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IDEAL QUASI-NORMAL CONVERGENCE AND RELATED NOTIONS

BY

LEV BUKOVSKÝ (Košice), PRATULANANDA DAS (Kolkata) and
JAROSLAV ŠUPINA (Košice)

Abstract. Recently the second author and D. Chandra (2013) began to study the notion of ideal quasi-normal convergence and some topological notions defined by this convergence. We show how some properties of those notions depend on the ideal; sometimes, they are also equivalent to some property of the ideal. Moreover, we exhibit non-trivial cases when the new notion introduced by the ideal quasi-normal convergence is equivalent to the corresponding original notions. Some relations between the new notions for different ideals are investigated as well. Then we extend the characterization of some of the notions by convergence properties of the topological space $C_p(X)$. Finally, we study the relation of the new convergences to the covering properties of the underlying topological space.

1. Introduction. In 1937, Henri Cartan [C1], [C2] began to study the convergence of a filter in a topological space. Soon after, in 1940, N. Bourbaki published the monograph [Bou] where the presentation of general topology is based on convergence of filters, including the simple and beautiful proof of Tychonoff's Theorem about the product of compact spaces. Later, several authors independently generalized the convergence of a sequence of reals to a convergence based on an ideal $\mathcal{I}$ of subsets of ω. Some of them may not have been familiar with the work of H. Cartan and N. Bourbaki and they called the convergence they introduced $\mathcal{I}$-convergence. For further information see, e.g., [KSW]. Actually, $\mathcal{I}$-convergence is Cartan's convergence with respect to the dual filter of $\mathcal{I}$.

Many authors consider natural modifications of several notions defined by the convergence of a sequence of reals or, more often, by the convergence of a sequence of real functions. See e.g. [BDK], [DC1] or [FS], where the reader can find other references to papers devoted to ideal convergence.

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There are natural questions: Are the new notions defined via ideal convergence different from the classical ones? Are the newly introduced notions non-void at all? When do the modified notions keep the basic properties of the classical ones?

We shall use standard notation and terminology of set theory and set-theoretic topology. All topological spaces are assumed to be Hausdorff.

The set of all natural numbers is denoted by ω, and $\mathbb{R}$ is the set of all reals. In the whole paper by an *ideal* we understand a hereditary family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ ($B \in \mathcal{I}$ for any $B \subseteq A \in \mathcal{I}$) that is closed under unions ($A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$), contains all finite subsets of ω and $\omega \notin \mathcal{I}$ (1). The smallest ideal is the *Fréchet ideal* $\mathrm{Fin} = [\omega]^{<\omega}$, the set of all finite subsets of ω.

For a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote $\mathcal{A}^d = \{\omega \setminus A : A \in \mathcal{A}\}$. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a *filter* if $\mathcal{F}^d$ is an ideal.

If $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal then $\mathcal{B} \subseteq \mathcal{I}$ is a *base* of $\mathcal{I}$ if for any $A \in \mathcal{I}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$. If $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a filter then a family $\mathcal{B} \subseteq \mathcal{F}$ is a base of $\mathcal{F}$ if $\mathcal{B}^d$ is a base of the dual ideal $\mathcal{F}^d$. We recall a folklore fact: a family $\mathcal{A} \subseteq [\omega]^\omega$ is a base of some filter if and only if $\mathcal{A}$ has the *finite intersection property*, briefly *f.i.p.*, i.e., if for any finitely many sets $A_0, \dots, A_n \in \mathcal{A}$, the intersection $A_0 \cap \dots \cap A_n$ is infinite.

For an ideal $\mathcal{I}$ we denote

$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \mathcal{A} \text{ is a base of } \mathcal{I}\}.$$

A set A is *almost contained* in a set B, written $A \subseteq^* B$, if $A \setminus B$ is finite. Assume that $\mathcal{A} \subseteq \mathcal{I}$ is such that every $B \in \mathcal{I}$ is almost contained in some $A \in \mathcal{A}$. Then $\mathcal{B} = \{A \cup F : A \in \mathcal{A} \wedge F \in [\omega]^{<\omega}\}$ is a base of $\mathcal{I}$. Moreover, if $\mathcal{A}$ is infinite, then $|\mathcal{B}| = |\mathcal{A}|$.

An ideal $\mathcal{I}$ is said to be a P-*ideal* if for any countable $\mathcal{A} \subseteq \mathcal{I}$ there exists a set $B \in \mathcal{I}$ such that $A \subseteq^* B$ for each $A \in \mathcal{A}$. Some authors then say that $\mathcal{I}$ has *property* (AP). If $A \subseteq \omega$ is such that $\omega \setminus A$ is infinite, then

$$\langle A \rangle^* = \{B \subseteq \omega : B \subseteq^* A\}$$

is a P-ideal with a countable base. Note that the Fréchet ideal Fin equals $\langle \emptyset \rangle^*$.

An infinite set $B \subseteq \omega$ is said to be a *pseudointersection* of a family $\mathcal{A} \subseteq [\omega]^\omega$ if $B \subseteq^* A$ for any $A \in \mathcal{A}$. We can introduce the dual notion: a set B is a *pseudounion* of the family $\mathcal{A}$ if $\omega \setminus B$ is infinite and $A \subseteq^* B$ for each $A \in \mathcal{A}$. Thus an ideal $\mathcal{I}$ is a P-ideal if and only if every countable subfamily of $\mathcal{I}$ has a pseudounion belonging to $\mathcal{I}$. If a pseudounion A of $\mathcal{I}$ belongs to $\mathcal{I}$, then $\mathcal{I} = \langle A \rangle^*$.

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(1) Note that some authors call such ideals proper and admissible.

It is common in the literature to say that an ideal $\mathcal{I}$ is *tall* if for any $B \in [\omega]^\omega$, there exists an $A \in \mathcal{I}$ such that $A \cap B$ is infinite. Thus, an ideal $\mathcal{I}$ has a pseudounion if and only if $\mathcal{I}$ is not tall. In the following we prefer to say that "$\mathcal{I}$ has a pseudounion", since we shall often specify it.

The *pseudointersection number* is the cardinal

$$\mathfrak{p} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ has f.i.p. and has no pseudointersection}\}.$$

Thus, if $\mathcal{I}$ is an ideal with $\operatorname{cof}(\mathcal{I}) < \mathfrak{p}$, then $\mathcal{I}$ has a pseudounion. Since $\mathfrak{p} > \aleph_0$, any ideal with a countable base has a pseudounion.

The opposite implication is false. Indeed, if $\langle \mathcal{A}, \mathcal{B} \rangle$ is a Hausdorff gap (for definition see e.g. [Bu] or [Sch1]), then any set $B \in \mathcal{B}$ is a pseudounion of the ideal

$$\mathcal{I} = \{C \subseteq \omega : (\exists A \in \mathcal{A})(C \subseteq^* A)\},$$

and no pseudounion of $\mathcal{I}$ belongs to $\mathcal{I}$. F. Rothberger [Ro] proved that there exists a $(\mathfrak{b}, \omega)$-gap (^{2}). In this case $\mathcal{I}$ is a P-ideal and $\operatorname{cof}(\mathcal{I}) = \mathfrak{b} \geq \mathfrak{p}$. Note that even the inequality $\mathfrak{b} > \mathfrak{p}$ is consistent with **ZFC**.

Recall that a sequence $\langle x_n : n \in \omega \rangle$ of elements of a topological space X *$\mathcal{I}$-converges* to $x \in X$, written $x_n \xrightarrow{\mathcal{I}} x$, if for every neighborhood U of x, we have $\{n \in \omega : x_n \notin U\} \in \mathcal{I}$, i.e., the function $\langle x_n : n \in \omega \rangle$ from ω into X converges modulo the filter $\mathcal{I}^d$ to x in the sense of H. Cartan. Similarly, we say that a sequence $\langle x_n : n \in \omega \rangle$ of reals is *$\mathcal{I}$-divergent to* ∞, written $x_n \xrightarrow{\mathcal{I}} \infty$, if $\{n : x_n < a\} \in \mathcal{I}$ for any real $a > 0$.

If $f : X \to \mathbb{R}$, we say that f is a *real function on* X. When X is understood from the context, we say simply that f is a real function. A sequence $\langle f_n : n \in \omega \rangle$ of real functions *$\mathcal{I}$-converges to a real function* f on X, written $f_n \xrightarrow{\mathcal{I}} f$, if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for each $x \in X$.

P. Das and D. Chandra [DC1] generalized the notions of [Ba] and [CL] as follows. A sequence $\langle f_n : n \in \omega \rangle$ of real functions on X *$\mathcal{I}$-quasi-normally converges* to a real function f on X, written $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ on X, if there exists a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of reals that $\mathcal{I}$-converges to 0 (a *control sequence*) and is such that $\{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$. A sequence *strongly $\mathcal{I}$-quasi-normally converges* to f if $\langle 2^{-n} : n \in \omega \rangle$ is a control sequence. We then write $f_n \xrightarrow{\mathrm{s}\mathcal{I}\mathrm{QN}} f$. Actually, here $\langle 2^{-n} : n \in \omega \rangle$ may be replaced by any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$.

Let us make a rather trivial observation. Each subsequence of a quasi-normally convergent sequence is quasi-normally convergent. However, if $\mathcal{I} \neq \mathrm{Fin}$, then it is easy to construct an $\mathcal{I}$-quasi-normally convergent sequence which has a subsequence that does not $\mathcal{I}$-quasi-normally converge.

(^{2}) For the definition of $\mathfrak{b}$, see e.g. [Bu].

R. Filipów and M. Staniszewski [FS] have shown that for a P-ideal, in the definition of $\mathcal{I}$-quasi-normal convergence one can replace the $\mathcal{I}$-convergence of $\langle \varepsilon_n : n \in \omega \rangle$ to 0 by the stronger condition $\lim_{n\to\infty} \varepsilon_n = 0$.

Note that for any sequence $\langle f_n : n \in \omega \rangle$ of real functions we have

(1) If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $f_n \xrightarrow{\mathcal{I}_1\mathrm{QN}} f$ implies $f_n \xrightarrow{\mathcal{I}_2\mathrm{QN}} f$, and $f_n \xrightarrow{\mathrm{s}\mathcal{I}_1\mathrm{QN}} f$ implies $f_n \xrightarrow{\mathrm{s}\mathcal{I}_2\mathrm{QN}} f$.

A sequence $\langle f_n : n \in \omega \rangle$ of real functions on X $\mathcal{I}$-*uniformly converges* to a real function f, written $f_n \xrightarrow{\mathcal{I}\text{-u}} f$, if

$$\{n \in \omega : (\exists x \in X)(|f_n(x) - f(x)| \geq \varepsilon)\} \in \mathcal{I}$$

for each $\varepsilon > 0$. The properties of $\mathcal{I}$-uniform convergence were studied in [BDK].

It is not surprising that some basic properties of $\mathcal{I}$-quasi-normal convergence and those of the notions defined by it are closely related to the properties of the ideal $\mathcal{I}$. In this note, we shall present some results of this type. We show that in some cases the new notions coincide with the original ones. Moreover, we show that some classical results can be "translated" to their ideal modifications.

2. Properties of the ideal and decompositions. In this section we show some relations between properties of the ideal and properties of some convergences.

The implication (i)→(ii) of the next theorem was announced in [DC1, Remark 2.1]. The proof of (iii)→(i) follows an idea of J. Jasinski and I. Recław [JR]. For $\kappa = \aleph_0$, the theorem was proved by R. Filipów and M. Staniszewski [FS, Corollary 5.4].

THEOREM 2.1. *The following are equivalent:*

(i) $\mathrm{cof}(\mathcal{I}) \leq \kappa$.

(ii) *There exists $\lambda \leq \kappa$ such that for any non-empty set X and for any sequence $\langle f_n : n \in \omega \rangle$ of real functions such that $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ on X, there exist sets X_ξ, $\xi < \lambda$, such that $X = \bigcup_{\xi<\lambda} X_\xi$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ. Moreover, if X is a topological space and f_n, $n \in \omega$, are continuous, then the sets X_ξ, $\xi < \lambda$, can be chosen to be closed.*

(iii) *There exists $\lambda \leq \kappa$ such that for any sequence $\langle f_n : n \in \omega \rangle$ of real functions such that $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ on $\mathcal{I}$, there exist sets X_ξ, $\xi < \lambda$, such that $\mathcal{I} = \bigcup_{\xi<\lambda} X_\xi$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ. Moreover, if $\mathcal{I}$ is considered as a topological subspace of $\mathcal{P}(\omega)$ and f_n, $n \in \omega$, are continuous, then the sets X_ξ, $\xi < \lambda$, can be chosen to be closed.*

Proof. Throughout the proof, f and f_n, $n \in \omega$, denote real functions on X.

Assume that $\operatorname{cof}(\mathcal{I}) = \lambda \le \kappa$ and $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X. Then there exists a sequence $\{\varepsilon_n\}_{n\in\omega}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ such that for every $x \in X$ the set

$$A_x = \{n : |f_n(x) - f(x)| \ge \varepsilon_n\}$$

belongs to $\mathcal{I}$. Assume that $\{C_\xi : \xi < \lambda\}$ is a base of $\mathcal{I}$. Thus for every $A \in \mathcal{I}$ there exists some $\xi < \lambda$ with $A \subseteq C_\xi$. We set

$$X_\xi = \{x \in X : A_x \subseteq C_\xi\}.$$

Hence $X = \bigcup_{\xi<\lambda} X_\xi$.

It is now easy to observe that $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ. Indeed, take an $\varepsilon > 0$. Let $B = \{n \in \omega : \varepsilon_n \ge \varepsilon\}$. Then $B \in \mathcal{I}$, since $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$. If $x \in X_\xi$, then $A_x \subseteq C_\xi$. Therefore $|f_n(x) - f(x)| < \varepsilon$ for each $n \in \omega \setminus (C_\xi \cup B)$. Evidently $C_\xi \cup B \in \mathcal{I}$. This proves our assertion.

Now, let X be a topological space, and suppose f_n, $n \in \omega$, are continuous. Assume that $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X. Set

$$X_\xi = \{x \in X : (\forall n, m \in \omega \setminus C_\xi)(|f_n(x) - f_m(x)| \le \varepsilon_n + \varepsilon_m)\}.$$

Clearly X_ξ is closed for all $\xi < \lambda$. If $x \in X$ then it readily follows that $x \in X_\xi$ for some $\xi < \lambda$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ.

Assume (iii). The Cantor set ${}^\omega 2$ is endowed with the product topology and is homeomorphic to the Cantor middle-third subset of $\mathbb{R}$. We can identify $\mathcal{P}(\omega)$ with ${}^\omega 2$ equipped with the product topology. We shall consider $\mathcal{I}$ as a topological space with the subspace topology of ${}^\omega 2$.

Define $f_n : \mathcal{I} \to \mathbb{R}$ by

$$f_n(A) = \begin{cases} 1, & n \in A, \\ 0, & n \notin A, \end{cases} \tag{2}$$

for any $A \in \mathcal{I}$. One can easily see that each f_n is a continuous function. Moreover, for any $0 < \varepsilon < 1$ and $A \in \mathcal{I}$ we have

$$\{n \in \omega : f_n(A) \ge \varepsilon\} = \{n \in \omega : f_n(A) = 1\} = A \in \mathcal{I}.$$

Therefore $f_n \xrightarrow{\mathcal{I}\text{QN}} 0$ with control $\langle 2^{-n} : n \in \omega\rangle$.

By (iii) there exist sets $X_\xi \subseteq \mathcal{I}$, $\xi < \lambda$, such that $\mathcal{I} = \bigcup_{\xi<\lambda} X_\xi$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on X_ξ for each $\xi < \lambda$. Therefore for any $\xi < \lambda$ we have

$$A_\xi = \{n \in \omega : (\exists A \in X_\xi)(f_n(A) \ge 1)\} \in \mathcal{I}.$$

Since $A = \{n : f_n(A) \ge 1\}$, we obtain $A \subseteq A_\xi$ for any $A \in X_\xi$. Hence $\{A_\xi : \xi < \lambda\}$ is a base of the ideal $\mathcal{I}$. ■

We ask about the converse to the implications in assertion (ii) or (iii). In [DC1] a converse was proved for ideals with countable basis. Later, R. Filipów and M. Staniszewski [FS, Proposition 5.1] proved the ideal generalization of the basic result of [Ba] and [CL].

THEOREM 2.2. *Let f, f_n, $n \in \omega$, be real functions defined on a set X. If there are sets $X_k \subset X$, $k \in \omega$, such that $X = \bigcup_{k\in\omega} X_k$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on X_k for every $k \in \omega$, then $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X.*

We prove another result concerning decomposition of the set X.

THEOREM 2.3. *For a set $C \subset \omega$, the following are equivalent:*

(i) *C is a pseudounion of the ideal $\mathcal{I}$.*

(ii) *For every set X and every sequence $\langle f_n : n \in \omega\rangle$ of real functions with $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X, there exist sets X_k, $k \in \omega$, such that $X = \bigcup_k X_k$ and $f_n \xrightarrow{\langle C\rangle^*\text{-u}} f$ on each X_k.*

(iii) *For every set X and every sequence $\langle f_n : n \in \omega\rangle$ of real functions with $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X, there exists a cardinal (maybe finite) κ and sets X_ξ, $\xi < \kappa$, such that $X = \bigcup_{\xi<\kappa} X_\xi$ and $f_n \xrightarrow{\langle C\rangle^*\text{-u}} f$ on each X_ξ.*

Proof. Throughout the proof, f and f_n, $n \in \omega$, denote real functions on X.

(i)→(ii) Suppose C is a pseudounion of $\mathcal{I}$. Assume that $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X with a control $\langle \varepsilon_n : n \in \omega\rangle$. Then $\mathcal{I}\text{-}\lim_{n\to\infty} \varepsilon_n = 0$ and for every $x \in X$ there is a set $A_x \in \mathcal{I}$ such that $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in \omega \setminus A_x$. We set

$$C_k = C \cup \{0, 1, \ldots, k\}, \qquad X_k = \{x \in X : (\forall n \notin C_k)(|f_n(x) - f(x)| < \varepsilon_n)\}.$$

Observe that for any $x \in X$ we have $A_x \subseteq C_k$ for some $k \in \omega$, i.e., $x \in X_k$. Hence $X = \bigcup_k X_k$. One can easily show that $f_n \xrightarrow{\langle C\rangle^*\text{-u}} f$ on each X_k.

Evidently (ii)→(iii).

(iii)→(i) Assume that (iii) holds true. Define $f_n : \mathcal{I} \to \mathbb{R}$ as in (2). As above, $f_n \xrightarrow{\mathcal{I}\text{QN}} 0$. Thus there exists a cardinal κ and sets X_ξ, $\xi < \kappa$, such that $\mathcal{I} = \bigcup_{\xi<\kappa} X_\xi$ and $f_n \xrightarrow{\langle C\rangle^*\text{-u}} f$ on each X_ξ.

For any $A \in \mathcal{I}$, there exists a $\xi < \kappa$ such that $A \in X_\xi$. Since $f_n \xrightarrow{\langle C\rangle^*\text{-u}} f$ on X_ξ, we obtain $A = \{n \in \omega : f_n(A) \geq 1\} \in \langle C\rangle^*$, i.e., $A \subseteq^* C$. ■

3. Equivalences with classical notions. If $A \subseteq \omega$ is infinite, then $\langle e_A(n) : n \in \omega\rangle$ is the increasing sequence such that $A = \{e_A(n) : n \in \omega\}$. We begin with a simple observation.

LEMMA 3.1. *Assume that* $\mathcal{I} \subseteq \mathcal{P}(\omega)$ *is an ideal with a pseudounion* C. *Let* $A = \omega \setminus C$.

- (a) *For any sequence* $\langle x_n : n \in \omega \rangle$ *of reals, if* $x_n \xrightarrow{\mathcal{I}} 0$ *then* $x_{e_A(n)} \to 0$.
- (b) *For any sequence* $\langle f_n : n \in \omega \rangle$ *of real functions defined on* X, *if* $f_n \xrightarrow{\mathcal{I}} 0$ *on* X, *then* $f_{e_A(n)} \to 0$ *on* X.
- (c) *For any sequence* $\langle f_n : n \in \omega \rangle$ *of real functions defined on* X, *if* $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} 0$ *on* X, *then* $f_{e_A(n)} \xrightarrow{\mathrm{QN}} 0$ *on* X.

Proof. Note that by the definition of a pseudounion the set A is infinite.

(a) Let $\langle x_n : n \in \omega \rangle$ be a sequence of reals such that $x_n \xrightarrow{\mathcal{I}} 0$. If $\varepsilon > 0$ then $B = \{n \in \omega : |x_n| \geq \varepsilon\} \in \mathcal{I}$. Thus $B \subseteq^* C$. If $|x_{e_A(n)}| \geq \varepsilon$ then $e_A(n) \in B \setminus C$. Therefore $|x_{e_A(n)}| \geq \varepsilon$ just for finitely many n.

Assertion (b) follows immediately from (a).

(c) Let $\langle f_n : n \in \omega \rangle$ be a sequence of real functions defined on X such that $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} 0$ with a control $\langle \varepsilon_n : n \in \omega \rangle$. By (a) we have $\lim_{n\to\infty} \varepsilon_{e_A(n)} = 0$. As above, if $x \in X$ then $\{n \in A : |f_n(x)| \geq \varepsilon_n\}$ is finite. ∎

Related results were obtained by R. Filipów and M. Staniszewski [FS, Theorem 7.14 and Corollaries 7.15–7.16]. However, they deal with an ideal of the form $\langle A \rangle^*$. None of the ideals they consider is tall. Therefore

PROBLEM 3.2. *Is there a tall ideal* $\mathcal{I}$ *such that for any* $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} 0$ *there exists a subsequence converging quasi-normally to* 0?

In [JR] the authors say that a topological space X has the $\mathcal{I}$-*ideal convergence property* if for any sequence $\langle f_n : X \to \mathbb{R} : n \in \omega \rangle$ of continuous functions $\mathcal{I}$-converging to 0 there exists $C \in \mathcal{I}$ such that $f_{e_A(n)} \to 0$ on X, where $A = \omega \setminus C$. They remark that if $\mathcal{I}$ is of the form $\langle B \rangle^*$, then every topological space has the $\mathcal{I}$-ideal convergence property. Actually, that is the statement of Lemma 3.1(b) for the ideal $\langle B \rangle^*$.

In [DC2], the authors investigated the following modifications of the notions of [BRR]. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. A topological space X is an $(\mathcal{I},\mathcal{J})$QN-*space* if for any sequence $\langle f_n : n \in \omega \rangle$ of continuous real functions $\mathcal{I}$-converging to 0 on X, we have $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} 0$. A topological space X is an $(\mathcal{I},\mathcal{J})$wQN-*space* if for any sequence $\langle f_n : n \in \omega \rangle$ of continuous real functions $\mathcal{I}$-converging to 0 on X, there exists a sequence $\langle m_n : n \in \omega \rangle$ of natural numbers such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$. Note that we can require that $m_n \xrightarrow{\mathcal{J}} \infty$. Indeed, take instead of $\langle f_n : n \in \omega \rangle$ the sequence $\langle |f_n| + 2^{-n} : n \in \omega \rangle$. Then for any $a > 0$ and any $x \in X$ we have

$$\{n : m_n \leq a\} \subseteq \{n : |f_{m_n}(x)| + 2^{-m_n} \geq 2^{-a}\} \in \mathcal{J}.$$

If from every $\mathcal{I}$-converging sequence one can choose a strongly $\mathcal{J}$-quasi-normally converging subsequence, we say that X is an $(\mathcal{I}, \mathrm{s}\mathcal{J})$wQN-*space.*

It is easy to see that if $\mathcal{I} \nsubseteq \mathcal{J}$, then there exists no $(\mathcal{I}, \mathcal{J})$QN-space. Thus, speaking about an $(\mathcal{I}, \mathcal{J})$QN-space, we always assume that $\mathcal{I} \subseteq \mathcal{J}$.

A $(\mathrm{Fin}, \mathcal{I})$QN-space is the $\mathcal{I}$QN-space of [DC1], and a $(\mathrm{Fin}, \mathrm{Fin})$QN-space is the QN-space of [BRR]; similarly for the other notions.

Note that

(3) *If $\mathcal{I}_2 \subseteq \mathcal{I}_1$ and $\mathcal{J}_1 \subseteq \mathcal{J}_2$, then any $(\mathcal{I}_1, \mathcal{J}_1)$*QN-*space is an $(\mathcal{I}_2, \mathcal{J}_2)$*QN-*space.*

Similar relations hold true for the other notions.

We show that for an ideal with a pseudounion we do not in fact obtain any new notions:

THEOREM 3.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals.*

(a) *If $\mathcal{I}$ has a pseudounion, then every $\mathcal{J}$*wQN-*space is an $(\mathcal{I}, \mathcal{J})$*wQN-*space.*

(b) *If $\mathcal{J}$ has a pseudounion, then every $\mathcal{J}$*QN-*space is a* QN-*space and every $(\mathcal{I}, \mathcal{J})$*wQN-*space is an $(\mathcal{I}, \mathrm{Fin})$*wQN-*space.*

Proof. (a) Let C be a pseudounion of $\mathcal{I}$. We set $A = \omega \setminus C$. Assume that X is a $(\mathrm{Fin}, \mathcal{J})$wQN-space and $f_n \xrightarrow{\mathcal{I}} 0$. Then by Lemma 3.1(b), $f_{e_A(n)} \to 0$. Hence a subsequence of $\langle f_{e_A(n)} : n \in \omega \rangle$ $\mathcal{J}$QN-converges to 0.

(b) Now, let C be a pseudounion of $\mathcal{J}$, and $A = \omega \setminus C$. Assume that X is a $\mathcal{J}$QN-space and $f_n \to 0$ on X. We set $g_{e_A(n)} = f_n$ and $g_n = 0$ otherwise. Evidently $g_n \to 0$ on X. Therefore $g_n \xrightarrow{\mathcal{J}\mathrm{QN}} 0$, and by Lemma 3.1(c) we obtain $g_{e_A(n)} = f_n \xrightarrow{\mathrm{QN}} 0$.

Now assume that X is an $(\mathcal{I}, \mathcal{J})$wQN-space and $f_n \xrightarrow{\mathcal{I}} 0$ on X. Then a subsequence of $\langle f_n : n \in \omega \rangle$ $\mathcal{J}$QN-converges to 0 on X. By Lemma 3.1(c), a further subsequence quasi-normally converges to 0 on X. Thus X is an $(\mathcal{I}, \mathrm{Fin})$wQN-space. ∎

Taking into account (3) we obtain

COROLLARY 3.4. *Let X be a topological space, and $\mathcal{I}$ an ideal with a pseudounion.*

(a) *X is an $\mathcal{I}$*QN-*space if and only if X is a* QN-*space.*

(b) *X is an $\mathcal{I}$*wQN-*space if and only if X is a* wQN-*space.*

We can strengthen Corollary 3.4(a) in a rather trivial way.

COROLLARY 3.5. *If $\mathcal{I} \subseteq \mathcal{J}$ and the ideal $\mathcal{J}$ has a pseudounion, then every $(\mathcal{I}, \mathcal{J})$*QN-*space is a* QN-*space.*

Proof. Any $(\mathcal{I},\mathcal{J})$QN-space is a $(\mathrm{Fin},\mathcal{J})$QN-space $=$ $\mathcal{J}$QN-space. The assertion follows from Corollary 3.4(a). ∎

Corollary 3.6. *If $\mathcal{I}=\langle C\rangle^*$ for some $C\subseteq\omega$, then X is a* QN*-space if and only if X is an $(\mathcal{I},\mathcal{I})$*QN*-space.*

Proof. Let $A=\omega\setminus C$. Assume that $f_n\xrightarrow{\mathcal{I}}0$. Then by Lemma 3.1(b) we obtain $f_{e_A(n)}\to 0$, and therefore $f_{e_A(n)}\xrightarrow{\mathrm{QN}}0$. Since $C\in\mathcal{I}$, we immediately see that $f_n\xrightarrow{\mathcal{I}\mathrm{QN}}0$. ∎

By J. Šupina [Su] every topological space is an $\mathcal{I}$QN-space if and only if the ideal $\mathcal{I}$ contains an isomorphic copy of Fin×Fin. Hence, in this case QN $\neq$ $\mathcal{I}$QN.

Problem 3.7. *For which ideal $\mathcal{I}$ not containing an isomorphic copy of* Fin×Fin *do we have $\mathcal{I}$*QN $\neq$ QN*? Similarly for $\mathcal{I}$*QN*- and $\mathcal{I}$*wQN*-spaces.*

J. Šupina [Su] recently showed that assuming $\mathfrak{p}=\mathfrak{c}$, for a γ-space X which is not a QN-space (the existence of such spaces was proved in [BRR]) there exists a tall ideal $\mathcal{I}$, not containing an isomorphic copy of Fin×Fin, such that X is an $\mathcal{I}$QN-space. We can even assume that $\mathcal{I}$ is a maximal ideal. Anyway, that is only a very partial answer to our Problem 3.7.

4. $(\mathcal{I},\mathcal{J})$QN, $(\mathcal{I},\mathcal{J})$wQN and properties of $\mathrm{C}_p(X)$. We recall that a topological space Y has the *Arkhangel'skiĭ's property* (α_1) if

> *for any $y\in Y$ and any sequence $\langle\langle y_{n,m}:m\in\omega\rangle:n\in\omega\rangle$ of sequences such that* $\lim_{m\to\infty}y_{n,m}=y$ *for all n, there is a sequence $\langle z_m:m\in\omega\rangle$ such that* $\lim_{m\to\infty}z_m=y$ *and $\{y_{n,m}:m\in\omega\}\subseteq^*\{z_m:m\in\omega\}$ for each n,*

and Y has *Arkhangel'skiĭ's property* (α_4) if

> *for any $y\in Y$ and any sequence $\langle\langle y_{n,m}:m\in\omega\rangle:n\in\omega\rangle$ of sequences such that* $\lim_{m\to\infty}y_{n,m}=y$ *for all n, there is a sequence $\langle m_n:n\in\omega\rangle$ such that* $\lim_{n\to\infty}y_{n,m_n}=y$.

If Y is a topological group, we can suppose that y is the neutral element.

The main results concerning wQN-spaces and QN-spaces (see e.g. [Bu], [BH] or [Sa]) are the Fremlin–Scheepers Theorem saying that a topological space X is a wQN-space if and only if the space $\mathrm{C}_p(X)$ of continuous real functions on X with the pointwise convergence topology has property (α_4), and the Bukovský–Haleš–Sakai–Scheepers Theorem saying that X is a QN-space if and only if $\mathrm{C}_p(X)$ has property (α_1).

For ideals $\mathcal{I}$ and $\mathcal{J}$ we can modify properties (α_1) and (α_4) for $\mathrm{C}_p(X)$ (or any space of real functions) as follows. $\mathrm{C}_p(X)$ has *property* $(\mathcal{I},\mathcal{J}\text{-}\alpha_1)$ if

> *for any sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions such that* $f_{n,m} \xrightarrow{\mathcal{I}} 0$ *for each* n*, there exists a sequence* $\langle B_n : n \in \omega\rangle \subseteq \mathcal{J}$ *with* $\bigcup_{n\in\omega} B_n = \omega$ *such that*

$$(\forall\varepsilon > 0)(\forall x \in X)(\exists A \in \mathcal{J})(\forall n, m)(m \notin A \cup B_n \to |f_{n,m}(x)| < \varepsilon), \tag{4}$$

and $\mathrm{C}_p(X)$ has *property* $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ if

> *for any sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions such that* $f_{n,m} \xrightarrow{\mathcal{I}} 0$ *for each* n*, there exists a sequence* $\langle m_n : n \in \omega\rangle$ *such that* $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$*.*

As in the definition of an $\mathcal{I}$wQN-space, we can assume that $m_n \xrightarrow{\mathcal{J}} \infty$: as above, replace the sequence $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ by $\langle\langle |f_{n,m}| + 2^{-m} : m \in \omega\rangle : n \in \omega\rangle$.

One can easily show that the condition $(\mathrm{Fin}, \mathrm{Fin}\text{-}\alpha_1)$ is equivalent to (α_1), and $(\mathrm{Fin}, \mathrm{Fin}\text{-}\alpha_4)$ is equivalent to (α_4).

Note that in the definition of $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ we may assume that $f_{n,m_n} \xrightarrow{\mathrm{s}\mathcal{J}\mathrm{QN}} 0$. Indeed, let $g_{n,m} = 2^n f_{n,m}$. Then $g_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n and therefore there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $g_{n,m_n} \xrightarrow{\mathcal{J}} 0$. Since

$$\{n : |g_{n,m_n}(x)| \le 1\} = \{n : |f_{n,m_n}(x)| \le 2^{-n}\},$$

we see that $f_{n,m_n} \xrightarrow{\mathrm{s}\mathcal{J}\mathrm{QN}} 0$.

Now, we prove the ideal version of Fremlin–Scheepers' characterization of wQN-spaces.

THEOREM 4.1. *If* X *is a topological space then the following are equivalent:*

(i) X *is an* $(\mathcal{I}, \mathrm{s}\mathcal{J})$wQN*-space.*
(ii) $\mathrm{C}_p(X)$ *has property* $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$*.*

Proof. (ii)→(i) Suppose $\mathrm{C}_p(X)$ has property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$ and $f_n \xrightarrow{\mathcal{I}} 0$. We set $f_{n,m} = 2^n f_m$. Then $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for every n. Therefore there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$. One can easily check that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control $\langle 2^{-n} : n \in \omega\rangle$.

(i)→(ii) Assume that X is an $(\mathcal{I}, \mathrm{s}\mathcal{J})$wQN-space and $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n. Set

$$g_m(x) = \sum_{n=0}^{\infty} \min\{2^{-n}, |f_{n,m}(x)|\}. \tag{5}$$

Let $x \in X$ and $\varepsilon > 0$. Let n_0 be such that $2^{-n_0+2} < \varepsilon$. For $n < n_0$ we have

$$A_n = \{m : |f_{n,m}(x)| \geq \varepsilon/(2n_0)\} \in \mathcal{I}.$$

If $m \notin \bigcup_{n<n_0} A_n$, then

$$g_m(x) < \sum_{n<n_0} \frac{\varepsilon}{2n_0} + \sum_{n\geq n_0} 2^{-n} < \varepsilon.$$

Thus $g_m \xrightarrow{\mathcal{I}} 0$.

Hence there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $g_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control $\langle 2^{-n} : n \in \omega\rangle$. Note that for any $x \in X$, if $g_m(x) < 2^{-n}$ then $|f_{n,m}(x)| < 2^{-n}$. Thus

$$\{n : |f_{n,m_n}(x)| \geq 2^{-n}\} \subseteq \{n : g_{m_n}(x) \geq 2^{-n}\} \in \mathcal{J},$$

and therefore $f_{n,m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$. ■

PROBLEM 4.2. *Does* $(\mathcal{I},\mathcal{J})\mathrm{wQN} \to (\mathcal{I},\mathcal{J}\text{-}\alpha_4)$ *hold true?*

Similarly, we prove the ideal version of the Bukovský–Haleš–Sakai–Scheepers characterization of QN-spaces.

THEOREM 4.3. *For any topological space* X *and any ideal* $\mathcal{I}$*, the following are equivalent:*

(i) X *is an* $(\mathcal{I},\mathcal{J})\mathrm{QN}$*-space.*
(ii) $\mathrm{C}_p(X)$ *has property* $(\mathcal{I},\mathcal{J}\text{-}\alpha_1)$*.*

Proof. (i)→(ii) Assume that $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ is a sequence of sequences of continuous real functions such that $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each $n \in \omega$. As above, if we define g_m by (5), then each g_m is continuous and $g_m \xrightarrow{\mathcal{I}} 0$. By (i) we know that $g_m \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with a control $\langle \varepsilon_m : m \in \omega\rangle$. Therefore for any $x \in X$ there exists a set $A_x \in \mathcal{J}$ such that if $m \notin A_x$ then $g_m(x) < \varepsilon_m$. Since $\varepsilon_m \xrightarrow{\mathcal{J}} 0$, there are $B_n \in \mathcal{J}$ such that $B_n \subseteq B_{n+1}$, $\bigcup_{n=0}^{\infty} B_n = \omega$ and $\varepsilon_m < 2^{-n}$ whenever $m \notin B_n$. Note that for any $x \in X$, $n \in \omega$ and $m \notin A_x \cup B_n$ we have $g_m(x) < 2^{-n}$. Therefore $|f_{n,m}(x)| \leq g_m(x) < 2^{-n}$.

Let $x \in X$, $\varepsilon > 0$, and n_0 with $2^{-n_0} < \varepsilon$. Then there exists a set $C_x \in \mathcal{I}$ such that $|f_{n,m}(x)| < \varepsilon$ for any $n < n_0$ and $m \notin C_x$. Thus for any $n \in \omega$, $m \notin A_x \cup B_n$, and $m \notin C_x$, we have $|f_{n,m}(x)| < \varepsilon$. Setting $A = A_x \cup C_x$ we obtain (4).

(ii)→(i) Assume that $f_n(x) \xrightarrow{\mathcal{I}} 0$ for each $x \in X$. We set $f_{n,m} = 2^n|f_m|$. Then $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n. Therefore there exist $B_n \in \mathcal{J}$ with $B_n \subseteq B_{n+1}$ and $\bigcup_{n=0}^{\infty} B_n = \omega$ such that (4) holds. We set

$$\varepsilon_m = 2^{-n+1} \quad \text{for } m \in B_n \setminus B_{n-1},$$

where $B_{-1} = \emptyset$.

Since $\{m : \varepsilon_m \geq 2^{-n}\} = B_{n+1}$, we obtain $\varepsilon_m \xrightarrow{\mathcal{J}} 0$. Let $x \in X$. By (4) there exists $A \in \mathcal{J}$ such that $f_{n,m}(x) = 2^n|f_m(x)| < 1$ for any n, m satisfying $m \notin A \cup B_n$. Evidently for any $m \notin B_0$ there exists an n such that $m \in B_{n+1} \setminus B_n$. If $m \notin A \cup B_0$ and $m \in B_{n+1} \setminus B_n$, we have $f_{n,m}(x) = 2^n|f_m(x)| < 1$ and $\varepsilon_m = 2^{-n}$. Thus $|f_m(x)| < 2^{-n} = \varepsilon_m$. Since $A \cup B_0 \in \mathcal{I}$ we obtain $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} 0$ with control $\langle \varepsilon_m : m \in \omega\rangle$. ∎

We can also introduce the ideal convergence modifications of properties (α_0) and $(\alpha_0)^*$ for $\mathrm{C}_p(X)$, which were introduced in [BH], as follows. $\mathrm{C}_p(X)$ has *property* $(\mathcal{I}, \mathcal{J}\text{-}\alpha_0)$ if

> *for any sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions such that* $f_{n,m} \xrightarrow{\mathcal{I}} 0$ *for each* n, *there exists a sequence* $\langle n_m : m \in \omega\rangle$ $\mathcal{J}$*-diverging to* ∞ *such that* $f_{n_m,m} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$,

and $\mathrm{C}_p(X)$ has *property* $(\mathcal{I}, \mathcal{J}\text{-}\alpha_0^*)$ if

> *for any sequence* $\langle f_n : n \in \omega\rangle$ *pointwise converging to* 0 *and any sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions such that* $f_{n,m} \xrightarrow{\mathcal{I}} f_n$ *for each* n, *there exists a sequence* $\langle n_m : m \in \omega\rangle$ $\mathcal{J}$*-diverging to* ∞ *such that* $f_{n_m,m} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.

Evidently properties $(\mathrm{Fin}, \mathrm{Fin}\text{-}\alpha_0)$ and $(\mathrm{Fin}, \mathrm{Fin}\text{-}\alpha_0^*)$ coincide with properties (α_0) and (α_0^*) of [BH], respectively.

The proof of Theorem 2.2 of [BH] can be modified to give a proof of the following assertion.

THEOREM 4.4. *For a topological space X the following are equivalent:*

(i) *X is an $(\mathcal{I}, \mathcal{J})$QN-space.*
(ii) *$\mathrm{C}_p(X)$ has property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_0)$.*
(iii) *$\mathrm{C}_p(X)$ has property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_0^*)$.*

5. $\mathcal{I}$-γ-covers. Let $\mathcal{I}$ be an ideal. A sequence $\langle U_n : n \in \omega\rangle$ of subsets of a topological space X is said to be an *$\mathcal{I}$-γ-cover* if $U_n \neq X$ for each n, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$.

Note that a γ-cover is usually a set of sets; however, dealing with ideals, we need to have an ordering of the elements of the cover. We shall identify a countable γ-cover with a Fin-γ-cover. One can easily observe that in this case the enumeration is inessential. The family of all open $\mathcal{I}$-γ-covers of a given topological space X will be denoted by $\mathcal{I}\text{-}\mathit{\Gamma}(X)$ or simply $\mathcal{I}\text{-}\mathit{\Gamma}$. Note that $\mathrm{Fin}\text{-}\mathit{\Gamma} = \mathit{\Gamma}$, where $\mathit{\Gamma}$ is the family of all countable open γ-covers.

A cover $\langle V_n : n \in \omega\rangle$ is called a *refinement* of the cover $\langle U_n : n \in \omega\rangle$ if $V_n \subseteq U_n$ for each $n \in \omega$. An $\mathcal{I}$-γ-cover $\langle U_n : n \in \omega\rangle$ is *shrinkable* if there exists a closed $\mathcal{I}$-γ-cover that is a refinement of $\langle U_n : n \in \omega\rangle$. We denote by $\mathcal{I}$-Γ^{sh} the family of all open shrinkable $\mathcal{I}$-γ-covers.

If $\langle U_n : n \in \omega\rangle$ and $\langle V_n : n \in \omega\rangle$ are $\mathcal{I}$-γ-covers, then $\langle U_n \cap V_n : n \in \omega\rangle$ is an $\mathcal{I}$-γ-cover. If $\langle U_n : n \in \omega\rangle$ and $\langle V_n : n \in \omega\rangle$ are shrinkable $\mathcal{I}$-γ-covers, then so is $\langle U_n \cap V_n : n \in \omega\rangle$. Finally, if $\langle U_n : n \in \omega\rangle$ is an $\mathcal{I}$-γ-cover and $U_n \subseteq V_n \neq X$ for each n, then $\langle V_n : n \in \omega\rangle$ is an $\mathcal{I}$-γ-cover. We shall use those facts without further mention.

For two families $\mathcal{A}$, $\mathcal{B}$ of sequences of subsets of X, we introduce (similarly to M. Scheepers [Sch2]) property $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ as follows: for every sequence $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of sequences from $\mathcal{A}$, there exists a sequence $\langle m_n : n \in \omega\rangle$ of natural numbers such that $\langle U_{n,m_n} : n \in \omega\rangle \in \mathcal{B}$. If a topological space X has property $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ we shall also say that X is an $\mathrm{S}_1(\mathcal{A},\mathcal{B})$-space. A topological space X (or a subset with subspace topology) with property $\mathrm{S}_1(\Omega, \Gamma)$ is a γ-space, where Ω is the family of all ω-covers (3) of X. The basic results concerning the existence of γ-spaces were proved by F. Galvin and A.W. Miller [GM].

As above, one can easily show that if X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space, then for every sequence $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of sequences of $\mathcal{I}$-γ-covers there exists a sequence $\langle m_n : n \in \omega\rangle$ of natural numbers such that $m_n \xrightarrow{\mathcal{J}} \infty$ and $\langle U_{n,m_n} : n \in \omega\rangle$ is a $\mathcal{J}$-γ-cover.

A similar assertion holds true for an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{sh}}, \mathcal{J}\text{-}\Gamma)$-space.

The $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-spaces with $\mathcal{I} = \mathcal{J}$ equal to the density ideal were studied by G. Di Maio and Lj. D. R. Kočinac [MK]. For the general case see [Da].

Evidently we have the monotonicity relation:

$$\text{If } \mathcal{I}_1 \supseteq \mathcal{I}_2 \text{ and } \mathcal{J}_1 \subseteq \mathcal{J}_2 \text{ then every } \mathrm{S}_1(\mathcal{I}_1\text{-}\Gamma, \mathcal{J}_1\text{-}\Gamma)\text{-space is an } \mathrm{S}_1(\mathcal{I}_2\text{-}\Gamma, \mathcal{J}_2\text{-}\Gamma)\text{-space.} \tag{6}$$

A simple observation shows that under some set-theoretical assumptions there exist non-trivial $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-spaces.

THEOREM 5.1. *A γ-space is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space for any ideals $\mathcal{I}$ and $\mathcal{J}$.*

Proof. By (6) it suffices to prove that any γ-space is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$-space.

Let $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ be a sequence of $\mathcal{I}$-γ-covers of X. Each $\mathcal{I}$-γ-cover $\langle U_{n,m} : m \in \omega\rangle$ is an ω-cover. Indeed, for any fixed n and for any $x \in X$, we have

$$A_x = \{m \in \omega : x \notin U_{n,m}\} \in \mathcal{I}.$$

---

(3) An open cover $\mathcal{A}$ of X is an *ω-cover* if for every finite $F \subseteq X$, there exists a set $U \in \mathcal{A}$ such that $F \subseteq U$.

If $F \subseteq X$ is finite then $\bigcup_{x \in F} A_x \in \mathcal{I}$. Therefore there exists $m \notin \bigcup_{x \in F} A_x$. Then $F \subseteq U_{n,m}$.

Since X is a γ-space, there exists a sequence $\langle m_n : n \in \omega \rangle$ of natural numbers such that $\langle U_{n,m_n} : n \in \omega \rangle$ is a γ-cover. ∎

PROBLEM 5.2. *Find an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$*-space that is not a* γ*-space.*

THEOREM 5.3. *Let* X *be a topological space. If the ideal* $\mathcal{J}$ *has a pseudounion, then the following are equivalent:*

(i) X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$*-space.*
(ii) X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$*-space.*

Proof. Let A be a pseudounion of $\mathcal{J}$, $C = \omega \setminus A$, $\mathcal{J} \neq \mathrm{Fin}$. Since both sets A and C are infinite, there exists an injection $\mu : A \to C$ such that $\mu(n) > n$ for any $n \in A$.

Let $\langle\langle U_{n,m} : m \in \omega \rangle : n \in \omega\rangle$ be a sequence of $\mathcal{I}$-γ-covers of X such that $U_{n+1,m} \subseteq U_{n,m}$ for any $n, m \in \omega$. Then there exists a sequence $\langle m_n : n \in \omega \rangle$ of natural numbers such that $\langle U_{n,m_n} : n \in \omega \rangle$ is a $\mathcal{J}$-γ-cover. We set

$$k_n = \begin{cases} m_n & \text{if } n \in C, \\ m_{\mu(n)} & \text{if } n \in A. \end{cases}$$

We claim that $\langle U_{n,k_n} : n \in \omega \rangle$ is a γ-cover.

Let $x \in X$. We set $L = \{n : x \notin U_{n,k_n}\}$. Since $K = \{n : x \notin U_{n,m_n}\} \subseteq^* A$, the set $C \cap L = C \cap K$ is finite. If $n \in A$ then $U_{\mu(n),m_{\mu(n)}} \subseteq U_{n,m_{\mu(n)}} = U_{n,k_n}$. Thus, if $n \in A \cap L$ then $\mu(n) \in C \cap K$. Since μ is an injection, the set $A \cap L$ is finite, and therefore the set L is finite as well. ∎

Therefore, similarly to Section 3 we obtain the equivalence of an ideal notion and a classical one.

COROLLARY 5.4. *Let the ideals* $\mathcal{I}$ *and* $\mathcal{J}$ *have a pseudounion. Then a topological space* X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$*-space if and only if* X *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space.*

6. $(\mathcal{I}, \mathcal{J})$QN, $(\mathcal{I}, \mathcal{J})$wQN and $\mathcal{I}$-γ-covers. In [BH] the authors found a characterization of wQN-spaces by covers, namely $\mathrm{wQN} \equiv \mathrm{S}_1(\Gamma^{\mathrm{sh}}, \Gamma)$. We can prove a similar result.

THEOREM 6.1. *If* X *is a normal topological space, then the following are equivalent:*

(i) X *is an* $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$*-space.*
(ii) X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{sh}}, \mathcal{J}\text{-}\Gamma)$*-space.*

Proof. (i)→(ii) Let $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ be a sequence of shrinkable open $\mathcal{I}$-γ-covers. Then there exists a sequence $\langle\langle V_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of closed $\mathcal{I}$-γ-covers refining the latter ones. Since X is a normal topological space, there exist continuous functions $f_{n,m} : X \to [0,1]$, $n, m \in \omega$, such that $f_{n,m}(x) = 0$ for $x \in V_{n,m}$ and $f_{n,m}(x) = 1$ for $x \in X \setminus U_{n,m}$. Then $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for any $n \in \omega$. By Theorem 4.1 any $(\mathcal{I}, s\mathcal{J})$wQN-space has property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_4)$, so there exists a sequence $\langle m_n : n \in \omega\rangle$ of natural numbers such that $f_{n,m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$. It is easy to see that $\langle U_{n,m_n} : n \in \omega\rangle$ is a $\mathcal{J}$-γ-cover.

(ii)→(i) Assume that $f_m \xrightarrow{\mathcal{I}} 0$. Since the topological space X is infinite, there exists a sequence $\langle G_n : n \in \omega\rangle$ of pairwise disjoint non-empty open sets. Choose a sequence $\langle x_n : n \in \omega\rangle$ such that $x_n \in G_n$. For any $n, m \in \omega$ we set

$$U_{n,m} = \{x \in X : |f_m(x)| < 2^{-n} \wedge x \neq x_m\}.$$

One can easily see that $\langle U_{n,m} : m \in \omega\rangle$ is a shrinkable $\mathcal{I}$-γ-cover of X. Thus there exists a sequence $\langle m_n : n \in \omega\rangle$ of natural numbers such that $\langle U_{n,m_n} : n \in \omega\rangle$ is a $\mathcal{J}$-γ-cover of X. For any $x \in X$ we have

$$\{n : f_{m_n}(x) \geq 2^{-n}\} \subseteq \{n : x \notin U_{n,m_n}\} \cup \{m_n\} \in \mathcal{J}.$$

Therefore $f_{m_n} \xrightarrow{\mathcal{J}} 0$ with control $\langle 2^{-n} : n \in \omega\rangle$. ■

As a corollary we obtain the ideal version of Scheepers' result [Sch3].

Corollary 6.2.

$$\mathrm{S}_1(\mathcal{I}\text{-}\varGamma, \mathcal{J}\text{-}\varGamma) \to \mathrm{S}_1(\mathcal{I}\text{-}\varGamma^{\mathrm{sh}}, \mathcal{J}\text{-}\varGamma) \equiv (\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN} \to (\mathcal{I}, \mathcal{J})\mathrm{wQN}.$$

Corollary 6.3. *Any γ-space is an $(\mathcal{I}, \mathcal{J})$wQN-space for any ideals $\mathcal{I}$ and $\mathcal{J}$.*

J. Šupina [Su], assuming a certain relationship between the ideals $\mathcal{I}$ and $\mathcal{J}$, showed that a suitable discrete $(\mathcal{I}, \mathcal{J})$wQN-space is not an $\mathrm{S}_1(\mathcal{I}\text{-}\varGamma^{\mathrm{sh}}, \mathcal{J}\text{-}\varGamma)$-space.

Problem 6.4. *Find a non-discrete topological $(\mathcal{I}, \mathcal{J})$wQN-space that is not an $\mathrm{S}_1(\mathcal{I}\text{-}\varGamma^{\mathrm{sh}}, \mathcal{J}\text{-}\varGamma)$-space.*

In [BH] and [Sa] the authors found a characterization of QN-spaces by γ-covers. By a simple modification of their proofs one can prove the ideal version of that result, which is actually a cover version of Theorem 4.4.

Theorem 6.5. *Let X be a normal topological space. The following statements are equivalent.*

(i) *X is an $(\mathcal{I}, \mathcal{J})$QN-space.*
(ii) *For every sequence $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of shrinkable open $\mathcal{I}$-γ-covers of X there exists a sequence $\langle n_m : m \in \omega\rangle$ $\mathcal{J}$-divergent to ∞ such that $\langle U_{n_m,m} : m \in \omega\rangle$ is a $\mathcal{J}$-γ-cover of X.*

(iii) *For every sequence* $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of shrinkable open* $\mathcal{I}$-γ-*covers there exists a sequence* $\langle B_n : n \in \omega\rangle$ *of sets from* $\mathcal{J}$ *such that*

$$(\forall x \in X)(\exists A \in \mathcal{J})(\forall n \in \omega)(m \notin A \cup B_n \to x \in U_{n,m}).$$

REMARK. Some authors require, in the definitions of an $(\mathcal{I},\mathcal{J})$wQN-space and properties $(\mathcal{I},\mathcal{J}$-$\alpha_4)$ and $\mathrm{S}_1(\mathcal{A},\ \mathcal{B})$, that the sequence $\langle m_n : n \in \omega\rangle$ be increasing. If we do so, then Theorems 4.1 and 6.1 and Corollaries 6.2 and 6.3 still remain valid.

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Lev Bukovský, Jaroslav Šupina
Institute of Mathematics
Faculty of Sciences
P. J. Šafárik University
Jesenná 5
040 01 Košice, Slovakia
E-mail: lev.bukovsky@upjs.sk
jaroslav.supina@upjs.sk

Pratulananda Das
Department of Mathematics
Jadavpur University
Jadavpur, Kolkata–32, West Bengal, India
E-mail: pratulananda@yahoo.co.in