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Generating iterated function systems for self-similar sets with a separation condition

by

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Abstract. Let $\{f_i\}_{i=1}^m$ be an iterated function system (IFS) for a self-similar set $E \subseteq \mathbb{R}^d$ (which is not a singleton) with the smallest integer $m \geq 2$. Suppose the distance of any two sets of the form $f_{i_1}(E)$ and $f_{i_2}(E)$ is strictly larger than the diameter of any $f_i(E)$. Then the semigroup of all generating IFSs for E, equipped with the composition as product, is finitely generated. This partially answers a question posed by Elekes, Keleti and Máthé.

1. Introduction. Self-similar sets provide basic models for Euclidean subspaces with fractal geometric structure, and iterated function systems (IFS) are a common tool for constructing them.

It is well-known that different collections of IFS can produce the same self-similar set. However, surprisingly, there are few results about the set of all generating IFSs for a given self-similar set E. When $E \subseteq \mathbb{R}$, this challenging problem was first considered by Feng and Wang [4] under various assumptions. Related work about generating IFSs for intersections of self-similar sets can be found e.g. in [1, 5, 8].

For the higher-dimensional case, the situation is somewhat different and more complicated, as rotations then play an essential role in IFS. As far as we know, the discussion in higher dimensions is limited either to homogeneous IFSs with the strong separation condition (SSC) [2] or to some special kind of planar self-similar sets [7]. These works do not involve more general self-similar sets.

In this paper, we will focus on generating IFSs for a class of self-similar sets in $\mathbb{R}^d$, allowing different linear parts but assuming the following sepa-

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ration condition:

$$\min_{1\le i\ne j\le m} d(f_i(E), f_j(E)) > \max_{1\le i\le m} \operatorname{diam} f_i(E), \tag{1.1}$$

where $\{f_i\}_{i=1}^m$ is a generating IFS for a self-similar set E with the smallest integer m.

Even though condition (1.1) is a little restrictive, we are able to generate a fairly rich set of examples such as Example 2 in [7].

Another motivation comes from Question 9.3 of Elekes, Keleti and Máthé [3]: Let $E \subseteq \mathbb{R}^d$ be a self-similar set generated by the IFS $\{f_i\}_{i=1}^m$ with the SSC, and let g be a contractive similitude such that $g(E) \subseteq E$. Is $g(E)$ a finite union of sets of the form $f_{\mathbf{i}}(E)$, where $\mathbf{i} \in \Omega^*$ and Ω^* is the set of all finite words over $\{1, \dots, m\}$?

The following theorem gives an affirmative answer to this question when the SSC is replaced by (1.1). The general case remains open.

Theorem 1.1. *Let $\{f_i\}_{i=1}^m$ be an IFS for a self-similar set $E \subseteq \mathbb{R}^d$ (which is not a singleton) with the smallest integer $m \ge 2$. If this IFS satisfies the extra condition* (1.1)*, then for any contractive similitude g with $g(E) \subseteq E$, there exists $\mathbf{i} \in \Omega^*$ such that $g(E) = f_{\mathbf{i}}(E)$.*

2. Proof of Theorem 1.1. First we introduce some notation.

By a *contractive similitude* we mean a map $S : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $|S(x) - S(y)| = \rho|x - y|$ for some *contractive ratio* $\rho \in (0, 1)$. It is well-known that $S(x)$ can be written as $\rho Rx + b$, where $x, b \in \mathbb{R}^d$ and R is a $d \times d$ orthogonal matrix.

Let $\{f_i\}_{i=1}^m$ be a family of contractive similitudes. Associated to them is a unique nonempty compact set E satisfying $E = \bigcup_{i=1}^m f_i(E)$. This set E is called a *self-similar set*, and the family $\{f_i\}_{i=1}^m$ is referred to as a generating *iterated function system* (IFS) for E. Furthermore, if the sets $f_1(E), \dots, f_m(E)$ are pairwise disjoint, we say that the IFS $\{f_i\}_{i=1}^m$ satisfies the *strong separation condition* (SSC).

The following notation for a sequence space is also needed.

Let $\Omega = \{1, \dots, m\}$, and let Ω^n be the set of words of length n in Ω. We use Ω^* and Ω^w to denote the set of all finite words and the set of all one-sided sequences in Ω, respectively.

Let $\mathbf{i}, \mathbf{j}$ be two words in Ω^*. We write $\mathbf{ij}$ for the juxtaposition of $\mathbf{i}$ and $\mathbf{j}$, and we write $f_{\mathbf{i}}$ or $f_{i_1 \dots i_n}$ for $f_{i_1} \circ \dots \circ f_{i_n}$ if $\mathbf{i} = i_1 \dots i_n$. Let $\mathbf{i}$ be a finite or infinite word. For a positive integer n, we use $\mathbf{i}|_n = i_1 \dots i_n$ to denote the truncation of $\mathbf{i}$ to its first n symbols.

Let ρ_g denote the contractive ratio of a contractive similitude g. For brevity, we write ρ_i and $\rho_{i_1 \dots i_n}$ instead of ρ_{f_i} and $\rho_{f_{i_1} \circ \dots \circ f_{i_n}}$, respectively.

The composition of contractive similitudes $g_1 \circ g_2$ is often written as $g_1 g_2$ or g_{12}.

We shall prove several results leading to Theorem 1.1; we reserve the letter E to denote a self-similar set in $\mathbb{R}^d$.

Lemma 2.1. *Let $\{f_i\}_{i=1}^m$ be an IFS for E. Suppose further that for any contractive similitude h with $h(E) \subsetneq E$, we have $h(E) \subseteq f_i(E)$ for some $i \in \Omega$. Let g be a contractive similitude satisfying $g(E) \subsetneq E$. Then $g(E) = f_{\mathbf{i}}(E)$ for some $\mathbf{i} \in \Omega^*$.*

Proof. We use an idea from [4, proof of Theorem 4.1].

Let ℓ be the largest positive integer such that $g(E) \subseteq f_{i_1 \dots i_\ell}(E)$ for some $i_1 \dots i_\ell \in \Omega^*$. Then either $(f_{i_1 \dots i_\ell})^{-1} g(E) = E$, or $(f_{i_1 \dots i_\ell})^{-1} g(E) \subseteq f_{i_{\ell+1}}(E)$ for some $i_{\ell+1} \in \Omega$. However, the latter is ruled out by the maximality of ℓ. ■

Lemma 2.2. *Let $\{f_i\}_{i=1}^m$ be an IFS for E satisfying* (1.1). *Let g be a contractive similitude with $g(E) \subsetneq E$. Assume that there exists a positive integer $\ell \geq 2$ such that $g^\ell(E) = f_i(E)$ for some $i \in \{1, \dots, m\}$. Then $\{f_i\}_{i=1}^m$ is not an IFS for E with the smallest integer m.*

Proof. It follows from $g^\ell(E) = f_i(E)$ that $g^{-1} f_i(E) = g^{\ell-1}(E) \subsetneq E$, hence there exists $j \neq i$ such that $g^{-1} f_i(E)$ intersects $f_j(E)$. Moreover, we claim that $g f_j(E) \subseteq f_i(E)$. Indeed, suppose not; then by (1.1) we have

$$\rho_g \cdot \operatorname{diam} f_j(E) = \operatorname{diam} g f_j(E) \geq \min_{k \neq i} d(f_i(E), f_k(E)) > \operatorname{diam} f_j(E),$$

a contradiction. Therefore, $f_j(E) \subseteq g^{-1} f_i(E) = g^{\ell-1}(E) \subseteq g(E)$. Combining this with $f_i(E) = g^\ell(E) \subseteq g(E)$ implies that $\{g, f_k\}_{k=1,\, k \neq i,j}^m$ is an IFS for E, which contradicts the minimality of m. ■

To prove Theorem 1.1, we also need the following two lemmas and one corollary.

Lemma 2.3 ([3, Lemma 4.8] simplified). *Let $\{f_i\}_{i=1}^m$ be an IFS for E with the SSC. There exist only finitely many contractive similitudes g for which $g(E) \subseteq E$ and $g(E)$ intersects at least two different sets in $\{f_1(E), \dots, f_m(E)\}$.*

Lemma 2.4. *Let $\mathbf{i}, \mathbf{j}$ be two different words in Ω^n. Then*

$$\begin{aligned} d(f_{\mathbf{i}}(E), f_{\mathbf{j}}(E)) &> \max\{\rho_{i_1 \dots i_{n-1}}, \rho_{j_1 \dots j_{n-1}}\} \cdot \max_i\{\rho_i\} \cdot \operatorname{diam} E \\ &\geq \max\{\operatorname{diam} f_{\mathbf{i}}(E), \operatorname{diam} f_{\mathbf{j}}(E)\}. \end{aligned}$$

Proof. We only need to prove the first inequality.

Let $\mathbf{i} = i_1 \dots i_n$ and $\mathbf{j} = j_1 \dots j_n$. Let $k_0 = \min\{1 \leq k \leq n \mid i_k \neq j_k\}$.

The desired statement follows instantly from (1.1) and

$$\begin{aligned} d(f_{\mathbf{i}}(E), f_{\mathbf{j}}(E)) &= \rho_{i_1\ldots i_{k_0-1}} d(f_{i_{k_0}\ldots i_n}(E), f_{j_{k_0}\ldots j_n}(E)) \\ &\ge \rho_{i_1\ldots i_{k_0-1}} d(f_{i_{k_0}}(E), f_{j_{k_0}}(E)) \\ &\ge \max\{\rho_{i_1\ldots i_{n-1}}, \rho_{j_1\ldots j_{n-1}}\} \cdot d(f_{i_{k_0}}(E), f_{j_{k_0}}(E)). \ \blacksquare \end{aligned}$$

We have the following immediate corollary of Lemma 2.4.

Corollary 2.5. *Let* $\mathbf{i}$ *be a nonempty word in* Ω^*. *Let* g *be a contractive similitude such that* $g(E)$ *is contained in* E *and intersects* $f_{\mathbf{i}}(E)$. *Assume further that the contractive ratio* ρ_g *is at most* $\rho_{\mathbf{i}}$. *Then* $g(E) \subseteq f_{\mathbf{i}}(E)$.

Proof. Suppose $\mathbf{i} \in \Omega^n$ for some positive integer n. If the conclusion does not hold, then by Lemma 2.4,

$$\begin{aligned} \rho_g \cdot \operatorname{diam} E = \operatorname{diam} g(E) &\ge \min_{\mathbf{j}\ne\mathbf{i},\, \mathbf{j}\in\Omega^n} d(f_{\mathbf{i}}(E), f_{\mathbf{j}}(E)) \\ &> \operatorname{diam} f_{\mathbf{i}}(E) = \rho_{\mathbf{i}} \cdot \operatorname{diam} E, \end{aligned}$$

a contradiction. ■

Proof of Theorem 1.1. By Lemma 2.1, it suffices to prove that there exists $i \in \Omega$ such that $g(E) \subseteq f_i(E)$.

Suppose this does not hold; then by (1.1),

$$\rho_g \cdot \operatorname{diam} E = \operatorname{diam} g(E) \ge \min_{\ell\ne k} d(f_\ell(E), f_k(E)) > \max_i\{\rho_i\} \cdot \operatorname{diam} E,$$

so

$$\rho_g > \max_i\{\rho_i\}. \tag{2.1}$$

We claim that for all $i \in \Omega$, there exists $j \in \Omega$ such that $gf_i(E) \subseteq f_j(E)$. Indeed, suppose not; then

$$\rho_g \cdot \operatorname{diam} f_i(E) = \operatorname{diam} gf_i(E) \ge \min_{\ell\ne k} d(f_\ell(E), f_k(E)) > \operatorname{diam} f_i(E),$$

which contradicts $\rho_g < 1$.

Next we will prove that there exists a unique $\mathbf{i} = i_1 \ldots i_n \ldots \in \Omega^w$ such that

$$gf_{i_1\ldots i_n}(E) \subseteq f_{i_1\ldots i_n}(E) \quad \text{for all positive integers } n. \tag{2.2}$$

We consider the following three cases.

Case 1: $\#\{i \mid gf_i(E) \subseteq f_i(E)\} \ge 2$, where $\#A$ is the number of elements in A. Without loss of generality, we can assume that $gf_i(E) \subseteq f_i(E)$ and $gf_j(E) \subseteq f_j(E)$ with $i \ne j$; then $\rho_g \cdot d(f_i(E), f_j(E)) = d(gf_i(E), gf_j(E)) \ge d(f_i(E), f_j(E))$, a contradiction.

Case 2: $\#\{i \mid gf_i(E) \subseteq f_i(E)\} = 0$. By the claim following (2.1), there exist different $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $gf_{i_\ell}(E) \subseteq f_{i_{\ell+1}}(E)$ for each

$\ell = 1, \dots, n$, where $i_{n+1} = i_1$ and $n \geq 2$ is a positive integer. Consequently,

$$\rho_g \cdot d(f_{i_\ell}(E), f_{i_{\ell+1}}(E)) = d(gf_{i_\ell}(E), gf_{i_{\ell+1}}(E)) \geq d(f_{i_{\ell+1}}(E), f_{i_{\ell+2}}(E))$$

for all $\ell = 1, \dots, n$ (with the convention $i_{n+2} = i_2$). Multiplying these inequalities yields $\rho_g^n \geq 1$, which is impossible.

Case 3: $\#\{i \mid gf_i(E) \subseteq f_i(E)\} = 1$. We can assume that $gf_{i_1}(E) \subseteq f_{i_1}(E)$ for some $i_1 \in \Omega$. The same argument as above gives $\#\{i \mid gf_{i_1 i}(E) \subseteq f_{i_1 i}(E)\} = 1$.

By induction, the existence of $\mathbf{i}$ is thus proved.

Next we prove that there exist $\mathbf{j} \in \Omega^*$ and a contractive similitude h_1 with $h_1(E) \subseteq E$ and $f_{\mathbf{j}} h_1 = h_1 f_{\mathbf{j}}$.

According to (2.1) and (2.2), for all positive integers k, $f_{\mathbf{i}|_k}^{-1} g f_{\mathbf{i}|_k}(E)$ is a subset of E and intersects at least two different sets in $\{f_1(E), \dots, f_m(E)\}$. By Lemma 2.3 and the pigeonhole principle, we can find positive integers $p < q$ such that $f_{\mathbf{i}|_p}^{-1} g f_{\mathbf{i}|_p} = f_{\mathbf{i}|_q}^{-1} g f_{\mathbf{i}|_q}$.

Suppose $\mathbf{i}|_q = \mathbf{i}|_p \mathbf{j}$ for some $\mathbf{j} \in \Omega^*$. Write $f_{\mathbf{i}|_p}^{-1} g f_{\mathbf{i}|_p} = h_1$; then $h_1(E) \subseteq E$ and $f_{\mathbf{j}} h_1 = h_1 f_{\mathbf{j}}$.

We will need the following claim:

($*$) There exist a positive integer $\ell \geq 2$ and a contractive similitude h with $h(E) \subseteq E$ such that $h^\ell(E) = f_{\mathbf{j}}(E)$.

As $\rho_{h_1} = \rho_g > \rho_{\mathbf{j}}$, there exists a positive integer $\ell_1 \geq 2$ such that $\rho_{h_1}^{\ell_1} \leq \rho_{\mathbf{j}} < \rho_{h_1}^{\ell_1 - 1}$. Moreover, from $f_{\mathbf{j}} h_1 = h_1 f_{\mathbf{j}}$ we conclude that $f_{\mathbf{j}}$ and h_1, hence $f_{\mathbf{j}}$ and $h_1^{\ell_1}$, share the same fixed point. Thus, $h_1^{\ell_1}(E)$ intersects $f_{\mathbf{j}}(E)$. These facts, together with Corollary 2.5, yield $h_1^{\ell_1}(E) \subseteq f_{\mathbf{j}}(E)$, or equivalently $f_{\mathbf{j}}^{-1} h_1^{\ell_1}(E) \subseteq E$.

Define $h_2 = f_{\mathbf{j}}^{-1} h_1^{\ell_1}$. Then $\rho_{h_1} < \rho_{h_2} \leq 1$ and $h_2(E) \subseteq E$. Concerning the value of ρ_{h_2}, there are two cases to consider:

(i) If $\rho_{h_2} = 1$, then $h_2(E) = E$.

(ii) If $\rho_{h_1} < \rho_{h_2} < 1$, then h_2, as a contractive similitude, has the same fixed point as $f_{\mathbf{j}}$.

Replacing h_1 with h_2 in the above procedure yields a positive integer $\ell_2 \geq 2$ with $\rho_{h_2}^{\ell_2} \leq \rho_{\mathbf{j}} < \rho_{h_2}^{\ell_2 - 1}$ and $h_2^{\ell_2}(E) \subseteq f_{\mathbf{j}}(E)$. Similarly, we can define $h_3 = f_{\mathbf{j}}^{-1} h_2^{\ell_2}$. Then $h_3(E) \subseteq E$.

If $\rho_{h_3} = 1$, then $h_3(E) = E$. If $\rho_{h_2} < \rho_{h_3} < 1$, we can repeat the argument of (ii) again.

We claim that repeated application of the above process gives $h_{N+1}(E) = E$ for some positive integer N. For otherwise, we can define contractive similitudes h_n for all positive integers n. However, since the contractive ratio of h_n is strictly both increasing and larger than $\max_i\{\rho_i\}$, each h_n intersects at least two different sets in $\{f_1(E),\ldots,f_m(E)\}$, contrary to Lemma 2.3.

We assume henceforth that $h_{N+1}(E)=E$, or equivalently $h_N^{\ell_N}(E)=f_{\mathbf{j}}(E)$. For simplicity, we write h and ℓ instead of h_N and ℓ_N, respectively. Then $h(E)\subseteq E$ and $h^{\ell}(E)=f_{\mathbf{j}}(E)$, proving $(*)$.

Finally, we will show that $\{f_i\}_{i=1}^m$ is not an IFS for E with the smallest integer m.

Let $\mathbf{j}=j_1\ldots j_n$. We distinguish two cases:

(I) If $n=1$, then $h^{\ell}(E)=f_{j_1}(E)$, and we are done by Lemma 2.2.

(II) If $n\ge 2$, then we claim that

$$h^{-1}f_{\mathbf{j}}(E)\subseteq f_{j_1\ldots j_{n-1}}(E). \tag{2.3}$$

First, it follows from

$$\begin{aligned} h^{-1}f_{\mathbf{j}}(E)\cap f_{j_1\ldots j_{n-1}}(E) &= h^{\ell-1}(E)\cap f_{j_1\ldots j_{n-1}}(E)\\ &\supseteq h^{\ell}(E)\cap f_{j_1\ldots j_{n-1}}(E)=f_{\mathbf{j}}(E)\end{aligned}$$

that $h^{-1}f_{\mathbf{j}}(E)$ intersects $f_{j_1\ldots j_{n-1}}(E)$. Secondly, from $\rho_h\ge\rho_g$ and (2.1) we have

$$\rho_{h^{-1}f_{\mathbf{j}}}=\frac{\rho_{j_1\ldots j_n}}{\rho_h}\le\frac{\rho_{j_1\ldots j_n}}{\rho_g}<\rho_{j_1\ldots j_{n-1}}.$$

Then (2.3) follows instantly from Corollary 2.5.

As $h^{-1}f_{\mathbf{j}}(E)\subseteq E$ and $\rho_{h^{-1}f_{\mathbf{j}}}>\rho_{\mathbf{j}}$, there exists $\mathbf{k}\in\Omega^n$ with $\mathbf{k}\ne\mathbf{j}$ such that $h^{-1}f_{\mathbf{j}}(E)$ intersects $f_{\mathbf{k}}(E)$. Due to (2.3), we have $\mathbf{k}=j_1\ldots j_{n-1}k_n$ with $k_n\ne j_n$. Furthermore,

$$hf_{\mathbf{k}}(E)\subseteq f_{\mathbf{j}}(E), \tag{2.4}$$

for otherwise, by Lemma 2.4,

$$\begin{aligned}\rho_h\cdot\operatorname{diam} f_{\mathbf{k}}(E) &\ge \min\{d(f_{\mathbf{j}}(E),f_{\mathbf{j}'}(E))\mid \mathbf{j}'\in\Omega^n,\ \mathbf{j}'\ne\mathbf{j},\ \mathbf{j}'|_{n-1}=j_1\ldots j_{n-1}\}\\ &> \rho_{j_1\ldots j_{n-1}}\cdot\max_i\{\rho_i\}\cdot\operatorname{diam} E\ge\operatorname{diam} f_{\mathbf{k}}(E),\end{aligned}$$

a contradiction.

It follows from (2.3), (2.4) and $f_{\mathbf{j}}h^{-1}=h^{-1}f_{\mathbf{j}}$ that

$$f_{j_n}h^{-1}(E)=(f_{j_1\ldots j_{n-1}})^{-1}f_{\mathbf{j}}h^{-1}(E)\subseteq E$$

and

$$f_{k_n}(E)=f_{j_n}(f_{\mathbf{j}}^{-1}f_{\mathbf{k}})(E)\subseteq f_{j_n}(f_{\mathbf{j}}^{-1}h^{-1}f_{\mathbf{j}})(E)=f_{j_n}h^{-1}(E),$$

which together with $f_{j_n}(E)\subseteq f_{j_n}h^{-1}(E)$ implies that $\{f_i\}_{i=1}^m$ is not an IFS for E with the smallest integer m. ■

Combined with a result of Morán, Theorem 1.1 has the following consequence.

Corollary 2.6. *Let E be the self-similar set like in Theorem* 1.1. *Then the semigroup of all generating IFSs for E* (*equipped with the composition as product*) *is finitely generated.*

Proof. Suppose that g is a contractive similitude with $g(E) \subseteq E$. Then by Theorem 1.1, $f_{\mathbf{i}}^{-1} \circ g$ is an isometry of E for some $\mathbf{i} \in \Omega^*$. This, together with the result of Morán [6] that the group of isometries of any self-similar set generated by an IFS with the SSC is finite, yields Corollary 2.6. ■

Remark 2.7. Theorem 1.1 fails when condition (1.1) is replaced by the SSC, since either [3, Theorem 6.2] or [4, Example 6.2] can serve as a counterexample. However, whether condition (1.1) can be relaxed needs further research.

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