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Intersection of Generic Rotations in Some Classical Spaces

by

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Summary. Consider an o-minimal structure on the real field $\mathbb{R}$ and two definable subsets A, B of the Euclidean space $\mathbb{R}^n$, of the unit sphere $\mathbb{S}^n$ or of the hyperbolic space $\mathbb{H}^n$, $n \geq 2$, which are of dimensions $k, l \leq n-1$, respectively. We prove that the dimension of the intersection $\sigma(A) \cap B$ is less than $\min\{k, l\}$ for a generic rotation σ of the ambient space; here we set $\dim \emptyset = -1$.

1. Introduction. This paper is devoted to generic intersections of rotations in the three classical spaces: the Euclidean space $\mathbb{R}^n$, the unit sphere $\mathbb{S}^n$ and the hyperbolic space $\mathbb{H}^n$ with $n \geq 2$. From now on, $\mathbb{X}^n$ will stand for one of those spaces.

By a *rotation* of $\mathbb{X}^n$ we mean an orientation preserving isometry of $\mathbb{X}^n$ whose set of fixed points is an $(n-2)$-dimensional subspace H of $\mathbb{X}^n$. Given such a subspace H and an angle, there are at most two rotations of $\mathbb{X}^n$ around H by that angle. Let $\mathfrak{R}_n$ denote the set of all rotations of $\mathbb{X}^n$ with its natural topology. It may be regarded as an algebraic submanifold of the linear group of all isometries of $\mathbb{X}^n$, which, as is easily seen, has dimension $2n-1$.

Consider an o-minimal expansion Σ of the field $\mathbb{R}$ of real numbers. By definable sets we shall always mean sets definable with parameters with respect to the structure Σ. This paper is related to the articles [3, 4], and its main result, stated below, partially generalizes the theorem on generic intersections from the former article, because the set of rotations does not have a group structure (under composition).

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THEOREM 1.1. *If $A, B \subset \mathbb{X}^n$ are non-empty definable subsets of dimensions $k, l < n$, respectively, then there exists a definable nowhere dense subset Z of $\mathfrak{R}_n$ such that*

$$\dim(\sigma(A) \cap B) < \min\{k, l\} \quad \textit{for all } \sigma \in \mathfrak{R}_n \setminus Z.$$

This theorem is inspired by and applied in [2], devoted to a concept of a small set, a refinement of the notion of a Tarski nullset.

2. Proof of Theorem 1.1. The proof given here is an adaptation of the proof of the theorem on generic intersections in [3]. We need an elementary proposition relying on definable cell decomposition, which can be found e.g. in [1, Chap. 4, Proposition 1.5].

PROPOSITION 2.1. *Let $f : V \to W$ be a definable map between non-empty definable sets. Then*

$$\dim f^{-1}f(v) \le k \textit{ for all } v \in V \;\Rightarrow\; \dim V \le k + \dim f(V),$$
$$\dim f^{-1}f(v) \ge k \textit{ for all } v \in V \;\Rightarrow\; \dim V \ge k + \dim f(V). \blacksquare$$

Now let

$$X := \mathbb{X}^n \times \mathbb{X}^n, \qquad \mathcal{X} := \{(\sigma, x, y) \in \mathfrak{R}_n \times X : y = \sigma(x)\},$$

and let

$$p : \mathcal{X} \to \mathfrak{R}_n \quad \text{and} \quad q : \mathcal{X} \to X$$

be the canonical projections. Clearly, the set $\mathcal{X}$ and the maps p and q are definable. It is not difficult to calculate the dimension of the fibres $q^{-1}(x, y)$:

$$\dim q^{-1}(x, y) = n - 1 \qquad \text{if } x \neq y, \tag{2.1}$$
$$\dim q^{-1}(x, y) = 2n - 3 \qquad \text{if } x = y. \tag{2.2}$$

Without loss of generality we may assume that $k \le l$. We can partition $A \times B$ into two sets:

$$A \times B = (A \times B \setminus \Delta) \cup (A \times B \cap \Delta),$$

where $\Delta \subset X$ is the diagonal of X.

Proposition 2.1 along with formulae (2.1) and (2.2) yields

$$\dim q^{-1}(A \times B \setminus \Delta) \le k + l + n - 1 \le k + 2n - 2 = k - 1 + \dim \mathfrak{R}_n,$$
$$\dim q^{-1}(A \times B \cap \Delta) \le k + 2n - 3 < k - 1 + \dim \mathfrak{R}_n.$$

Hence

$$\dim \mathcal{E} \le k - 1 + \dim \mathfrak{R}_n \quad \text{where} \quad \mathcal{E} := q^{-1}(A \times B) \subset \mathcal{X}.$$

Again it follows from Proposition 2.1 that the definable subset

$$Z := \{\sigma \in \mathfrak{R}_n : \dim(p^{-1}(\sigma) \cap \mathcal{E}) > k - 1\}$$

is of dimension $< \dim \mathfrak{R}_n$, and thus is a nowhere dense subset of $\mathfrak{R}_n$.

Further, let $\pi : \mathit{\Delta} \to \mathbb{X}^n$ be the projection onto the first factor. Then

$$\begin{aligned}\sigma(A) \cap B &= \pi((\sigma(A) \times B) \cap \mathit{\Delta}) \\ &= \pi \circ (\sigma \times \mathrm{Id}_{\mathbb{X}^n})((A \times B) \cap \{(x, \sigma(x)) : x \in \mathbb{X}^n\}),\end{aligned}$$

whence the sets $\sigma(A) \cap B$ and

$$\{\sigma\} \times [(A \times B) \cap \{(x, \sigma(x)) : x \in \mathbb{X}^n\}] = p^{-1}(\sigma) \cap \mathcal{E}$$

are definably homeomorphic. Therefore

$$\dim(\sigma(A) \cap B) \le k - 1 \quad \text{for all } \sigma \in \mathfrak{R}_n \setminus Z,$$

which completes the proof. ∎

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