FUNDAMENTA
MATHEMATICAE
237 (2017)

Large regular Lindelöf spaces with points G_δ

by

Toshimichi Usuba (Tokyo)

Abstract. By analyzing Dow's construction, we introduce a general construction of regular Lindelöf spaces with points G_δ. Using this construction, we prove the following: Suppose that either (1) there exists a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$, (2) CH and $\square(\omega_2)$ hold, or (3) CH holds and there exists a Kurepa tree. Then there exists a regular Lindelöf space with points G_δ and of size $> 2^\omega$. This shows that, under CH, the non-existence of such a Lindelöf space has a large cardinal strength. We also prove that every c.c.c. forcing adding a new real creates a regular Lindelöf space with points G_δ and of size at least $(2^{\omega_1})^V$.

1. Introduction. Arhangel'skii [1] proved that every first countable Hausdorff Lindelöf space has cardinality at most 2^ω, and then he asked whether "first countable" can be replaced by "with points G_δ", where a topological space X is said to be *with points* G_δ if for every $x \in X$, the set $\{x\}$ is a G_δ-set in X.

The first answer to this question was Shelah's consistency result [9]. Using forcing methods, Shelah proved the consistency of the existence of a regular Lindelöf space with points G_δ and of size $\omega_2 = (2^\omega)^+$. Afterwards, Gorelic [3] refined Shelah's result. These results show that the existence of a large regular Lindelöf space with points G_δ is consistent with ZFC, but it is unknown whether the non-existence of such a Lindelöf space is consistent or not.

While Shelah's and Gorelic's spaces are constructed by forcing methods (1), recently Dow [2] has found a relatively simple construction using the combinatorial principle $\diamondsuit^*$:

2010 *Mathematics Subject Classification*: Primary 03E35, 54A35.
Key words and phrases: Kurepa tree, Lindelöf space, points G_δ, P-space, square principle.
Received 14 November 2014; revised 4 August 2016.
Published online 27 December 2016.

(1) Under $V = L$, Knight [7] constructed a regular Lindelöf space with point G_δ and of size ω_ω without forcing methods, but his proof contains some errors, and the author does not know whether it can be fixed or not.

DOI: 10.4064/fm296-8-2016

Theorem 1.1 (Dow [2]). *Suppose $\diamondsuit^*$ holds, that is, there is a sequence $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ such that:*

(1) $\mathcal{A}_\alpha \in [\mathcal{P}(\alpha)]^\omega$.
(2) *For every $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club in ω_1.*

Then there exists a zero-dimensional Hausdorff Lindelöf space of size 2^{ω_1} and with points G_δ.

In this paper, by analyzing Dow's construction, we introduce another simple and general construction of a regular Lindelöf space with points G_δ. In particular, we show that, under CH, the non-existence of a large regular Lindelöf space with points G_δ has a large cardinal strength.

Let X be a topological space. For $x \in X$, let $\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open sets with $\{x\} = \bigcap \mathcal{U}\}$. The *pseudocharacter* of X, $\psi(X)$, is $\sup\{\psi(x, X) : x \in X\}$. So X is with points G_δ if and only if $\psi(X) \leq \omega$.

The following proposition is a crux of this paper:

Proposition 1.2. *Let Y be a regular Lindelöf space of pseudocharacter $\leq \omega_1$ such that for each $y \in Y$, if $\psi(y, Y) = \omega_1$ then there exists a sequence $\langle O^y_\alpha : \alpha < \omega_1 \rangle$ of clopen sets with the following properties:*

(1) $O^y_\alpha \supseteq O^y_{\alpha+1}$.
(2) $O^y_\alpha = \bigcap_{\beta<\alpha} O^y_\beta$ *if α is limit.*
(3) $\bigcap_{\alpha<\omega_1} O^y_\alpha = \{y\}$.

Let Z be any uncountable regular Lindelöf space with points G_δ (e.g., the Cantor space ${}^\omega 2$). Then there exists a regular Lindelöf space X with points G_δ and of size $\max\{|Y|, |Z|\}$.

A topological space is said to be a *P-space* if every G_δ-set is open. If Y is a regular P-space of pseudocharacter $\leq \omega_1$, then it is easy to see that Y satisfies the assumptions of Proposition 1.2. Now the following theorem is immediate from the proposition.

Theorem 1.3. *Suppose that there exists a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$. Then there exists a regular Lindelöf space with points G_δ and of size $> 2^\omega$.*

For a regular uncountable cardinal κ, the principle $\square(\kappa)$ is the assertion that there exists a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ such that:

- $C_\alpha \subseteq \alpha$ is a club in α.
- For every $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.
- There is no club C in κ with $C \cap \alpha = C_\alpha$ for every $\alpha \in \lim(C)$.

The principle $\square(\kappa)$ was introduced in Todorčević [12] where it is shown using a construction from Jensen [5] that $\square(\kappa)$ holds for every regular uncountable cardinal κ which is not weakly compact in L. We prove that if $\square(\omega_2)$ holds or there exists a Kurepa tree, then there exists a large regular Lindelöf space of pseudocharacter $\leq \omega_1$ which satisfies the assumptions of Proposition 1.2. Hence we have:

THEOREM 1.4. *Suppose CH. If $\square(\omega_2)$ holds, or there exists an ω_1-Kurepa tree, then there exists a regular Lindelöf space with points G_δ and of size $> 2^\omega$.*

This theorem tells us that the consistency of CH with non-existence of a large regular Lindelöf space with points G_δ implies the consistency of the existence of a weakly compact cardinal.

We also prove the following theorem, which shows that simple forcings, such as Cohen or Random forcing, create a large regular Lindelöf space with points G_δ.

THEOREM 1.5. *Let $\mathbb{P}$ be a c.c.c. forcing notion which adds a new real. Then $\mathbb{P}$ forces that "there exists a regular Lindelöf space of size at least $(2^{\omega_1})^V$ and with points G_δ".*

2. A construction of large Lindelöf spaces with points G_δ. In this section, we prove Proposition 1.2 using a construction based on ideas from Dow's paper [2].

Fix spaces Y and Z satisfying the assumptions of Proposition 1.2. Let $Y_0 = \{y \in Y : \psi(y, Y) \leq \omega\}$ and $Y_1 = \{y \in Y : \psi(y, Y) = \omega_1\}$. For each $y \in Y_1$, fix a sequence $\langle O^y_\alpha : \alpha < \omega_1\rangle$ of clopen sets which satisfies assumptions (1)–(3) of Proposition 1.2.

LEMMA 2.1. *For $y \in Y_1$, the family $\{O^y_\alpha : \alpha < \omega_1\}$ is a local base at y in Y. In particular, for every countable family $\mathcal{U}$ of open neighborhoods of y, there is $\alpha < \omega_1$ with $O^y_\alpha \subseteq \bigcap \mathcal{U}$.*

Proof. If not, then there exists an open neighborhood V of y such that $O_y^\alpha \not\subseteq V$ for every $\alpha < \omega_1$. Then $\{O^y_\alpha \setminus V : \alpha < \omega_1\}$ is a $\subseteq$-decreasing sequence of non-empty closed sets with $\bigcap_{\alpha<\omega_1}(O^y_\alpha \setminus V) = \emptyset$. This is impossible since Y is Lindelöf. ∎

Let $X = Y_0 \cup (Y_1 \times Z)$. Clearly $|X| = \max\{|Y|, |Z|\}$. We will define a required topology on X.

For a subset $A \subseteq Y$, let $[\![A]\!] = (A \cap Y_0) \cup ((A \cap Y_1) \times Z)$.

Note that for $y \in Y_1$ and $\gamma < \omega_1$, the family $\{[\![O^y_\alpha]\!] \setminus [\![O^y_{\alpha+1}]\!] : \gamma \leq \alpha < \omega_1\}$ is a partition of $[\![O^y_\gamma]\!] \setminus (\{y\} \times Z)$.

Fix an injection $\pi : \omega_1 \to Z$. Let $R = \mathrm{Range}(\pi)$. For $z \in R$, let $\delta_z = \pi^{-1}(z)$.

Now, for $y \in Y_1$, $\gamma < \omega_1$, and an open $W \subseteq Z$, let $O(y,\gamma,W)$ be the set

$$(\{y\} \times W) \cup \bigcup \{[\![O^y_{\delta_z}]\!] \setminus [\![O^y_{\delta_z+1}]\!] : z \in R \cap W,\ \delta_z \geq \gamma\}.$$

Note that $O(y,\gamma,W) \subseteq [\![O^y_\gamma]\!]$ and $(\{y\} \times Z) \cap O(y,\gamma,W) = \{y\} \times W$.

The topology on X will be generated by the family

$$\{[\![V]\!] : V \subseteq Y \text{ is open}\} \cup \{O(y,\gamma,W) : y \in Y_1, \gamma < \omega_1, W \subseteq Z \text{ is open}\}.$$

We will prove that the above family actually generates a topology on X. The following lemmas are easy to check:

Lemma 2.2. *For open sets $V_0, V_1 \subseteq Y$, we have $[\![V_0]\!] \cap [\![V_1]\!] = [\![V_0 \cap V_1]\!]$.*

Lemma 2.3. *Let $y \in Y_1$, $\gamma_0, \gamma_1 < \omega_1$, and $W_0, W_1 \subseteq Z$ be open. Then $O(y, \max\{\gamma_0,\gamma_1\}, W_0 \cap W_1) \subseteq O(y,\gamma_0,W_0) \cap O(y,\gamma_1,W_1)$.*

Lemma 2.4. *Let $y \in Y_1$, $V \subseteq Y$ be open, and $z \in Z$. If $\langle y,z\rangle \in [\![V]\!]$, then $y \in V$ and $\langle y,z\rangle \in O(y,\gamma,Z) \subseteq [\![V]\!]$ for some $\gamma < \omega_1$.*

Proof. We have $y \in V$. By Lemma 2.1, there is $\gamma < \omega_1$ such that $O^y_\gamma \subseteq V$. Then it is clear that $\langle y,z\rangle \in O(y,\gamma,Z) \subseteq [\![O^y_\gamma]\!] \subseteq [\![V]\!]$. ∎

Lemma 2.5. *Let $y_0 \in Y_0$, $y_1 \in Y_1$, $\gamma < \omega_1$, and $W \subseteq Z$ be open. If $y_0 \in O(y_1,\gamma,W)$, then there is an open $V \subseteq Y$ with $y_0 \in [\![V]\!] \subseteq O(y_1,\gamma,W)$.*

Proof. By the definition of $O(y_1,\gamma,W)$, there is a unique $z \in R \cap W$ with $\delta_z \geq \gamma$ and $y_0 \in [\![O^y_{\delta_z}]\!] \setminus [\![O^y_{\delta_z+1}]\!]$. Let $V = O^y_{\delta_z} \setminus O^y_{\delta_z+1}$. Then V is open in Y, $y_0 \in [\![V]\!]$, and $[\![V]\!] \subseteq [\![O^y_{\delta_z}]\!] \setminus [\![O^y_{\delta_z+1}]\!] \subseteq O(y_1,\gamma,W)$. ∎

Lemma 2.6. *Let $y_1 \in Y_1$, $\gamma < \omega_1$, and $W \subseteq Z$ be open. Let $y \in Y_1$ and $z \in Z$. If $\langle y,z\rangle \in O(y_1,\gamma,W)$, then there are $\gamma' < \omega_1$ and open $W' \subseteq Z$ such that $\langle y,z\rangle \in O(y,\gamma',W') \subseteq O(y_1,\gamma,W)$.*

Proof. If $y = y_1$, then $z \in W$. Hence $\langle y,z\rangle \in O(y,\gamma,W) = O(y_1,\gamma,W)$.

Suppose $y \neq y_1$. Then there is a unique $z \in R \cap W$ with $\delta_z \geq \gamma$ and $\langle y,z\rangle \in [\![O^{y_1}_{\delta_z}]\!] \setminus [\![O^{y_1}_{\delta_z+1}]\!] = [\![O^{y_1}_{\delta_z} \setminus O^{y_1}_{\delta_z+1}]\!]$. By Lemma 2.4, there is $\gamma' < \omega_1$ with $\langle y,z\rangle \in O(y,\gamma',Z) \subseteq [\![O^{y_1}_{\delta_z} \setminus O^{y_1}_{\delta_z+1}]\!] \subseteq O(y_1,\gamma,W)$. ∎

These lemmas show that the above family in fact generates a topology on X. For $y \in Y_0$, the family $\{[\![V]\!] : V \subseteq Y$ is an open neighborhood of $y\}$ is a local base at y, and, for $y \in Y_1$ and $z \in Z$, the family $\{O(y,\gamma,W) : \gamma < \omega_1$, $W \subseteq Z$ is open with $z \in W\}$ is a local base at $\langle y,z\rangle$.

We now show that X is regular. First we check that X is Hausdorff:

- For $y \in Y_1$ and $z_0, z_1 \in Z$ with $z_0 \neq z_1$, take disjoint open sets $W_0, W_1 \subseteq Z$ with $z_i \in W_i$. Then $\langle y,z_i\rangle \in O(y,0,W_i)$ and $O(y,0,W_0) \cap O(y,0,W_1) = \emptyset$.
- For $y_0, y_1 \in Y_0$ with $y_0 \neq y_1$, take disjoint open sets $V_0, V_1 \subseteq Y$ with $y_i \in V_i$. We have $y_i \in [\![V_i]\!]$ and $[\![V_0]\!] \cap [\![V_1]\!] = \emptyset$.

- For $y_0 \in Y_0$ and $\langle y_1, z\rangle \in Y_1 \times Z$, take $\alpha < \omega_1$ with $y_0 \notin O^{y_1}_\alpha$. Then $y_0 \in [\![Y \setminus O^{y_1}_\alpha]\!]$, $\langle y_1, z\rangle \in O(y_1, \alpha, Z)$, and $[\![Y \setminus O^{y_1}_\alpha]\!] \cap O(y_1, \alpha, Z) = \emptyset$.
- For $\langle y_0, z_0\rangle, \langle y_1, z_1\rangle \in Y_1 \times Z$ with $y_0 \neq y_1$, pick $\alpha < \omega_1$ with $y_0 \notin O^{y_1}_\alpha$. We have $\langle y_0, z_0\rangle \in [\![Y \setminus O^{y_1}_\alpha]\!]$, $\langle y_1, z_1\rangle \in O(y_1, \alpha, Z)$, and $[\![Y \setminus O^{y_1}_\alpha]\!] \cap O(y_1, \alpha, Z) = \emptyset$.

To see that X is regular, take an open set $O \subseteq X$ and $x \in O$. We will find an open O' with $x \in O' \subseteq \overline{O'} \subseteq O$.

- If $x \in Y_0$, then there is an open neighborhood $V \subseteq Y$ of x with $[\![V]\!] \subseteq O$. Since Y is regular, we can find an open neighborhood V' of V with $x \in V' \subseteq \overline{V'} \subseteq V$. Now it is easy to see that the closure of $[\![V']\!]$ in X is contained in O.
- If $x = \langle y, z\rangle \in Y_1 \times Z$, there are $\gamma < \omega_1$ and an open neighborhood $W \subseteq Z$ of z with $\langle y, z\rangle \in O(y, \gamma, W) \subseteq O$. As Z is regular, we can find an open neighborhood $W' \subseteq W$ of z with $\overline{W'} \subseteq W$. We can check that the closure of $O(y, \gamma, W')$ is contained in $O(y, \gamma, W)$.

Next we prove that our space X is with points G_δ.

LEMMA 2.7. *For $y \in Y_0$, there are open sets $O_n \subseteq X$ $(n < \omega)$ such that $\{y\} = \bigcap_{n<\omega} O_n$.*

Proof. Since $y \in Y_0$, we can find open sets $V_n \subseteq Y$ $(n < \omega)$ such that $\{y\} = \bigcap_{n<\omega} V_n$. Then it is clear that $\{y\} = \bigcap_{n<\omega} [\![V_n]\!]$. ∎

LEMMA 2.8. *For $y \in Y_1$ and $z \in Z$, there are $\gamma < \omega_1$ and open sets $W_n \subseteq Z$ $(n < \omega)$ such that $\{\langle y, z\rangle\} = \bigcap_{n<\omega} O(y, \gamma, W_n)$.*

Proof. Take a large $\gamma < \omega_1$ such that if $z \in R$ then $\gamma > \delta_z$. Since Z is with points G_δ, there are open sets $W_n \subseteq Z$ $(n < \omega)$ with $\{z\} = \bigcap_{n<\omega} W_n$. We see that $\{\langle y, z\rangle\} = \bigcap_{n<\omega} O(y, \gamma, W_n)$.

First, it is clear that $(\{y\} \times Z) \cap \bigcap_{n<\omega} O(y, \gamma, W_n) = \{\langle y, z\rangle\}$.

Let $y_0 \in Y_0$ and $y_1 \in Y_1 \setminus \{y\}$. We will show that $(\{y_0\} \cup (\{y_1\} \times Z)) \cap O(y, \gamma, W_n) = \emptyset$ for some $n < \omega$.

For $y_1 \in Y_1$, if $y_1 \notin O^y_0$ then we are done. Suppose $y_1 \in O^y_0$. Then there is $z_1 \in R$ with $y_1 \in O^y_{\delta_{z_1}} \setminus O^y_{\delta_{z_1}+1}$. Pick $n < \omega$ with $z_1 \notin W_n \setminus \{z\}$. Then $(\{y_1\} \times Z) \cap O(y, \gamma, W_n) = \emptyset$. Indeed, if not then there is a unique $w \in R \cap W_n$ with $\delta_w \geq \gamma$ and $(\{y_1\} \times Z) \cap ([\![O^y_{\delta_w}]\!] \setminus [\![O^y_{\delta_w+1}]\!]) \neq \emptyset$. So w must be z_1. Since $w \in W_n$ but $z_1 \notin W_n \setminus \{z\}$, we see that z_1 must be z. Hence $\delta_z = \delta_w > \gamma$, contrary to the choice of γ.

The same argument shows that $y_0 \notin O(y, \gamma, W_n)$ for some $n < \omega$. ∎

Finally, we have to verify that X is Lindelöf.

LEMMA 2.9. *Let $y \in Y_1$. Le $\mathcal{U}$ be a family of open sets in X with $\{y\} \times Z \subseteq \bigcup \mathcal{U}$. Then there are an open neighborhood $V \subseteq Y$ of y and a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $[\![V]\!] \subseteq \bigcup \mathcal{U}'$.*

Proof. We may assume each element U of $\mathcal{U}$ has the form $O(y, \gamma_U, W_U)$.

By the definition of the topology of X, the subspace $\{y\} \times Z$ is homeomorphic to Z. Since Z is Lindelöf and $Z \subseteq \bigcup\{W_U : U \in \mathcal{U}\}$, there are countably many $U_n \in \mathcal{U}$ $(n < \omega)$ with $Z \subseteq \{W_{U_n} : n < \omega\}$. Let $\gamma = \sup\{\gamma_{U_n} : n < \omega_1\} < \omega_1$ and $V = O^y_\gamma$. Then V is an open neighborhood of y in Y. We will show that $[\![V]\!] \subseteq \bigcup_{n<\omega} U_n$.

First, clearly, $\{y\} \times Z \subseteq \bigcup_{n<\omega} U_n$. Take $y_0 \in V \cap Y_0$ and $y_1 \in V \cap Y_1$ with $y_1 \neq y$. For y_1, since $y_1 \in V = O^y_\gamma$, there is $z \in R$ with $\delta_z \geq \gamma$ and $y_1 \in O^y_{\delta_z} \setminus O^y_{\delta_z+1}$. Then there is $n < \omega$ with $z \in W_{U_n}$. Now we have $\{y_1\} \times Z \subseteq [\![O^y_{\delta_z}]\!] \setminus [\![O^y_{\delta_z+1}]\!] \subseteq O(y, \gamma, W_{U_n}) \subseteq O(y, \gamma_{U_n}, W_{U_n}) = U_n$. Similarly, for $y_0 \in Y_0$, there is $z \in R$ with $\delta_z \geq \gamma$ and $y_0 \in O^y_{\delta_z} \setminus O^y_{\delta_z+1}$. Then $y_0 \in O(y, \gamma, W_{U_n}) \subseteq O(y, \gamma_{U_n}, W_{U_n}) = U_n$. ∎

Lemma 2.10. *The space X is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. For $y \in Y_1$, by Lemma 2.9, there are a countable subfamily $\mathcal{U}_y \subseteq \mathcal{U}$ and an open neighborhood $V_y \subseteq Y$ of y with $[\![V_y]\!] \subseteq \bigcup \mathcal{U}_y$.

For $y \in Y_0$, pick $U_y \in \mathcal{U}_y$ with $y \in U_y$. Then we can find an open neighborhood $V_y \subseteq Y$ of y with $[\![V_y]\!] \subseteq U_y$.

Now $\{V_y : y \in Y\}$ is an open cover of Y. Since Y is Lindelöf, we can find countably many $y^0_n \in Y_0$ and $y^1_n \in Y_1$ $(n < \omega)$ with $Y \subseteq \bigcup_{n<\omega} V_{y^0_n} \cup \bigcup_{n<\omega} V_{y^1_n}$. Then it is clear that $\{U_{y^0_n} : n < \omega\} \cup \bigcup\{\mathcal{U}_{y^1_n} : n < \omega\}$ is a countable subfamily of $\mathcal{U}$ which covers X. ∎

We have completed the proof of Proposition 1.2. We now give some related remarks.

Remark 2.11. If Y and Z are Hausdorff and zero-dimensional, then our space X is also Hausdorff and zero-dimensional.

Remark 2.12. Lemma 2.1 tells us that if Y is a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$, then $\chi(Y) \leq \omega_1$, and we have $|Y| \leq 2^{\omega_1}$ by well-known Arhangel'skii's inequality.

Remark 2.13. The existence of a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$ is independent from ZFC.

Juhász and Weiss [6] showed that under $V = L$, there exists a regular Lindelöf P-LOTS of weight ω_1 and size $> 2^\omega$.

On the other hand, if κ is weakly compact, then there is no regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$ in $V^{\mathrm{Col}(\omega_1, <\kappa)}$, where $\mathrm{Col}(\omega_1, <\kappa)$ is a standard σ-closed poset collapsing κ to ω_2. A Lindelöf space is said to be *indestructible* if every σ-closed forcing preserves the Lindelöf property of the space. Scheepers and Tall [8] showed that every regular Lindelöf P-space is indestructible, and Tall and the author [10] proved that if κ is weakly compact, then, in $V^{\mathrm{Col}(\omega_1, <\kappa)}$, there is no indestructibly

Lindelöf space of size ω_2 and of pseudocharacter $\leq \omega_1$. In $V^{\mathrm{Col}(\omega_1,<\kappa)}$, we have $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Hence in this model, there is no regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$.

Remark 2.14. In contrast to the previous remark, the author does not know whether the existence of a large regular Lindelöf space Y satisfying the assumptions of Proposition 1.2 is independent from ZFC.

3. Tree spaces. In this section, we prove Theorem 1.4. We show that if there is some special kind of trees, then we can construct a regular Lindelöf space which satisfies the assumptions in Proposition 1.2.

First we introduce some basic definitions about trees. Let α be an ordinal. We identify a subset $T \subseteq {}^{\leq\alpha}2$ with the tree $\langle T, \subseteq\rangle$. We say that a tree $T \subseteq {}^{\leq\alpha}2$ is *normal* if T fulfills the following conditions:

- T is downward closed, that is, for every $s, t \in {}^{\leq\alpha}2$, if $s \subseteq t$ and $t \in T$ then $s \in T$.
- For every $t \in T$, if T has an immediate successor, then t has two immediate successors in T, that is, $t^\frown 0, t^\frown 1 \in T$.

For $\beta \leq \alpha$, let $T_\beta = T \cap {}^{\beta}2$. A *branch* of a tree $T \subseteq {}^{\leq\alpha}2$ is a maximal chain of T. Finally, the *Cantor tree* is the tree ${}^{\leq\omega}2$.

The following is the main result in this section.

Proposition 3.1. *Let $T \subseteq {}^{<\omega_2}2$ be a normal tree such that:*

(1) *T has no branch of size ω_2.*
(2) *T does not contain an isomorphic copy of the Cantor tree.*

Suppose T has λ many branches. Then there exists a zero-dimensional Hausdorff Lindelöf space of pseudocharacter $\leq \omega_1$ and of size $\max\{|T|, \lambda\}$ *such that for every $y \in Y$ with $\psi(y, Y) = \omega_1$, there exists a sequence of clopen sets $\langle O^y_\alpha : \alpha < \omega_1\rangle$ satisfying assumptions* (1)–(3) *of Proposition* 1.2.

Theorem 1.4 follows from this proposition. It is clear that an ω_1-Kurepa tree has no branch of size ω_2, and contains no isomorphic copy of the Cantor tree. For the $\Box(\omega_2)$ assumption, Todorčević [12] showed that $\Box(\omega_2)$ implies that there is an ω_2-Aronszajn tree which does not contain an isomorphic copy of the Cantor tree. If such an ω_2-Aronszajn tree exists, then it is easy to construct an ω_2-Aronszajn normal tree $T \subseteq {}^{<\omega_2}2$ which does not contain an isomorphic copy of the Cantor tree. An ω_2-Aronszajn tree cannot have a branch of size ω_2. Thus, applying Proposition 1.2, we obtain the required spaces.

Now we start the proof of Proposition 3.1. Arguments which will be used in this proof are essentially due to Dow [2], and the referee pointed out to us that this space can be seen as a variant of the space considered by

Todorčević [13, §6], who constructed a space using a special ω_1-Aronszajn tree, while our space will be obtained using a tree without branches of size ω_2 by a similar construction.

Fix a normal tree $T \subseteq {}^{<\omega_2}2$ which satisfies the assumptions in the proposition. For $i < 2$, let $E^2_i = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_i\}$.

Set

- $T^* = \bigcup_{\alpha<\omega_2} T_{\alpha+1}$,
- $S = \{t \in {}^{<\omega_2}2 : \mathrm{dom}(t) \in E^2_0,\ \forall\alpha < \mathrm{dom}(t)\,(t{\restriction}\alpha \in T) \text{ but } t \notin T\}$,
- $U = \{\rho \in {}^{<\omega_2}2 : \mathrm{dom}(\rho) \in E^2_1,\ \forall\alpha < \mathrm{dom}(\rho)\,(\rho{\restriction}\alpha \in T)\}$.

Let $Y = T^* \cup S \cup U$. We know that $|Y| = \max\{|T|, \lambda\}$. We will define the desired topology τ on Y.

For $t \in {}^{<\omega_2}2$, let $[t] = \{u \in T^* \cup S \cup U : t \subseteq u\}$. Note the following:

- For $t \in T$, if $\mathrm{dom}(t)$ is a limit ordinal with countable cofinality, then $t \notin Y$ and $[t] = [t^\frown 0] \cup [t^\frown 1]$.
- For $t \in S$, we have $t \notin T$ and $[t] = \{t\}$.

Now the topology τ of Y is generated by the family

$$\{[t{\restriction}(\xi+1)] \setminus ([t^\frown 0] \cup [t^\frown 1]) : t \in T^* \cup S \cup U,\ \xi \in \mathrm{dom}(t)\}.$$

It is routine to check that this family actually generates a topology on Y and the topology is Hausdorff. Moreover, every element of the above family is clopen. Hence Y is zero-dimensional. We also note that for $t \in Y$ and $\xi \in \mathrm{dom}(t)$, $[t{\restriction}(\xi+1)]$ is clopen.

We have to check that $\psi(Y) \le \omega_1$, and if $\psi(y, Y) = \omega_1$ for $y \in Y$ then there is a sequence $\langle O^y_\alpha : \alpha < \omega_1\rangle$ as required.

If $t \in T^*$, then $\{t\}$ is open in Y. If $t \in S$, then $[t^\frown 0] \cup [t^\frown 1] = \emptyset$ and the family $\{[t{\restriction}(\xi+1)] : \xi \in \mathrm{dom}(t)\}$ is a local base at t. Since $\mathrm{cf}(\mathrm{dom}(t)) = \omega$, we can find an increasing sequence $\langle \xi_n : n < \omega\rangle$ with limit $\mathrm{dom}(t)$. Then $\{t\} = \bigcap_{n<\omega}[t{\restriction}(\xi_n+1)]$. Hence $\psi(t, Y) = \omega$.

For $t \in U$, we have $\psi(t, Y) = \omega_1$. We will find a sequence $\langle O^t_\alpha : \alpha < \omega_1\rangle$ satisfying assumptions (1)–(3) of Proposition 1.2. Fix a club $C \subseteq \mathrm{dom}(t)$ in $\mathrm{dom}(t)$ with order type ω_1, and such that if $\xi \in C$ is not a limit point of C, then ξ is a successor ordinal. Let $\langle \xi_\alpha : \alpha < \omega_1\rangle$ be the increasing enumeration of all elements of C. For $\alpha < \omega_1$, let $O^t_\alpha = [t{\restriction}\xi_\alpha] \setminus ([t^\frown 0] \cup [t^\frown 1])$. It is enough to check that each O^t_α is clopen. This property is clear if α is successor. If α is limit, then $\mathrm{cf}(\alpha) = \omega$, and $t{\restriction}\alpha \notin T^* \cup S \cup U$. Hence $[t{\restriction}\xi_\alpha] = [(t{\restriction}\xi_\alpha)^\frown 0] \cup [(t{\restriction}\xi_\alpha)^\frown 1]$. Now $[(t{\restriction}\xi_\alpha)^\frown 0]$, $[(t{\restriction}\xi_\alpha)^\frown 1]$, and $([t^\frown 0] \cup [t^\frown 1])$ are clopen. So $[t{\restriction}\xi_\alpha] \setminus ([t^\frown 0] \cup [t^\frown 1])$ is clopen.

Lemma 3.2. *The space Y is Lindelöf.*

Proof. First note that $T \cup S \cup U$ is a normal tree. Let $\mathcal{U}$ be an open cover of Y. Let $T_{\mathcal{U}}$ be the set of all $t \in T \cup S \cup U$ such that there is no countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ with $[t] \subseteq \bigcup \mathcal{U}'$. If $T_{\mathcal{U}}$ is empty, then there is a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ with $[\emptyset] \subseteq \mathcal{U}'$, and it is clear that $\mathcal{U}'$ is a countable cover of Y. Thus it is enough to see that $T_{\mathcal{U}}$ is empty. Suppose to the contrary that $T_{\mathcal{U}}$ is non-empty. We will derive a contradiction in several steps.

1. It is clear that if $t \in T_{\mathcal{U}}$ and $\xi \in \mathrm{dom}(t)$, then $t{\restriction}\xi \in T_{\mathcal{U}}$. Hence $T_{\mathcal{U}}$ is downward closed.

2. We now prove that $T_{\mathcal{U}}$ does not have a maximal element. Suppose to the contrary that $t \in T_{\mathcal{U}}$ is maximal.

(i) If $t \in S$, then $[t] = \{t\}$, so there is $O \in \mathcal{U}$ with $[t] \subseteq O$. This contradicts $t \in T_{\mathcal{U}}$.

(ii) If $t \in T \cup U$ and t has no immediate successor in T, then $[t] \subseteq \{t\}$ and we can derive a contradiction as in case (i).

(iii) If $t \in T \cup U$ and t has an immediate successor in T, then $t^\frown 0$, $t^\frown 1 \in T$ since T is normal. We know $t^\frown 0, t^\frown 1 \notin T_{\mathcal{U}}$. But then $\mathcal{U}$ has a countable subfamily which covers $[t] \subseteq \{t\} \cup [t^\frown 0] \cup [t^\frown 1]$, which is a contradiction.

3. Every $t \in S \cap T_{\mathcal{U}}$ is a maximal element of $T_{\mathcal{U}}$, so that $T_{\mathcal{U}}$ is a subtree of $T \cup U$.

4. We prove that $T_{\mathcal{U}}$ is branching. Suppose to the contrary that $\{u \in T_{\mathcal{U}} : t \subseteq u\}$ is a chain for some $t \in T_{\mathcal{U}}$. Let $\langle t_i : i < \alpha \rangle$ be the increasing enumeration of $\{u \in T_{\mathcal{U}} : t \subseteq u\}$. Let $t_\alpha = \bigcup_{i<\alpha} t_i$. By step 2, α is a limit ordinal and $t_\alpha \notin T_{\mathcal{U}}$. Since T has no branch of size ω_2, we deduce that $\alpha < \omega_2$, hence $t_\alpha \in T \cup S \cup U$.

Claim 3.3. *For every $i \le \alpha$, there is a countable subfamily $\mathcal{U}_i \subseteq \mathcal{U}$ such that $[t_0] \setminus [t_i] \subseteq \bigcup \mathcal{U}_i$.*

Proof. We proceed by induction on $i \le \alpha$. The case $i = 0$ is trivial.

Case 1: i is successor, say $i = j + 1$. Then $\mathrm{dom}(t_i)$ is successor, so $t_i \in T^*$. By the induction hypothesis, there is a countable $\mathcal{U}_j \subseteq \mathcal{U}$ with $[t_0] \setminus [t_j] \subseteq \bigcup \mathcal{U}_j$. Let $t_i^\dagger = t_j^\frown(1 - t_i(\mathrm{dom}(t_j)))$. Then t_i and $t_i^\dagger$ are immediate successors of t_j in T. Since $t_i^\dagger \notin T_{\mathcal{U}}$, there is a countable $\mathcal{U}' \subseteq \mathcal{U}$ with $[t_i^\dagger] \subseteq \bigcup \mathcal{U}'$. Pick $O \in \mathcal{U}$ with $\{t_i\} \in O$. Since $[t_0] \setminus [t_i] = ([t_0] \setminus [t_j]) \cup [t_i^\dagger] \cup \{t_j\}$, we have $[t_0] \setminus [t_i] \subseteq \bigcup \mathcal{U}' \cup \bigcup \mathcal{U}_j \cup \{O\}$. Hence $\mathcal{U}_i = \mathcal{U}' \cup \mathcal{U}_j \cup \{O\}$ is as required.

Case 2: i is limit with countable cofinality. Take an increasing sequence $\langle i_n : n < \omega \rangle$ with limit i. For each $n < \omega$, take a countable $\mathcal{U}_{i_n} \subseteq \mathcal{U}$ which covers $[t_0] \setminus [t_{i_n}]$. Now $[t_0] \setminus [t_i] = \bigcup_{n<\omega}([t_0] \setminus [t_{i_n}])$. So $\mathcal{U}_i = \bigcup_{n<\omega} \mathcal{U}_{i_n}$ is as required.

Case 3: i is limit with uncountable cofinality. We notice that $t_i \in U \subseteq Y$. Take $O \in \mathcal{U}$ with $t_i \in O$. Then there is $\xi < \mathrm{dom}(t_i)$ with $[t_i{\restriction}(\xi + 1)] \setminus ([t_i^\frown 0] \cup [t_i^\frown 1]) \subseteq O$. Take $k < i$ with $\xi + 1 \le \mathrm{dom}(t_k)$, and take a countable

$\mathcal{U}_k \subseteq \mathcal{U}$ with $[t_0] \setminus [t_k] \subseteq \bigcup \mathcal{U}_k$. Then $[t_0] \setminus [t_i] \subseteq \bigcup \mathcal{U}_k \cup \{O\}$, so $\mathcal{U}_k \cup \{O\}$ is as required. $\blacksquare_{\text{Claim}}$

Now we can find a countable $\mathcal{U}_\alpha \subseteq \mathcal{U}$ with $[t_0] \setminus [t_\alpha] \subseteq \bigcup \mathcal{U}_\alpha$. Since $t_\alpha \in T \cup S \cup U$ but $t_\alpha \notin T_{\mathcal{U}}$, there is a countable $\mathcal{V} \subseteq \mathcal{U}$ with $[t_\alpha] \subseteq \bigcup \mathcal{V}$. Then $[t_0] \subseteq \bigcup \mathcal{U}_\alpha \cup \bigcup \mathcal{V}$, contrary to $t_0 \in T_{\mathcal{U}}$.

5. Now we observe that $T_{\mathcal{U}} \subseteq T \cup U$ is branching. To derive a contradiction, we take $\{t_s : s \in {}^{<\omega}2\} \subseteq T_{\mathcal{U}}$ such that for $s \in {}^{<\omega}2$, $t_s \subseteq t_{s^\frown 0} \cap t_{s^\frown 1}$ and $t_{s^\frown 0} \perp t_{s^\frown 1}$. For $x \in {}^\omega 2$, let $t_x = \bigcup_{n<\omega} t_{x \restriction n}$. Then the assignment from $2^{\leq \omega}$ to $2^{<\omega_2}$ defined by $x \mapsto t_x$ is an order preserving embedding, so $\{t_x : x \in {}^{\leq\omega}2\}$ is isomorphic to the Cantor tree. Note that $t_x \in (T \cup S) \setminus U$. By our assumption on T, there is $x \in {}^\omega 2$ with $t_x \notin T$, so $t_x \in S$. Then we can pick $\xi \in \mathrm{dom}(t_x)$ such that $[t_x \restriction (\xi+1)] \setminus ([t_x^\frown 0] \cup [t_x^\frown 1]) \subseteq U$ for some $U \in \mathcal{U}$. Take $n < \omega$ with $t \restriction (\xi+1) \subseteq t_{x \restriction n}$, and take $s \in {}^{<\omega}2$ with $x \restriction n \subseteq s$ and $s \nsubseteq x$. Then $[t_s] \subseteq U$, which is a contradiction. $\blacksquare$

REMARK 3.4. Suppose CH, and fix a cardinal $\lambda > \omega_1$. Then there is a forcing notion which is σ-closed, satisfies the ω_2-c.c., and forces that "there exists an ω_1-tree which has just λ many branches and $2^{\omega_1} \geq \lambda$" (see Todorčević [11, pp. 282–283]). Combining this fact with Propositions 1.2 and 3.1, we have the consistency of the statement CH + $2^{\omega_1} > \omega_\omega$ + "there exists a regular Lindelöf space with points G_δ and of size ω_ω".

4. Proof of Theorem 1.5.

In this section we identify ${}^\omega 2$ with the Cantor space. First we prove the following lemma which might be folklore:

LEMMA 4.1. *Let $\mathbb{P}$ be a forcing notion which preserves ω_1, and $T \subseteq {}^{<\omega_1}2$ be a normal tree. Then, in $V^{\mathbb{P}}$, if T contains an isomorphic copy of the Cantor tree, then $({}^\omega 2)^V$ contains a perfect subset of $({}^\omega 2)^{V^{\mathbb{P}}}$.*

Proof. Let G be a $(V, \mathbb{P})$-generic filter and work in $V[G]$. Suppose that T contains an isomorphic copy of Cantor tree in $V[G]$. Then there is an order preserving embedding $f : {}^{\leq\omega}2 \to T$. We may assume that $f(x) = \bigcup_{n<\omega} f(x \restriction n)$ for every $x \in {}^\omega 2$, and there is $\alpha < \omega_1$ with $f(x) \in T_\alpha$ for every $x \in {}^\omega 2$. Now we identify the set ${}^\alpha 2$ as the Tychonoff product of α many two-point discrete spaces $2 = \{0, 1\}$. Let $X = f``({}^\omega 2)$. Note that $X \subseteq T_\alpha \subseteq ({}^\alpha 2)^V$. Clearly X does not have an isolated point. Furthermore, f is a continuous map from ${}^\omega 2$ to ${}^\alpha 2$, and X is the image of ${}^\omega 2$ by f, thus X is closed in ${}^\alpha 2$. Hence X is a perfect subset of ${}^\alpha 2$.

Next fix a bijection $\pi : \alpha \to \omega$ with $\pi \in V$. Then π induces the natural homeomorphism $\pi : {}^\alpha 2 \to {}^\omega 2$. Hence $Y = \pi``X \subseteq {}^\omega 2$ is a perfect subset of ${}^\omega 2$. For each $x \in ({}^\alpha 2)^V$, we have $\pi(x) \in V$. Since $X \subseteq ({}^\alpha 2)^V$, it follows that $Y \subseteq ({}^\omega 2)^V$. Hence $({}^\omega 2)^V$ contains a perfect subset Y of ${}^\omega 2$. $\blacksquare$

Now we are ready to prove Theorem 1.5. We use the following theorem:

THEOREM 4.2 (Groszek–Slaman [4]). *Let $\mathbb{P}$ be a forcing notion such that:*

(1) *$\mathbb{P}$ preserves ω_1.*
(2) *$\mathbb{P}$ adds a new real.*
(3) *For every set $x \in V^{\mathbb{P}}$ of ordinals with $|x|^{V^{\mathbb{P}}} \le \omega$, there is $y \in V$ with $x \subseteq y$ and $|y|^V \le \omega$.*

Then $({}^{\omega}2)^V$ does not contain a perfect subset of $({}^{\omega}2)^{V^{\mathbb{P}}}$.

Proof of Theorem 1.5. Let $\mathbb{P}$ be as in the statement. Then $\mathbb{P}$ satisfies conditions (1)–(3) in Theorem 4.2. Take a $(V,\mathbb{P})$-generic G and work in $V[G]$. Let T be the tree $({}^{<\omega_1}2)^V$. Then, in $V[G]$, T is a normal tree with height ω_1 and T has at least $(2^{\omega_1})^V$ uncountable branches. By Lemma 4.1 and Theorem 4.2, T does not contain an isomorphic copy of the Cantor tree. Then we can construct a required space by Propositions 1.2 and 3.1. ∎

REMARK 4.3. It is obvious that c.c.c. in Theorem 1.5 can be replaced by being proper.

Acknowledgements. The author is grateful to Rodrigo R. Dias for supporting his visit to a conference within the First Brazilian Congress of Young Researchers in Pure and Applied Mathematics at University of São Paulo in December 2014. The author would also like to thank Alan Dow, Franklin D. Tall, Stevo Todorčević, and the referee for many valuable comments and suggestions.

This research was supported by JSPS KAKENHI, Nos. 15K17587 and 15K04984.

References

[1] A. V. Arhangel'skii, *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dokl. 10 (1969), 951–955.

[2] A. Dow, *A new Lindelöf space with points G_δ*, Comment. Math. Univ. Carolin. 56 (2015), 223–230.

[3] I. Gorelic, *The Baire category and forcing large Lindelöf spaces with points G_δ*, Proc. Amer. Math. Soc. 118 (1993), 603–607.

[4] M. J. Groszek and T. A. Slaman, *A basis theorem for perfect sets*, Bull. Symbolic Logic 4 (1998), 204–209.

[5] R. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), 229–308.

[6] I. Juhász and W. Weiss, *On a problem of Sikorski*, Fund. Math. 100 (1978), 223–227.

[7] R. W. Knight, *A topological application of flat morasses*, Fund. Math. 194 (2007), 45–66.

[8] M. Scheepers and F. D. Tall, *Lindelöf indestructibility, topological games and selection principles*, Fund. Math. 210 (2010), 1–46; Errata, ibid. 224 (2014), 203–204.

[9] S. Shelah, *On some problems in general topology*, in: Set Theory (Boise, ID, 1992–1994), Contemp. Math. 192, Amer. Math. Soc., Providence, RI, 1996, 91–101.

[10] F. D. Tall and T. Usuba, *Lindelöf spaces with small pseudocharacter and an analog of Borel's conjecture for subsets of* $[0,1]^{\aleph_1}$, Houston J. Math. 40 (2014), 1299–1309.

[11] S. Todorčević, *Trees and linearly ordered sets*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 235–293.

[12] S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. 159 (1987), 261–294.

[13] S. Todorčević, *Some applications of S and L combinatorics*, in: The Work of Mary Ellen Rudin (Madison, WI, 1991), Ann. New York Acad. Sci. 705, New York Acad. Sci., New York, 1993, 130–167.

Toshimichi Usuba
Faculty of Science and Engineering
Waseda University
3-4-1 Okubo, Shinjuku
Tokyo 169-8555, Japan
E-mail: usuba@waseda.jp