, x_j x_{i'}] = x_j x_{i'}, \quad [T_i, x_i x_{1'}] = -2 x_i x_{1'}$$

yields $\phi(T_i) = 0$. For $k \in \mathbf{Y}_0 \setminus \{s, l\}$, applying ϕ to

$$[x_s x_{l'}, x_k x_{s'}] = x_k x_{l'}, \quad [x_s x_{l'}, x_l x_{s'}] = T_{ls}$$

yields

$$d^{(kl)} - d^{(ks)} = (-1)^{s+k} d^{(sl)}, \quad (-1)^{l+s} d^{(ls)} + d^{(sl)} = 0. \tag{3.1}$$

Clearly, the system of $m - 1$ equations

$$\lambda_1 - (-1)^{1+i} \lambda_i = d^{(1i)}, \quad i = 2, \ldots, m,$$

has a solution, still denoted by λ_i, $i = 2, \ldots, m$. By (3.1), we conclude that $\lambda_s - (-1)^{s+l} \lambda_l = d^{(sl)}$.

Set $D = -\sum_{r \in \mathbf{Y}_0} \lambda_r x^{\omega - \langle r' \rangle} \partial_r$ and $\psi = \phi - \operatorname{ad} D$. A direct computation shows $\psi(x_s x_{l'}) = 0$. Clearly, we also have $\psi(T_i) = 0$ for all $i \in \mathbf{Y}_0 \setminus \{1\}$. Hence $\psi(\mathfrak{G}_0) = 0$. ∎

PROPOSITION 3.2. *Suppose $\phi(\mathfrak{G}_0) = 0$. Then $\phi(\mathfrak{N}) = \phi(\mathfrak{M}) = 0$.*

Proof. For (2.2) and $\phi(\mathfrak{G}_{-1}) = \phi(\mathfrak{G}_0) = 0$, by Lemma 2.1(i) and Proposition 2.3(i), we have $\phi(\mathfrak{N}) = 0$. Notice that, for distinct $i, j, k, t \in \mathbf{Y}_0$,

$$[x^{(2\varepsilon_i)} x_{t'}, x_{i'} x_{j'} x_{k'}] = x^{(\varepsilon_i)} x_{t'} x_{j'} x_{k'}. \tag{3.2}$$

From Proposition 2.3(iv), we consider the following cases:

CASE 1: $m = 4$ and the $\mathbb{Z}$-degree of ϕ is -2. Suppose that $\phi(x_{i'} x_{j'} x_{k'}) = a_t \partial_{t'}$, $a_t \in \mathbb{F}$. From (3.2) and Proposition 2.3(vi), we have $a_t = 0$.

CASE 2: The $\mathbb{Z}$-degree of ϕ is 2. Suppose that

$$\phi(x_{i'}x_{j'}x_{k'}) = x_{i'}x_{j'}x_{k'} \sum_{s\in\mathbf{Y}_0\setminus\{i,j,k\}} a_s^{i,j,k} x_{s'}\partial_{s'},$$

where $a_s^{i,j,k} \in \mathbb{F}$. From (3.2) and Proposition 2.3(vi), we have

$$x^{(\varepsilon_i)}x_{t'}x_{j'}x_{k'} \sum_{s\in\mathbf{Y}_0\setminus\{i,j,k,t\}} a_s^{i,j,k} x_{s'}\partial_{s'} - x^{(\varepsilon_i)}x_{i'}x_{j'}x_{k'}a_t^{i,j,k}x_{t'}\partial_{i'} = 0.$$

It follows that $a_s^{i,j,k} = 0$ for all $s \in \mathbf{Y}_0 \setminus \{i,j,k\}$.

Thus we obtain $\phi(\mathfrak{M})$=0. ∎

LEMMA 3.3. *Suppose $\phi(\mathfrak{G}_0) = 0$. Let $b=\sum_{r=1}^{s} c_r p^r$ with $c_s \not\equiv 0 \pmod p$, $0 \le c_r < p$. For distinct $i,k \in \mathbf{Y}_0$ and $a < b$, if $\phi(x^{(a\varepsilon_i)}x_{k'}) = 0$, then $\phi(x^{(b\varepsilon_i)}x_{k'}) = 0$.*

Proof. Note that $\binom{b}{p^s} \not\equiv 0 \pmod p$ and $\binom{b}{p^s-1} \equiv 0 \pmod p$. A direct computation shows that, for $j \in \mathbf{Y}_0 \setminus \{i,k\}$,

$$[x^{(\varepsilon_k)}x_{i'}, x^{(p^s\varepsilon_i)}x_{k'}] = x^{(p^s\varepsilon_i)}x_{i'} - x^{((p^s-1)\varepsilon_i+\varepsilon_k)}x_{k'}.$$

Obviously, $\phi(x^{(p^s\varepsilon_i)}x_{i'} - x^{((p^s-1)\varepsilon_i+\varepsilon_k)}x_{k'}) = 0$ since $p^s < b$ and $x^{(\varepsilon_k)}x_{i'} \in \mathfrak{G}_0$. Notice that

$$[x^{((b-p^s+1)\varepsilon_i)}x_{k'}, x^{(p^s\varepsilon_i)}x_{i'} - x^{((p^s-1)\varepsilon_i+\varepsilon_k)}x_{k'}] = \binom{b}{p^s} x^{(b\varepsilon_i)}x_{k'}.$$

Thus $\phi(x^{(b\varepsilon_i)}x_{k'}) = 0$ for $b - p^s + 1 < b$. ∎

PROPOSITION 3.4. *Suppose $\phi(\mathfrak{G}_0) = 0$. There exists $a_{k_i}^i \in \mathbb{F}$ such that*

$$\left(\phi - \frac{1-(-1)^m}{2} \sum_{i\in\mathbf{Y}_0} \sum_{k_i=1}^{t_i-1} a_{k_i}^i \Phi_i^{(k_i)}\right)(\mathfrak{L}) = 0.$$

Proof. From [BLY, Remark 2.6], we know that

$$\{x^{(c\varepsilon_i)}x_{j'} \mid i,j \in \mathbf{Y}_0,\ i \ne j,\ 0 \le c \le p^{t_i-1}\}$$

is contained in the algebra generated by $\mathfrak{L} \cup \mathfrak{N}$, and from Lemma 3.3, we only consider the elements in $\mathfrak{L}$ of the form $x^{(p^q\varepsilon_i)}x_{k'}$ for distinct $i,k \in \mathbf{Y}_0$, $1 \le q \le t_i - 1$. Suppose $\phi \in \mathrm{Der}_{(t,\theta)}(\mathfrak{G},\mathcal{W})$. By Lemma 2.1(i), Proposition 2.3(ii) and (2.2), we only have to consider the following two cases:

CASE 1: $b = 2-t = p^q$. We can assume $\phi(x^{(b\varepsilon_i)}x_{k'}) = \sum_{t\in\mathbf{Y}_0} c_t^{ik} x_{k'}x_{t'}\partial_{t'}$, $c_t^{ik} \in \mathbb{F}$. For $j \in \mathbf{Y}_0 \setminus \{i,k\}$, applying ϕ to

$$[x_{i'}x_{j'}x_{k'}, x^{(b\varepsilon_i)}x_{k'}] = 0,$$

by Proposition 3.2 we have $c_i^{ik} = c_j^{ik} = 0$, that is, $\phi(x^{(b\varepsilon_i)}x_{k'}) = 0$.

CASE 2: $b = m - t = p^q$. We can assume that m is odd and $\phi(x^{b\varepsilon_i}x_{k'}) = a^{ik}x^{\omega}\partial_k$, $a^{ik} \in \mathbb{F}$. For $j \in \mathbf{Y}_0 \setminus \{i,k\}$, applying ϕ to

$$[x^{(\varepsilon_k)}x_{j'}, x^{(b\varepsilon_i)}x_{k'}] = x^{(b\varepsilon_i)}x_{j'},$$

we have $a^{ik} = a^{ij}$ for all $j \in \mathbf{Y_0} \setminus \{i,k\}$.

Set $a^i_q = a^{ik}$, $k \in \mathbf{Y}_0 \setminus \{i\}$, and $\varphi = \phi + \sum_{i\in\mathbf{Y}_0} a^i_q \Phi^{(q)}_i$. Obviously, $\varphi(x^{(b\varepsilon_i)}x_{k'}) = 0$. Thus, our assertion holds. ∎

THEOREM 3.5. *We have*

$$\mathrm{Der}(\mathfrak{G}, \mathcal{W}) = \begin{cases} \mathrm{ad}\,\mathcal{W}, & m \text{ is even,} \\ \mathrm{ad}\,\mathcal{W} \oplus \mathrm{span}_{\mathbb{F}}\{\Phi_i^{(k_i)} \mid i \in \mathbf{Y}_0,\ 1 \le k_i < t_i\}, & m \text{ is odd.} \end{cases}$$

Proof. This is a direct consequence of Lemma 2.1 and Propositions 3.1, 3.2 and 3.4. ∎

THEOREM 3.6.

(i) $H^0(\mathfrak{G}; \mathcal{W})$ *is* 0*-dimensional if* m *is odd, and* 1*-dimensional spanned by* $\sum_{i\in\mathbf{Y}_0}(-1)^{i+1}x^{\omega-\langle i'\rangle}\partial_i$ *if* m *is even.*

(ii) $H^1(\mathfrak{G}; \mathcal{W})$ *is* 0*-dimensional if* m *is even, and* $(t_1 + \cdots + t_m - m)$*-dimensional if* m *is odd.*

Proof. Obviously, $H^0(\mathfrak{G}; \mathcal{W}) \subset \mathcal{W}_{(\theta)} \cap \mathrm{ann}_{\mathcal{W}}(\mathfrak{G}_{-1}) = \mathcal{G}_{(\theta)}$. For $D \in H^0(\mathfrak{G}; \mathcal{W})$, by Proposition 2.3(v), we can assume that

$$D = \sum_{i\in\mathbf{Y}_0} \frac{1 + (-1)^m}{2} b_i x^{\omega-\langle i'\rangle}\partial_i, \quad b_i \in \mathbb{F}.$$

For distinct $k, l \in \mathbf{Y}_0$, we have $(-1)^k b_k = (-1)^l b_l$ for $i, j \in \mathbf{Y}_0$, since $[x_k x_{l'}, D] = 0$. Clearly, $g \cdot D = 0$, where $g \in \mathfrak{L} \cup \mathfrak{M} \cup \mathfrak{N}$. Thus

$$\sum_{i\in\mathbf{Y}_0} (-1)^{i+1}x^{\omega-\langle i'\rangle}\partial_i \in H^0(\mathfrak{G}; \mathcal{W}),$$

when m is even. The second conclusion follows directly from Theorem 3.5. ∎

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Wei Bai, Wende Liu (corresponding author)
School of Mathematical Sciences
Harbin Normal University
150025 Harbin, China
E-mail: baiwei_2005@126.com
wendeliu@ustc.edu.cn