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DYNAMICS OF COMMUTING HOMEOMORPHISMS OF CHAINABLE CONTINUA

BY

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Abstract. A chainable continuum, X, and homeomorphisms, $p, q : X \to X$, are constructed with the following properties:

(1) $p \circ q = q \circ p$,
(2) p, q have simple dynamics,
(3) $p \circ q$ is a positively continuum-wise fully expansive homeomorphism that has positive entropy and is chaotic in the sense of Devaney and in the sense of Li and Yorke.

1. Introduction. In this paper, a chainable continuum, X, and homeomorphisms, $p, q : X \to X$, are constructed with the following properties:

(1) $p \circ q = q \circ p$,
(2) p, q have simple dynamics,
(3) $p \circ q$ is a positively continuum-wise fully expansive homeomorphism that has positive entropy and is chaotic in the sense of Devaney and in the sense of Li and Yorke.

A *continuum* is a compact, connected metric space. A continuum is *chainable* (also known as *arc-like*) if it can be expressed as the inverse limit of arcs. Let $f : X \to X$ be an onto map. We say that f has *simple dynamics* if there exist exactly two fixed points $a, r \in X$ such that

(1) for every $\epsilon > 0$ and $z \in X - \{r\}$, there exists a number N_z^ϵ such that $\mathrm{d}(f^n(z), a) < \epsilon$ for all $n \geq N_z^\epsilon$,
(2) for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that if $\mathrm{d}(x, a) < \delta_\epsilon$ then $\mathrm{d}(f^n(x), a) < \epsilon$ for each $n \geq 0$.

Here, a is called the *attractor* for f and r is the *repellor* for f. An example of a function with simple dynamics is $f(x) = x^2$ defined on the interval $[0, 1]$. In this case 0 is the attractor and 1 is the repellor. It will be shown that maps with simple dynamics are true to their name in that they are not transitive, sensitive or chaotic in the sense of either Li and Yorke or Devaney (see [13]

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and [5]). However, it will be shown that it is possible to get a very chaotic function from the composition of two commuting maps with simple dynamics. It was shown by Cánovas and Linero that if f and g are commuting maps of the interval that share a periodic point that is not a power of 2 for f, then the composition must have positive entropy [3], and if f and g are commuting piecewise monotonic maps, then the entropy of the composition is less than or equal to the sum of the individual entropies [4]. Furthermore, it was shown by Sun, Xi and Chen that under certain hypotheses, if f and g are commuting maps of a tree that share a periodic point, then the entropy of the composition is also positive [15].

2. Definitions and terminology. The term *chaos* to describe a dynamical system has been defined in several nonequivalent ways. Here several definitions, measures and types of chaotic dynamical systems will be given.

Let $f : X \to X$ be a map on a compact metric space X. We say that f has *sensitive dependence on initial conditions* (or *f is s.d.i.c.*) if there is a constant $c > 0$ such that for every $x \in X$ and open set U that contains x, there exists a $y \in U$ and an integer $n \geq 0$ such that $\mathrm{d}(f^n(x), f^n(y)) > c$. We say that f is *transitive* on X if whenever U and V are open sets of X, there exists an integer $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$. Then a map f is *chaotic in the sense of Devaney* if

(1) f has sensitive dependence on initial conditions,
(2) f is transitive,
(3) the periodic points of f are dense in X.

A subset S of X is called a *scrambled set* of f if S has at least two elements and any distinct $x, y \in S$ satisfy the following:

(1) $\limsup_{n\to\infty} \mathrm{d}(f^n(x), f^n(y)) > 0$,
(2) $\liminf_{n\to\infty} \mathrm{d}(f^n(x), f^n(y)) = 0$,
(3) $\limsup_{n\to\infty} \mathrm{d}(f^n(x), f^n(p)) > 0$ for any periodic point p of f.

If there exists an uncountable scrambled set of f, then f is said to be *chaotic in the sense of Li and Yorke.*

One measure of the "chaos" of a dynamical system is *entropy*. The following definition of entropy is due to Bowen (see [16]). Suppose that $f : X \to X$ is a map of a compact space and n is a nonnegative integer. Define

$$d_n^+(x, y) = \max_{0\leq i<n} \mathrm{d}(f^i(x), f^i(y)).$$

A finite subset E_n of X is said to be (n, ϵ)-*separated* with respect to f if $d_n^+(x, y) > \epsilon$ whenever x and y are distinct elements of E_n. Let $s_n(\epsilon, X, f)$ denote the largest cardinality of any (n, ϵ)-separated subset of X with respect to f. Then put

$$s(\epsilon, X, f) = \limsup_{n\to\infty} \frac{\log s_n(\epsilon, X, f)}{n}.$$

The *entropy* of f on X is then defined as

$$\mathrm{Ent}(f, X) = \lim_{\epsilon\to 0} s(\epsilon, X, f).$$

The next type of function has very chaotic properties: A map $f : X \to X$ is *positively continuum-wise expansive* if there exists a constant $c > 0$ such that for every nondegenerate subcontinuum A, there is a nonnegative integer n such that $\mathrm{diam}(f^n(A)) \geq c$. Next if A and B are sets, define $\mathrm{d}(A, B) = \inf\{\mathrm{d}(x, y) \mid x \in A \text{ and } y \in B\}$. Then define

$$\mathrm{d}_S(A, B) = \sup\{d(A, y) \mid y \in B\}.$$

Finally define the *Hausdorff distance* to be

$$\mathrm{d}_H(A, B) = \max\{\mathrm{d}_S(A, B), \mathrm{d}_S(B, A)\}.$$

A stronger version of positively continuum-wise expansive is the following: f is a *continuum-wise fully expansive* (*CF-expansive*) homeomorphism if for any $\epsilon, \delta > 0$ there exists a positive integer N such that if A is a subcontinuum of X such that $\mathrm{diam}(A) > \delta$ then either

(1) $\mathrm{d}_H(f^n(A), X) < \epsilon$ for all $n \geq N$, or
(2) $\mathrm{d}_H(f^{-n}(A), X) < \epsilon$ for all $n \geq N$.

A homeomorphism or a map f is *positively continuum-wise fully expansive* (*PCF-expansive*) if condition (1) is satisfied for every nondegenerate subcontinuum A. The shift homeomorphism of the inverse limit of a tent map on I is a PCF-expansive homeomorphism (see [18]). However, in [11], Kato constructed a CF-expansive homeomorphism that is not PCF-expansive and whose inverse is not PCF-expansive.

One method for constructing complicated compact spaces and maps on those spaces is through inverse limits. Let X be a topological space and $f : X \to X$ be a map. The *inverse limit* of (X, f) is a new topological space:

$$\widehat{X} = \varprojlim\{X, f\}_{i=1}^{\infty} = \{\langle x_i\rangle_{i=1}^{\infty} \mid x_i \in X \text{ and } f(x_{i+1}) = x_i\}.$$

$\widehat{X}$ has the subspace topology induced on it by $\prod_{i=1}^{\infty} X$. If $\langle x_i\rangle_{i=1}^{\infty}, \langle y_i\rangle_{i=1}^{\infty} \in \widehat{X}$ then define the metric on $\widehat{X}$ by

$$\widehat{\mathrm{d}}(\langle x_i\rangle_{i=1}^{\infty}, \langle y_i\rangle_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{\mathrm{d}(x_i, y_i)}{2^i}$$

where d is the metric on X. Also, let $\pi_i : \widehat{X} \to X$ be the ith coordinate map. For more on inverse limits see [6], [7], or [17].

Define the *shift homeomorphism* $\widehat{f} : \widehat{X} \to \widehat{X}$ by

$$\widehat{f}(\langle x_i \rangle_{i=1}^{\infty}) = \langle f(x_i) \rangle_{i=1}^{\infty}.$$

Also, notice that

$$\widehat{f}^{-1}(\langle x_1, x_2, x_3, \ldots \rangle) = \langle x_2, x_3, x_4, \ldots \rangle.$$

The shift homeomorphism, $\widehat{f}$, often has the same dynamical properties as the bonding map f; see [10] and [19].

3. Results and relationships on chaotic maps. There are several results examining the relationship between expansiveness, entropy and chaos in the sense of Devaney and in the sense of Li and Yorke. The following is Theorem 4.1 in [8]:

THEOREM 3.1. *If $f : X \to X$ is a continuum-wise expansive homeomorphism, then* $\mathrm{Ent}(f) > 0$.

The following is Lemma 4.3 in [9]:

THEOREM 3.2. *If $f : I \to I$ is a continuum-wise expansive map of the interval, then f has sensitive dependence on initial conditions.*

The next theorem follows from Theorem 3.1 in [19]:

THEOREM 3.3. *Suppose $\widehat{X} = \varprojlim\{X, f\}$, $h : X \to X$, $h \circ f = f \circ h$ and $\widehat{h} : \widehat{X} \to \widehat{X}$ is defined by $\widehat{h}(\langle x_i \rangle_{i=1}^{\infty}) = \langle h(x_i) \rangle_{i=1}^{\infty}$. Then* $\mathrm{Ent}(\widehat{h}) = \mathrm{Ent}(h)$.

The following is Theorem C of [12]:

THEOREM 3.4. *Let $f : X \to X$ be a map on a compact metric space. Then f is chaotic in the sense of Devaney if and only if the shift homeomorphism $\widehat{f}$ of $\varprojlim\{X, f\}$ is chaotic in the sense of Devaney.*

The following is Corollary 2.4 of [2]:

THEOREM 3.5. *If $f : X \to X$ is a map on a compact metric space with positive entropy, then f is chaotic in the sense of Li and Yorke.*

The following theorem is a corollary to Lemma 2 in [1]:

THEOREM 3.6. *Let $f : I \to I$ be a transitive map of the interval. Then the set of periodic points is dense in I.*

PROPOSITION 3.7. *Let $f : X \to X$ be PCF-expansive. Then f is transitive.*

Proof. Let U and V be open sets. Pick $x \in V$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) = \{y \in X \mid \mathrm{d}(x, y) < \epsilon\} \subset V$. Also, there exists a nondegenerate subcontinuum $A \subset U$. Since f is positively continuum-wise fully expansive, there exists $n \geq 0$ such that $\mathrm{d}_H(f^n(A), X) < \epsilon$. Thus,

$$\emptyset \neq f^n(A) \cap V \subset f^n(U) \cap V. \ \blacksquare$$

Corollary 3.8. *Let $f : I \to I$ be a PCF-expansive map on the interval. Then f is chaotic in the sense of Devaney and in the sense of Li and Yorke.*

Theorem 3.9. *Let $\widehat{X} = \varprojlim\{X, f\}$ and $h : X \to X$ be such that h and f are both onto and commute. If h is PCF-expansive, then $\widehat{h} : \widehat{X} \to \widehat{X}$, defined by $\widehat{h}(\langle x_i\rangle_{i=1}^{\infty}) = \langle h(x_i)\rangle_{i=1}^{\infty}$, is PCF-expansive.*

Proof. Let $\epsilon > 0$ and $\widehat{A}$ be a subcontinuum of $\widehat{X}$. Then there exist subcontinua $\{A_i\}_{i=1}^{\infty}$ of X such that $\widehat{A} = \varprojlim\{A_k, f|_{A_k}\}_{k=1}^{\infty}$. Also, there exists an m such that

$$\sum_{i=m+1}^{\infty} \frac{\operatorname{diam}(X)}{2^i} < \frac{\epsilon}{2}.$$

Since h is positively continuum-wise fully expansive there exists an N_i such that $\mathrm{d}_H(h^n(A_i), X) < \epsilon/2$ for each $n \geq N_i$. Let $N = \max\{N_1, \ldots, N_m\}$. Then for every $n \geq N$ we have

$$\begin{aligned}
\widehat{\mathrm{d}}_H(h^n(\widehat{A}), \widehat{X}) &= \sum_{i=1}^{\infty} \frac{\mathrm{d}_H(h^n(A_i), X)}{2^i} \\
&= \sum_{i=1}^{m} \frac{\mathrm{d}_H(h^n(A_i), X)}{2^i} + \sum_{i=m+1}^{\infty} \frac{\mathrm{d}_H(h^n(A_i), X)}{2^i} \\
&< \frac{\epsilon}{2}\sum_{i=1}^{m} \frac{1}{2^i} + \frac{\epsilon}{2} \leq \epsilon. \ \blacksquare
\end{aligned}$$

4. Functions with simple dynamics. In this section, we see that functions with simple dynamics do not have any chaotic properties.

Proposition 4.1. *Suppose that $f : X \to X$ is a map with simple dynamics with repellor r and attractor a. Then f does not contain any scrambled sets.*

Proof. Let x and y be distinct elements of X.

Case 1: $x = r$. Since $y \in X - \{r\}$, $\lim_{n\to\infty} f^n(y) = a$. Therefore,

$$\liminf_{n\to\infty} \mathrm{d}(f^n(x), f^n(y)) = \mathrm{d}(r, a) \neq 0.$$

Hence, x and y cannot be in the same scrambled set.

Case 2: $y = r$. The proof is similar to Case 1.

Case 3: $x, y \in X - \{r\}$. Then $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(y) = a$. So

$$\limsup_{n\to\infty} \mathrm{d}(f^n(x), f^n(y)) = \mathrm{d}(a, a) = 0.$$

Hence, x and y cannot be in the same scrambled set. $\blacksquare$

COROLLARY 4.2. *If f has simple dynamics, then f does not have chaos in the sense of Li and Yorke.*

PROPOSITION 4.3. *Suppose that $f : X \to X$ is a map with simple dynamics with repellor r and attractor a. Then f does not have sensitive dependence on initial conditions.*

Proof. Pick any $\epsilon > 0$ and let $\delta > 0$ be such that if $\mathrm{d}(x,y) < \delta$ then $\mathrm{d}(f^n(x), f^n(y)) < \epsilon/2$ for all $n \geq 0$. Let $B_a = \{x \in X \mid \mathrm{d}(x,a) < \delta\}$. Pick any distinct $x, y \in B_a$. Then for every $n \geq 0$, $\mathrm{d}(f^n(x), a) < \epsilon/2$ and $\mathrm{d}(f^n(y), a) < \epsilon/2$. Thus by the triangle inequality, $\mathrm{d}(f^n(y), f^n(x)) < \epsilon$ for all $n \geq 0$. Consequently, f is not sensitive. ∎

PROPOSITION 4.4. *Suppose that $f : X \to X$ is a map with simple dynamics with repellor r and attractor a. Then f has only two periodic points, a and r.*

Proof. It will be shown that the only periodic points are the fixed points a, r. Suppose on the contrary that $x \in X - \{a, r\}$ is periodic. Then there exists an m such that $f^m(x) = x$. Furthermore, $f(\{x, f(x), \ldots, f^{m-1}(x)\}) = \{x, f(x), \ldots, f^{m-1}(x)\}$ and $a \notin \{x, f(x), \ldots, f^{m-1}(x)\}$. Let $\epsilon = \mathrm{d}(a, \{x, f(x), \ldots, f^{m-1}(x)\})$, which is positive. Then $\mathrm{d}(f^n(x), a) \geq \epsilon$ for all n, which is impossible. ∎

COROLLARY 4.5. *Suppose that $f : X \to X$ is a map with simple dynamics with repellor r and attractor a where X does not have the trivial topology and $|X| > 2$. Then f does not have a dense set of periodic points.*

PROPOSITION 4.6. *Suppose that $f : X \to X$ is a map with simple dynamics with repellor r and attractor a. Then f is not transitive.*

Proof. Let $\epsilon = \mathrm{d}(a,r)/3$, $V = \{x \in X \mid \mathrm{d}(x,r) < \epsilon\}$ and $B_a = \{x \in X \mid \mathrm{d}(x,a) < \epsilon\}$. Then $V \cap B_a = \emptyset$. Let $\delta_\epsilon > 0$ be such that if $\mathrm{d}(x,a) < \delta_\epsilon$ then $\mathrm{d}(f^n(x), a) < \epsilon$ for all $n \geq 0$. Define $U = \{x \in X \mid \mathrm{d}(x,a) < \delta_\epsilon\}$. Then $f^n(U) \subset B_a$ for all n. Hence, $f^n(U) \cap V = \emptyset$ for all $n \geq 0$. So f is not transitive. ∎

LEMMA 4.7. *Suppose that $f, g : X \to X$ are commuting onto maps such that f has simple dynamics with attractor a and repellor r. Then $g(a) = a$ and $g(r) = r$.*

Proof. First notice that $f(g(a)) = g(f(a)) = g(a)$ and $f(g(r)) = g(f(r)) = g(r)$. Thus, $g(a)$ and $g(r)$ are fixed points of f. Therefore, $g(a), g(r) \in \{a, r\}$.

For the purpose of a contradiction, suppose that $g(a) = r$. Let $0 < \epsilon < (1/4)\mathrm{d}(a,r)$. Since X is compact, g is uniformly continuous. Hence, there exists a $\delta_\epsilon > 0$ such that if $\mathrm{d}(x,y) < \delta_\epsilon$ then $\mathrm{d}(g(x), g(y)) < \epsilon$. Pick

$y \in X - \{r\}$ such that $g(y) \notin \{a, r\}$. Since a is an attractor, there exists an N_1 such that $\mathrm{d}(f^n(y), a) < \delta_\epsilon$ for all $n \geq N_1$. So $\mathrm{d}(g(f^n(y)), r) = \mathrm{d}(g(f^n(y)), g(a)) < \epsilon$ for all $n \geq N_1$. Also since $g(y) \neq r$, there exists an N_2 such that $\mathrm{d}(f^n(g(y)), a) < \epsilon$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then $\mathrm{d}(f^N(g(y)), a) < (1/4)\mathrm{d}(a, r)$ and $\mathrm{d}(f^N(g(y)), r) < (1/4)\mathrm{d}(a, r)$. So by the triangle inequality, $\mathrm{d}(a, r) < (1/2)\mathrm{d}(a, r)$, which is impossible since a and r are distinct.

To show that $g(r) = r$ suppose on the contrary that there exists $z \in X - \{r\}$ such that $g(z) = r$. Choose ϵ and δ_ϵ in a similar way. Then there exists an M such that $\mathrm{d}(a, f^n(z)) < \delta_\epsilon$ for all $n \geq M$. Hence,

$$\begin{aligned}\mathrm{d}(a, r) &= \mathrm{d}(g(a), f^M(r)) = \mathrm{d}(g(a), f^M(g(z))) \\ &= \mathrm{d}(g(a), g(f^M(z))) < \epsilon < (1/4)\mathrm{d}(a, r).\end{aligned}$$

Again, this is impossible since a and r are distinct. Since g is onto, $g(r) = r$. ∎

THEOREM 4.8. *Let $\widehat{X} = \varprojlim\{X, f\}$ and $h : X \to X$ be such that h and f are both onto and commute. If h has simple dynamics with attractor a and repellor r, then $\widehat{h} : \widehat{X} \to \widehat{X}$, defined by $\widehat{h}(\langle x_i\rangle_{i=1}^\infty) = \langle h(x_i)\rangle_{i=1}^\infty$, has simple dynamics with attractor $\widehat{a} = \langle a\rangle_{i=1}^\infty$ and repellor $\widehat{r} = \langle r\rangle_{i=1}^\infty$.*

Proof. Since $h(a) = a$ and $h(r) = r$, it follows that $f(a) = a$ and $f(r) = r$ by Lemma 4.7. Thus $\widehat{a} = \langle a\rangle_{i=1}^\infty$ and $\widehat{r} = \langle r\rangle_{i=1}^\infty$ are elements of $\widehat{X}$ and fixed points of $\widehat{h}$. Pick $\epsilon > 0$ and let $\widehat{x} = \langle x_i\rangle_{i=1}^\infty \in \widehat{X} - \{\widehat{r}\}$. Then each x_i is in $X - \{r\}$. Let $\epsilon > 0$. Then there exists an m such that

$$\sum_{i=m+1}^{\infty} \frac{\mathrm{diam}(X)}{2^i} < \frac{\epsilon}{2}.$$

For each $i \in \{1, \ldots, m\}$ there exists N_i such that $\mathrm{d}(h^n(x_i), a) < \epsilon/2$ for all $n \geq N_i$. Let $N = \max\{N_1, \ldots, N_m\}$. Then for every $n \geq N$,

$$\begin{aligned}\widehat{\mathrm{d}}(\widehat{h}^n(\widehat{x}), \widehat{a}) &= \sum_{i=1}^{\infty} \frac{\mathrm{d}(h^n(x_i), a)}{2^i} = \sum_{i=1}^{m} \frac{\mathrm{d}(h^n(x_i), a)}{2^i} + \sum_{i=m+1}^{\infty} \frac{\mathrm{d}(h^n(x_i), a)}{2^i} \\ &< \frac{\epsilon}{2}\sum_{i=1}^{m} \frac{1}{2^i} + \frac{\epsilon}{2} \leq \epsilon.\end{aligned}$$

Also, since h has simple dynamics, there exists $\delta_{\epsilon/2} > 0$ such that if $\mathrm{d}(x, a) < \delta_{\epsilon/2}$, then $\mathrm{d}(h^n(x), a) < \epsilon/2$ for all $n \geq 0$. Let $\widehat{\delta}_\epsilon = \delta_{\epsilon/2}/2^m$. Then if $\widehat{\mathrm{d}}(\widehat{x}, \widehat{a}) < \widehat{\delta}_\epsilon$,

$$\sum_{i=1}^{m} \frac{\mathrm{d}(x_i, a)}{2^i} < \frac{\delta_{\epsilon/2}}{2^m}.$$

Thus $\mathrm{d}(x_i, a) < \delta_{\epsilon/2}$ for all $i \in \{0, 1, \dots, m\}$. So $\mathrm{d}(h^n(x_i), a) < \epsilon/2$ for all n and $i \leq m$. Thus by a similar argument, $\widehat{\mathrm{d}}(h^n(\widehat{x}), \widehat{a}) < \epsilon$ for all n and $\widehat{h}$ has simple dynamics. ∎

The following propositions show that $f(x) = x^2$ and $g(x) = \sqrt{x}$ have simple dynamics on $[0, 1]$.

PROPOSITION 4.9. *Let $f : I \to I$ be a map such that $f(0) = 0$, $f(1) = 1$ and for $x \in (0, 1)$, $f(x) < x$. Then f has simple dynamics with attractor* 0 *and repellor* 1.

PROPOSITION 4.10. *Let $f : I \to I$ be a map such that $f(0) = 0$, $f(1) = 1$ and for $x \in (0, 1)$, $f(x) > x$. Then f has simple dynamics with attractor* 1 *and repellor* 0.

5. Main result. Before we construct the chainable continuum and homeomorphisms with the desired properties, we need some results to help prove that the constructions have the stated properties:

Suppose that $\{x_n\}_{n=0}^{\infty}$ is an increasing sequence in $(0, 1)$ that converges to 1 and $\{x_n\}_{n=0}^{-\infty}$ is a decreasing sequence in $(0, 1)$ that converges to 0. Let $A_n = [x_n, x_{n+1}]$. Then we say that $\{A_n\}_{n=-\infty}^{\infty}$ is a *bi-infinite partition of* $(0, 1)$.

PROPOSITION 5.1. *Suppose that $f : [a, b] \to \mathbb{R}$ is a map such that there exists a finite set $Y \subset [a, b]$ and $c > 0$ such that $f'(x) > c$ for every $x \in [a, b] - Y$. If $[x, y] \subset [a, b]$ then*

$$\mathrm{diam}(f([x, y])) \geq c\,\mathrm{diam}([x, y]).$$

PROPOSITION 5.2. *Suppose that $f : [a, b] \to \mathbb{R}$ is a map such that there exists a finite set $Y \subset [a, b]$ and $c > 0$ such that $-f'(x) > c$ for every $x \in [a, b] - Y$. If $[x, y] \subset [a, b]$ then*

$$\mathrm{diam}(f([x, y])) \geq c\,\mathrm{diam}([x, y]).$$

THEOREM 5.3. *Let $\{A_n\}_{n=-\infty}^{\infty}$, where $A_n = [x_n, x_{n+1}]$, be a bi-infinite partition of $(0, 1)$ and $f : [0, 1] \to [0, 1]$ be a map that is differentiable on $D = \bigcup_{n=-\infty}^{\infty}(\mathrm{int}(A_n) - C_n)$ where C_n is a finite subset of $\mathrm{int}(A_n)$ with the following properties:*

(1) *there exists $c > 1$ such that for every $n \in \mathbb{Z}$ one of the following is true:*
    (a) *$f'(x) \geq c$ for every $x \in \mathrm{int}(A_n) - C_n$,*
    (b) *$-f'(x) \geq c$ for every $x \in \mathrm{int}(A_n) - C_n$,*
(2) *$A_{n-1} \cup A_n \cup A_{n+1} \subset f(A_n)$ for each n,*
(3) *$f(\{x_n\}_{n=-\infty}^{\infty}) = \{x_n\}_{n=-\infty}^{\infty}$.*

Then f is PCF-expansive.

Proof. We first prove

CLAIM 1. *Let A be a subarc of I such that x_k is an endpoint of A. Then one of the following must be true:*

(1) *there exists an m such that x_m is an endpoint of $f(A)$,*
(2) $x_{k-1} \in A$,
(3) $x_{k+1} \in A$.

Suppose that (2) and (3) are false. Then $A \subset A_{k-1}$ or $A \subset A_k$. Since $f|_{A_{k-1}}$ and $f|_{A_k}$ are one-to-one and $f(\{x_n\}_{n=-\infty}^{\infty}) = \{x_n\}_{n=-\infty}^{\infty}$, it follows that $x_m = f(x_k)$ is an endpoint of $f(A)$ for some m.

CLAIM 2. *Let A be a subarc of I such that x_k is an endpoint of A. Then there exists a natural number n such that $|f^n(A) \cap \{x_k\}_{k=-\infty}^{\infty}| \geq 2$.*

Suppose on the contrary that $|f^n(A) \cap \{x_k\}_{k=-\infty}^{\infty}| < 2$ for all n. Then by induction and Claim 1, there exists a sequence of integers $\{k_n\}_{n=1}^{\infty}$ such that $k_1 = k$ and x_{k_n} is an endpoint of $f^{n-1}(A)$. Thus, $f^n(A) \subset A_{k_{n+1}-1}$ or $f^n(A) \subset A_{k_{n+1}}$ for each n. Therefore by induction and Propositions 5.1 and 5.2,

$$\operatorname{diam}(f^n(A)) \geq c^n \operatorname{diam}(A).$$

Since $\operatorname{diam}(A) > 0$, there exists an n such that $\operatorname{diam}(f^n(A)) > 1$. However, this contradicts $f^n(A) \subset I$.

CLAIM 3. *If A is a subarc of I, then there exists a p such that $f^p(A) \cap \{x_k\}_{k=-\infty}^{\infty} \neq \emptyset$.*

Suppose on the contrary that $f^p(A) \cap \{x_k\}_{k=-\infty}^{\infty} = \emptyset$ for all p. Then $f^p(A) \subset A_{n_p}$ for some n_p. Therefore by induction, and Propositions 5.1 and 5.2,

$$\operatorname{diam}(f^p(A)) \geq c^p \operatorname{diam}(A).$$

Since $\operatorname{diam}(A) > 0$, there exists an n such that $\operatorname{diam}(f^n(A)) > 1$. However, this contradicts $f^n(A) \subset I$.

CLAIM 4. *If A is a subarc of I, then there exists a q such that $|f^q(A) \cap \{x_k\}_{k=-\infty}^{\infty}| \geq 2$.*

By Claim 3, there exist p and k such that $x_k \in f^p(A)$. Thus there exists a subarc A' of $f^p(A)$ such that x_k is an endpoint of A'. Hence by Claim 2, there exists an n such that $|f^n(A') \cap \{x_k\}_{k=-\infty}^{\infty}| \geq 2$. Let $q = n + p$. Then $|f^q(A) \cap \{x_k\}_{k=-\infty}^{\infty}| \geq 2$.

CLAIM 5. *Given $\epsilon > 0$ and k, there exists N_ϵ^k such that $\mathrm{d}_H(f^n(A_k), I) < \epsilon$ for every $n \geq N_\epsilon^k$.*

There exist integers k_1, k_2 such that $x_{k_1} < \epsilon$ and $1 - x_{k_2} < \epsilon$. Let $N_\epsilon^k = \max\{|k - k_1|, |k - k_2|\}$. Then it follows from condition (2) of the hypothesis of the theorem and induction that $[x_{k_1}, x_{k_2}] \subset f^n(A_k)$ for all $n \geq N_\epsilon^k$. Thus

$$\mathrm{d}_H(f^n(A_k), I) \leq \mathrm{d}_H([x_{k_1}, x_{k_2}], I) < \epsilon.$$

To prove the theorem, let $\epsilon > 0$ and A be any subarc of I. Then by Claim 4, there exist q and k such that $A_k \subset f^q(A)$. Then by Claim 5, there exists N_ϵ^k such that $\mathrm{d}_H(f^n(A_k), I) < \epsilon$ for every $n > N_\epsilon^k$. Thus for every $n \geq N_\epsilon^k + q$, $\mathrm{d}_H(f^n(A), I) < \epsilon$. ∎

Now for the construction of the chainable continuum and homeomorphisms with simple dynamics: Let $H : [5/16, 3/8] \to [0, 1]$ be defined by

$$H(x) = -7x + 45/16,$$

$L : [3/8, 5/8] \to [0, 1]$ be defined by

$$L(x) = (5/2)x - 3/4,$$

and $g : [0, 1] \to [0, 1]$ be defined by

$$g(x) = \begin{cases} 2x & \text{if } x \leq 1/3, \\ (1/2)x + 1/2 & \text{if } x > 1/3. \end{cases}$$

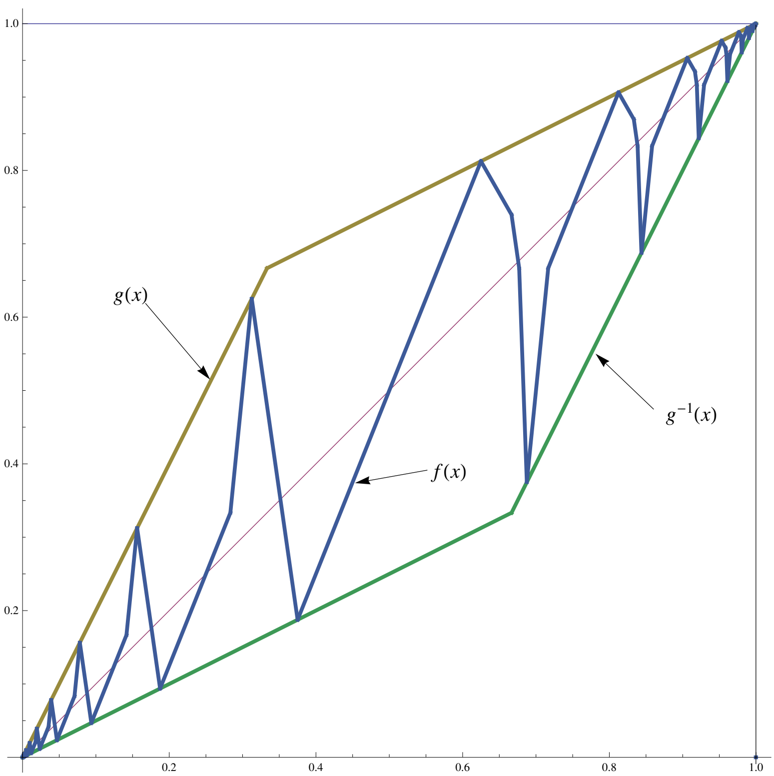

Fig. 1. $f(x)$, $g(x)$, and $g^{-1}(x)$

Notice that g is a homeomorphism and that

$$g^{-1}(x) = \begin{cases} (1/2)x & \text{if } x \leq 2/3, \\ 2x - 1 & \text{if } x > 2/3. \end{cases}$$

Next let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ g^n \circ H \circ g^{-n}(x) & \text{if } x \in (g^{n-1}(5/8), g^n(3/8)], \\ g^n \circ L \circ g^{-n}(x) & \text{if } x \in (g^n(3/8), g^n(5/8)], \\ 1 & \text{if } x = 1, \end{cases}$$

where $n \in \mathbb{Z}$ (see Figure 1). Then

$$f(x)=\begin{cases} 0 & \text{if } x = 0, \\ -7(x - g^n(3/8)) + g^{n-1}(3/8) & \text{if } x \in (g^{n-1}(5/8), g^n(3/8)] \\ & \quad \text{and } n < 0, \\ (5/2)(x - g^n(3/8)) + g^{n-1}(3/8) & \text{if } x \in (g^n(3/8), g^{n+1}(17/60)] \\ & \quad \text{and } n < 0, \\ 10(x - g^n(5/8)) + g^{n+1}(5/8) & \text{if } x \in (g^{n+1}(17/60), g^n(5/8)] \\ & \quad \text{and } n < 0, \\ -7x + 45/16 & \text{if } x \in (5/16, 3/8], \\ (5/2)x - 3/4 & \text{if } x \in (3/8, 5/8], \\ (-7/4)(x - g^{n-1}(5/8)) + g^n(5/8) & \text{if } x \in (g^{n-1}(5/8), g^{n-1}(2/3)] \\ & \quad \text{and } n > 0, \\ -7(x - g^{n-1}(2/3)) + g^{n-1}(71/96) & \text{if } x \in (g^{n-1}(2/3), g^{n-1}(65/96)] \\ & \quad \text{and } n > 0, \\ -28(x - g^n(3/8)) + g^{n-1}(3/8) & \text{if } x \in (g^{n-1}(65/96), g^n(3/8)] \\ & \quad \text{and } n > 0, \\ 10(x - g^n(3/8)) + g^{n-1}(3/8) & \text{if } x \in (g^n(3/8), g^{n-1}(43/60)] \\ & \quad \text{and } n > 0, \\ (5/2)(x - g^n(5/8)) + g^{n+1}(5/8) & \text{if } x \in (g^{n-1}(43/60), g^n(5/8)] \\ & \quad \text{and } n > 0, \\ 1 & \text{if } x = 1. \end{cases}$$

Notice that if $x \in (g^{n-1}(5/8), g^n(3/8)]$ for some n, then

$$\begin{aligned} g \circ f(x) &= g \circ f|_{(g^{n-1}(5/8), g^n(3/8)]}(x) = g \circ g^n \circ H \circ g^{-n}(x) \\ &= g^{n+1} \circ H \circ g^{-n-1}(g(x)) \\ &= f|_{(g^n(5/8), g^{n+1}(3/8)]} \circ g(x) = f \circ g(x). \end{aligned}$$

Similarly, it can be shown that if $x \in (g^n(3/8), g^n(5/8)]$ then

$$g \circ f(x) = g \circ f_{(g^n(3/8), g^n(5/8)]}(x) = f_{(g^{n+1}(3/8), g^{n+1}(5/8)]} \circ g(x) = f \circ g(x).$$

Thus $f \circ g = g \circ f$.

Corollary 5.4. *$f : I \to I$ is a PCF-expansive map.*

Proof. Let $x_{2n} = g^n(3/8)$ and $x_{2n+1} = g^n(5/8)$. Then

$$f(x_{2n}) = f(g^n(3/8)) = g^{n-1}(3/8) = x_{2n-2},$$
$$f(x_{2n+1}) = f(g^n(5/8)) = g^{n+1}(5/8) = x_{2n+3}.$$

Also, $f(\{x_{2n}\}_{n=-\infty}^{\infty}) = \{x_{2n}\}_{n=-\infty}^{\infty}$ and $f(\{x_{2n+1}\}_{n=-\infty}^{\infty}) = \{x_{2n+1}\}_{n=-\infty}^{\infty}$. So $f(\{x_n\}_{n=-\infty}^{\infty}) = \{x_n\}_{n=-\infty}^{\infty}$.

Next, let $A_{2n} = [x_{2n}, x_{2n+1}]$ and $A_{2n+1} = [x_{2n+1}, x_{2n+2}]$. Then

$$f(A_{2n}) = f([x_{2n}, x_{2n+1}]) = [x_{2n-2}, x_{2n+3}],$$
$$f(A_{2n+1}) = f([x_{2n+1}, x_{2n+2}]) = [x_{2n}, x_{2n+3}].$$

Hence $A_{n-1} \cup A_n \cup A_{n+1} \subset f(A_n)$ for each n. Finally, $f'(x) \geq 7/4$ for all $x \in A_{2n}$ and $-f'(x) \geq 7/4$ for all $x \in A_{2n+1}$. Thus f satisfies the hypothesis of Theorem 5.3. ∎

Define $\widehat{X} = \varprojlim\{X, f\}_{i=1}^{\infty}$ and let $\widehat{f}$ be the shift homeomorphism on $\widehat{X}$. Let $p = g^2 \circ f$ and $q = g^{-2}$ (see Figure 2). (Notice that g is a homeomorphism, so g^{-2} is also.) Then $p \circ q = q \circ p = f$.

Proposition 5.5. *The maps $p, q : I \to I$ have simple dynamics.*

Proof. Notice that $p(0) = q(0) = 0$, $p(1) = q(1) = 1$, $p(x) > x$ for all $x \in (0,1)$ and $q(x) < x$ for all $x \in (0,1)$. Thus, p and q have simple dynamics by Propositions 4.9 and 4.10. ∎

Define $\widehat{p}, \widehat{q} : \widehat{X} \to \widehat{X}$ by $\widehat{p}(\langle x_i \rangle_{i=1}^{\infty}) = \langle p(x_i) \rangle_{i=1}^{\infty}$ and $\widehat{q}(\langle x_i \rangle_{i=1}^{\infty}) = \langle q(x_i) \rangle_{i=1}^{\infty}$. The following is the main result of this paper:

Theorem 5.6. *$\widehat{p}, \widehat{q} : \widehat{X} \to \widehat{X}$ are homeomorphisms of a chainable continuum such that*

(1) *$\widehat{p}$ and $\widehat{q}$ commute,*
(2) *$\widehat{p}$ and $\widehat{q}$ have simple dynamics,*
(3) *$\widehat{f} = \widehat{q} \circ \widehat{p}$ is a positively continuum-wise expansive homeomorphism that is chaotic in the sense of Devaney and in the sense of Li and Yorke and has positive entropy.*

Proof. Since $p \circ q = q \circ p = f$, it follows that $\widehat{p} \circ \widehat{q} = \widehat{q} \circ \widehat{p} = \widehat{f}$ and thus $\widehat{p}$ and $\widehat{q}$ are homeomorphisms. By Proposition 5.5, p and q have simple dynamics, therefore by Theorem 4.8, $\widehat{p}$ and $\widehat{q}$ have simple dynamics. Since f is positively continuum-wise fully expansive, f is chaotic in the sense of

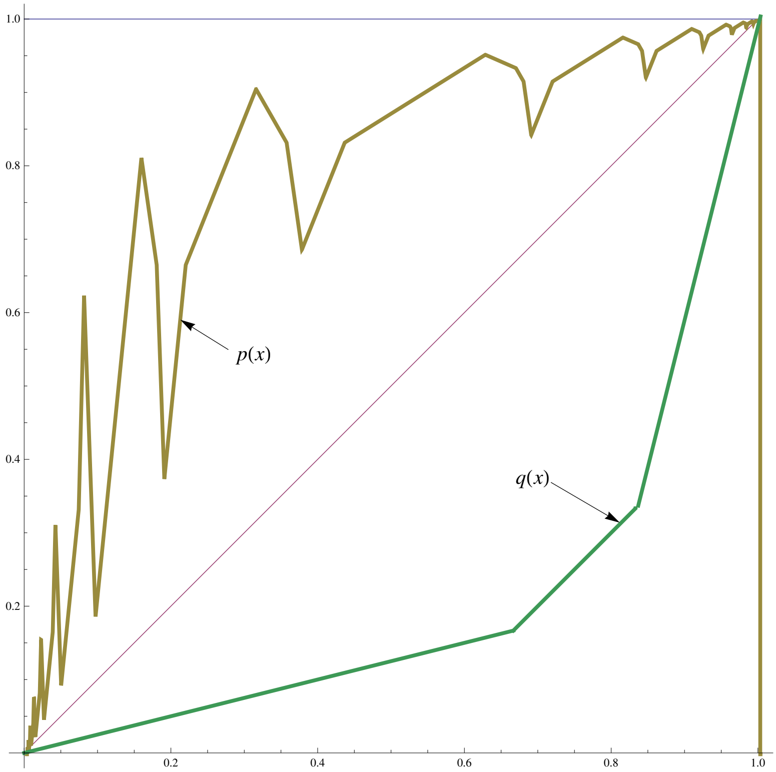

Fig. 2. $p(x)$ and $q(x)$

Devaney and in the sense of Li and Yorke by Corollary 3.8. Now, it follows from Theorem 3.9 that $\widehat{f}$ is positively continuum-wise fully expansive and hence has positive entropy and chaos in the sense of Devaney and in the sense of Li and Yorke by Theorems 3.3–3.5. ■

A continuum X is *decomposable* if there exist proper subcontinua A and B such that $A \cup B = X$. A continuum is *indecomposable* if it is not decomposable. The following theorem by Kato in [11] shows that $\widehat{X}$ is indecomposable:

THEOREM 5.7. *If $h : X \to X$ is a PCF-expansive homeomorphism, then X must be indecomposable.*

A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable. $\widehat{X}$ is not hereditarily indecomposable because it contains an arc. This can be shown by first noticing that $f|_{[3/8,5/8]}$ maps $[3/8, 5/8]$ one-to-one onto $[3/16, 13/16]$. Let $A_1 = [3/16, 13/16]$ and for $i > 1$ let $A_i = f^{-i+1}(A_1)$. Then $A_i \subset A_{i-1}$ and $f|_{A_i}$ is one-to-one for each $i > 1$. Hence, $A = \varprojlim\{A_i, f|_{A_{i+1}}\}_{i=1}^{\infty}$ is a subarc of $\widehat{X}$.

COROLLARY 5.8. *$p, q : I \to I$ are maps of the interval I such that*

(1) *p and q commute,*
(2) *p and q have simple dynamics,*
(3) *$f = q \circ p$ is a positively continuum-wise expansive map that is chaotic in the sense of Devaney and in the sense of Li and Yorke and has positive entropy.*

However, it is known that self-homeomorphisms of the interval do not admit any of the stated types of chaos. See, for example, [14] and [16].

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