

THE C_p -STABLE CLOSURE OF THE CLASS OF
SEPARABLE METRIZABLE SPACES

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Abstract. Denote by $\mathbf{C}_p[\mathfrak{M}_0]$ the C_p -stable closure of the class \mathfrak{M}_0 of all separable metrizable spaces, i.e., $\mathbf{C}_p[\mathfrak{M}_0]$ is the smallest class of topological spaces that contains \mathfrak{M}_0 and is closed under taking subspaces, homeomorphic images, countable topological sums, countable Tychonoff products, and function spaces $C_p(X, Y)$. Using a recent deep result of Chernikov and Shelah, we prove that $\mathbf{C}_p[\mathfrak{M}_0]$ coincides with the class of all Tychonoff spaces of cardinality strictly less than \beth_{ω_1} . Being motivated by the theory of generalized metric spaces, we also characterize other natural C_p -type stable closures of \mathfrak{M}_0 .

1. Introduction. All topological spaces considered in this paper are assumed to be Tychonoff. Many natural and important classes \mathcal{X} of topological spaces usually are preserved by basic topological operations, which motivated us in [3] to introduce the following stability property: We say that a class \mathcal{X} of topological spaces is *stable* if \mathcal{X} is closed under taking subspaces, homeomorphic images, countable topological sums, and countable Tychonoff products. For two classes \mathcal{X} and $\tilde{\mathcal{X}}$ of topological spaces, we shall say that $\tilde{\mathcal{X}}$ is an *extension* of \mathcal{X} if $\mathcal{X} \subset \tilde{\mathcal{X}}$. Among all stable extensions of \mathcal{X} there is the smallest one denoted by $[\mathcal{X}]$ and called the *stable closure* of \mathcal{X} .

It is clear that the class \mathfrak{M}_0 of all separable metrizable spaces is stable. One of the most natural generalizations of \mathfrak{M}_0 is the class of all cosmic spaces. Recall (see [9]) that a topological space X is called *cosmic* (respectively, a σ -space) if it is regular and has a countable (respectively, a σ -locally finite) network. A family \mathcal{N} of subsets of X is a *network* for X if for any $x \in X$ and any open neighborhood U of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset U$. The class \mathfrak{C} of all cosmic spaces is stable [10], and so are many other classes of generalized metric spaces, in particular, the classes of σ -spaces, \aleph_0 -spaces, \aleph -spaces, \mathfrak{P}_0 -spaces and \mathfrak{P} -spaces (see [10, 9, 2, 7]). Clearly, the class \mathfrak{T} of

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all topological spaces is stable, but the class \mathfrak{L} of all Lindelöf spaces is not. Note that any cosmic space is Lindelöf.

For topological spaces X and Y , we denote by $C_k(X, Y)$ and $C_p(X, Y)$ the space $C(X, Y)$ of all continuous functions from X into Y endowed with the compact-open topology and the topology of pointwise convergence, respectively. Having in mind the stability of the classes of \aleph_0 -spaces and \mathfrak{P}_0 -spaces under taking function spaces $C_k(X, Y)$ (see [10, 2]), we defined in [3] a class \mathcal{Z} of topological spaces to be C_k -stable if \mathcal{Z} is stable and for any Lindelöf $X \in \mathcal{Z}$ and any $Y \in \mathcal{Z}$ the function space $C_k(X, Y)$ belongs to \mathcal{Z} . In [3] we proved that the smallest C_k -stable extension of \mathfrak{M}_0 coincides with the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M}_0)$ of all topological spaces which can be embedded into $C_k(X, Y)$ for suitable separable metric spaces X and Y . These results motivate the following definition.

DEFINITION 1.1. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be classes of topological spaces. The class \mathcal{Z} is called $C_p^{\mathcal{X}, \mathcal{Y}}$ -stable if \mathcal{Z} is stable and for any $X \in \mathcal{Z} \cap \mathcal{X}$ and $Y \in \mathcal{Z} \cap \mathcal{Y}$ the function space $C_p(X, Y)$ belongs to \mathcal{Z} . The smallest $C_p^{\mathcal{X}, \mathcal{Y}}$ -stable extension of \mathcal{Z} is denoted by $\mathbf{C}_p^{\mathcal{X}, \mathcal{Y}}[\mathcal{Z}]$ and called the $C_p^{\mathcal{X}, \mathcal{Y}}$ -stable closure of \mathcal{Z} .

In this paper we shall describe the $C_p^{\mathcal{X}, \mathcal{Y}}$ -stable closures of \mathfrak{M}_0 for the cases of \mathcal{X} and \mathcal{Y} equal to \mathfrak{M}_0 or \mathfrak{T} .

In case $\mathcal{X} = \mathcal{Y} = \mathfrak{T}$ the class $\mathbf{C}_p^{\mathfrak{T}, \mathfrak{T}}[\mathcal{Z}]$ will be denoted by $\mathbf{C}_p[\mathcal{Z}]$ and called the C_p -stable closure of \mathcal{Z} .

Our first theorem (which will be proved in Section 2) essentially follows from Michael’s results [10].

THEOREM 1.2. *The classes $\mathbf{C}_p^{\mathfrak{T}, \mathfrak{M}_0}[\mathfrak{M}_0]$ and $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{M}_0}[\mathfrak{M}_0]$ coincide with the class of all cosmic spaces.*

The class $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{T}}[\mathfrak{M}_0]$ is strictly larger than $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{M}_0}[\mathfrak{M}_0] = \mathbf{C}_p^{\mathfrak{T}, \mathfrak{M}_0}[\mathfrak{M}_0]$ and admits the following description. For a topological space X , let $X^{<\omega} = \bigoplus_{n \in \omega} X^n$ be the topological sum of all its finite powers. We denote by $SC_p(X^{<\omega})$ the space of all real-valued separately continuous functions $f : X^{<\omega} \rightarrow \mathbb{R}$, endowed with the topology of pointwise convergence. For a class \mathcal{X} of topological spaces, we denote by $\mathbf{SC}_p(\mathcal{X}^{<\omega})$ the class of topological spaces which embed into function spaces $SC_p(X^{<\omega})$ with $X \in \mathcal{X}$.

THEOREM 1.3. *The class $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{T}}[\mathfrak{M}_0]$ coincides with $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$.*

Theorem 1.3 motivates studying the class $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ in more detail. It turns out that this class contains all cosmic spaces, all metrizable spaces of weight $\leq \mathfrak{c}$ and also all generalized ordered spaces whose order topology is second countable. Let us recall the necessary definitions related to generalized ordered spaces.

By a *linearly ordered space* we understand a linearly ordered set (X, \leq) endowed with the *order topology* generated by the subbase consisting of open half-intervals $(\leftarrow, a) = \{x \in X : x < a\}$ and $(a, \rightarrow) = \{x \in X : x > a\}$ where $a \in X$. By [6, 3.12.4(d)], the weight $w(X)$ of a linearly ordered space X coincides with its network weight $nw(X)$. A subset $C \subset X$ is *order convex* if for any points $x \leq y$ in C the order interval $[x, y] = \{z \in X : x \leq z \leq y\}$ is contained in C .

A *generalized ordered space* (briefly, a *GO-space*) is a topological space X endowed with a linear order \leq such that the open order convex subsets of X form a base of the topology of X . Standard examples of *GO-spaces* are the Sorgenfrey line and the Michael line (see [6, 1.2.2 and 5.1.32]). Also the real line \mathbb{R}_d endowed with the discrete topology is a *GO-space*.

By the *order topology* on a *GO-space* X we understand the topology on X generated by the linear order. The *linear weight* $lw(X)$ of a *GO-space* X is the weight of its order topology. In particular, $lw(\mathbb{R}_d) = \omega$. It follows that $w(X) \geq nw(X) \geq lw(X)$ for each *GO-space* X . Note that $w(\mathbb{R}_d) = \mathfrak{c} > lw(\mathbb{R}_d) = \aleph_0$.

THEOREM 1.4. *The class $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ contains all cosmic spaces, all metrizable spaces of weight $\leq \mathfrak{c}$, and all generalized ordered spaces of countable linear weight.*

An upper bound on $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is given by the class of spaces with countable *i-weight*. We recall that the *i-weight* $iw(X)$ of a Tychonoff space X is the smallest cardinal κ for which there is an injective continuous map $i : X \rightarrow Y$ into a Tychonoff space Y of weight $w(Y) \leq \kappa$. It is clear that $iw(X) \leq w(X)$, and if $iw(X) \leq \aleph_0$ then $|X| \leq \mathfrak{c}$. The following theorem is proved in Section 5.

THEOREM 1.5. *Each space $X \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0] = \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ has weight $w(X) \leq \mathfrak{c}$, cardinality $|X| \leq \mathfrak{c}$ and *i-weight* $iw(X) \leq \aleph_0$.*

EXAMPLE 1.6. By [8], the Banach space $X = \ell_1(\mathfrak{c})$ endowed with the weak topology is an \aleph -space of cardinality $|X| = \mathfrak{c}$, weight $w(X) = 2^\mathfrak{c}$ and *i-weight* $iw(X) = \aleph_0$. By Theorem 1.5, the space X does not belong to $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0] = \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$. Let us recall [9, §11] that a regular topological space X is an \aleph -space if X has a σ -locally finite k -network. A family \mathcal{N} of subsets of X is called a *k-network* in X if for any open set $U \subset X$ and compact subset $K \subset U$ there is a finite subfamily $\mathcal{F} \subset \mathcal{N}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Since each compact space of countable *i-weight* is metrizable, Theorem 1.5 implies:

COROLLARY 1.7. *Each compact space in $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0] = \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is metrizable.*

This corollary shows that $\mathbf{C}_p^{\mathfrak{M}_0, \aleph_1}[\mathfrak{M}_0]$ can be considered as a new class of generalized metric spaces.

Now we describe the C_p -stable closure $\mathbf{C}_p[\mathfrak{M}_0] := \mathbf{C}_p^{\aleph_1, \aleph_1}[\mathfrak{M}_0]$ of \mathfrak{M}_0 . It turns out that $\mathbf{C}_p[\mathfrak{M}_0]$ is huge and coincides with the class of all Tychonoff spaces of cardinality (or weight) strictly smaller than \beth_{ω_1} . Here $\beth_0 = \aleph_0$ and $\beth_\alpha = \sup\{2^{\beth_\beta} : \beta < \alpha\}$ for any ordinal $\alpha > 0$.

THEOREM 1.8. *The C_p -stable closure $\mathbf{C}_p[\mathfrak{M}_0]$ of the class \mathfrak{M}_0 of all separable metrizable spaces coincides with the class of all Tychonoff spaces of cardinality strictly less than \beth_{ω_1} .*

We prove this theorem in Section 6 using a recent deep result of Chernikov and Shelah [4].

2. Proof of Theorem 1.2. We shall deduce Theorem 1.2 from the following characterization of cosmic spaces due to Michael [10].

FACT 2.1 ([10]). *For a Tychonoff space X , the following conditions are equivalent:*

- (1) X is cosmic;
- (2) X is a continuous image of a separable metrizable space;
- (3) the function space $C_p(X) := C_p(X, \mathbb{R})$ is cosmic;
- (4) for any metrizable separable space Y , the function space $C_p(X, Y)$ is cosmic.

For a Tychonoff space X , we denote by $\delta : X \rightarrow C_p(C_p(X))$ the canonical map assigning to each $x \in X$ the Dirac measure δ_x concentrated at x . The Dirac measure $\delta_x : C_p(X) \rightarrow \mathbb{R}$ assigns to each $f \in C_p(X)$ its value $f(x)$. The following important fact is well-known and can be found in [1, 0.5.5].

FACT 2.2. *For any Tychonoff space X , the canonical map $\delta : X \rightarrow C_p(C_p(X))$ is a topological embedding.*

Proof of Theorem 1.2. Let X be a cosmic space. By Fact 2.1, the space $C_p(X)$ is the image of a separable metrizable space M under a continuous map $\xi : M \rightarrow C_p(X)$. It follows that the dual map

$$\xi^* : C_p(C_p(X)) \rightarrow C_p(M), \quad \xi^*(f) = f \circ \xi,$$

is a topological embedding of $C_p(C_p(X))$ into $C_p(M) \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{M}_0}[\mathfrak{M}_0]$. Now Fact 2.2 implies that $X \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{M}_0}[\mathfrak{M}_0] \subset \mathbf{C}_p^{\aleph_1, \mathfrak{M}_0}[\mathfrak{M}_0]$. Therefore, $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{M}_0}[\mathfrak{M}_0]$ contains all cosmic spaces.

To complete the proof it is enough to show that the class \mathfrak{C} of all cosmic spaces is $C_p^{\aleph_1, \mathfrak{M}_0}$ -stable. But this immediately follows from the stability of \mathfrak{C} and from Fact 2.1. ■

3. Proof of Theorem 1.3. For topological spaces X_1, \dots, X_n, Y, Z , we denote by $SC_p(X_1 \times \dots \times X_n, Y)$ the space of all separately continuous functions $f : X_1 \times \dots \times X_n \rightarrow Y$ endowed with the topology of point-wise convergence. It is well-known that $SC_p(X_1 \times \dots \times X_n, Y)$ is canonically homeomorphic to $C_p(X_1, SC_p(X_2 \times \dots \times X_n, Y))$. In particular, $SC_p(X \times Y, Z)$ is canonically homeomorphic to $C_p(X, C_p(Y, Z))$. In the following, $SC_p(X_1 \times \dots \times X_n, \mathbb{R})$ will be denoted by $SC_p(X_1 \times \dots \times X_n)$. For $n = 1$ the space $SC_p(X_1)$ coincides with $C_p(X_1)$.

We shall prove by induction that for every $n \in \mathbb{N}$ and every $X \in \mathfrak{M}_0$ the function space $SC_p(X^n)$ belongs to $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$. For $n = 1$ the space $SC_p(X) = C_p(X)$ belongs to $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$ by definition. Assume that for some $n \in \mathbb{N}$ we have proved that $SC_p(X^n)$ belongs to $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$. Taking into account that $SC_p(X^{n+1})$ is (canonically) homeomorphic to $C_p(X, SC_p(X^n))$, we conclude that $SC_p(X^{n+1}) \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$. Since $SC_p(X^{<\omega})$ is homeomorphic to $\prod_{n \in \mathbb{N}} SC_p(X^n)$, we conclude that $SC_p(X^{<\omega}) \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$, and hence $SC_p(\mathfrak{M}_0^{<\omega}) \subset \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$.

To prove the reverse inclusion it is enough to check that $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is closed under taking countable topological sums, countable Tychonoff products and taking function spaces with metrizable separable domain.

To see that $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is closed under taking countable topological sums, it is enough to prove that for any non-empty spaces $X_n \in \mathfrak{M}_0$, $n \in \omega$, the topological sum $\bigoplus_{n \in \omega} SC_p(X_n^{<\omega})$ embeds into $SC_p(X^{<\omega})$ for some $X \in \mathfrak{M}_0$. Consider $X = \bigoplus_{n \in \omega} X_n$ and the topological embedding

$$e : \bigoplus_{n \in \omega} SC_p(X_n^{<\omega}) \hookrightarrow SC_p(X^{<\omega})$$

assigning to each $f \in SC_p(X_n^{<\omega})$, $n \in \omega$, the function $\tilde{f} \in SC_p(X^{<\omega})$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X_n^{<\omega}, \\ 0 & \text{if } x \in X^{<\omega} \setminus X_n^{<\omega}. \end{cases}$$

As $X \in \mathfrak{M}_0$, we conclude that $SC_p(X^{<\omega})$ and hence $\bigoplus_{n \in \omega} SC_p(X_n^{<\omega})$ belong to $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$.

Next we prove that for any separable metrizable space X and any $Y \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ we have $C_p(X, Y) \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$. The space Y can be identified with a subspace of $SC_p(Z^{<\omega})$ for some separable metrizable space Z . Then $C_p(X, Y)$ can be identified with a subspace of $C_p(X, SC_p(Z^{<\omega}))$ which is canonically homeomorphic to

$$C_p\left(X, \prod_{n \in \mathbb{N}} SC_p(Z^n)\right) = \prod_{n \in \mathbb{N}} C_p(X, SC_p(Z^n)) = \prod_{n \in \mathbb{N}} SC_p(X \times Z^n).$$

As $X \times Z^n$ is canonically homeomorphic to a retract $X \times \{x_0\}^{n-1} \times Z^n$ of

$X^n \times Z^n = (X \times Z)^n$, we obtain

$$C_p(X, SC_p(Z^{<\omega})) = \prod_{n \in \mathbb{N}} SC_p(X \times Z^n) \hookrightarrow \prod_{n \in \mathbb{N}} SC_p((X \times Z)^n) = SC_p((X \times Z)^{<\omega}).$$

Since $X \times Z \in \mathfrak{M}_0$, we arrive at

$$C_p(X, Y) \hookrightarrow SC_p((X \times Z)^{<\omega}) \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega}).$$

Finally, we show that $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is closed under taking countable Tychonoff products. Fix any $X_n \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$, $n \in \omega$. As $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is closed under taking countable topological sums, $X = \bigoplus_{n \in \omega} X_n$ belongs to $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$. Since $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ is also closed under taking function spaces with separable metrizable domain, the function space $C_p(\omega, X)$ belongs to $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$. As

$$\prod_{n \in \omega} X_n \subset X^\omega = C_p(\omega, X) \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega}),$$

we conclude that $\prod_{n \in \omega} X_n \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$.

4. Proof of Theorem 1.4. We divide the proof of Theorem 1.4 into three lemmas. The first of them follows from Theorem 1.2 and the obvious inclusion $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{M}_0}[\mathfrak{M}_0] \subset \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{T}}[\mathfrak{M}_0]$.

LEMMA 4.1. *The class $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{T}}[\mathfrak{M}_0]$ contains all cosmic spaces.*

The next lemma will also play an important role in the proof of Theorem 1.8. In this lemma an ordered field is endowed with the topology generated by the linear order.

LEMMA 4.2. *For every ordered field F , the space*

$$SC_p(F \times F, F) \cong C_p(F, C_p(F, F))$$

contains a discrete subspace D of cardinality $|F|$.

Proof. Consider the discontinuous separately continuous function $\mathbf{sp}^F : F \times F \rightarrow F$ defined by

$$\mathbf{sp}^F(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ 2xy/(x^2 + y^2) & \text{otherwise.} \end{cases}$$

For any $a, b \in F$, consider the shifted function $\mathbf{sp}_{a,b}^F : F \times F \rightarrow F$ defined by $\mathbf{sp}_{a,b}^F(x, y) := \mathbf{sp}^F(x - a, y - b)$, and observe that the subspace $D = \{\mathbf{sp}_{a,a}^F : a \in F\}$ is discrete in $SC_p(F \times F, F) \cong C_p(F, C_p(F, F))$. ■

LEMMA 4.3. *Each metrizable space X of weight $\leq \mathfrak{c}$ belongs to the class $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{T}}[\mathfrak{M}_0]$.*

Proof. Let $sp := \mathbf{sp}^{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the classical discontinuous separately continuous function. By Lemma 4.2, the subspace $D = \{sp_{a,a} : a \in \mathbb{R}\}$ is discrete in $SC_p(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Moreover, the subspace $H = \{t \cdot sp_{a,a} : a \in \mathbb{R}, t \in [0, 1]\}$ of $SC_p(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is homeomorphic to a hedgehog with \mathfrak{c} spines. It follows from $SC_p(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \cong C_p(\mathbb{R}, C_p(\mathbb{R})) \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$ that $H \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$, and hence $H^\omega \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$. By [6, 4.4.9], each metrizable space of weight $\leq \mathfrak{c}$ embeds into H^ω . Consequently, $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$ contains all metrizable spaces of weight $\leq \mathfrak{c}$. ■

LEMMA 4.4. *The class $\mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{I}}[\mathfrak{M}_0]$ contains all generalized ordered spaces of countable linear weight.*

Proof. Let (X, \leq, τ) be a generalized ordered space with countable linear weight. Consider the following two subsets of X :

$$X_\ell = \{x \in X : (\leftarrow, x] \text{ is open in } X\},$$

$$X_r = \{x \in X : [x, \rightarrow) \text{ is open in } X\}.$$

Let τ_{\leq} be the topology on X generated by the linear order. By our assumption, this topology is second countable, so, by [6, 6.3.2(c)], the linearly ordered space (X, τ_{\leq}) is order homeomorphic to a subspace of \mathbb{R} . Therefore we can assume that $X \subset \mathbb{R}$ and that τ is generated by the subbase

$$\tau_{\leq} \cup \{(\leftarrow, x] : x \in X_\ell\} \cup \{[x, \rightarrow) : x \in X_r\}.$$

Denote by \tilde{X}_ℓ and \tilde{X}_r the sets X_ℓ and X_r respectively endowed with the separable metrizable topology inherited from \mathbb{R} .

Besides the Euclidean topology $\tilde{\tau}_{\leq}$ generated by the linear order on \mathbb{R} , we shall consider the following three *GO*-topologies on \mathbb{R} :

- the topology $\tilde{\tau}_\ell$ generated by the subbase $\tilde{\tau}_{\leq} \cup \{(-\infty, x] : x \in X_\ell\}$,
- the topology $\tilde{\tau}_r$ generated by the subbase $\tilde{\tau}_{\leq} \cup \{[x, +\infty) : x \in X_r\}$,
- the topology $\tilde{\tau}$ generated by the subbase $\tilde{\tau}_\ell \cup \tilde{\tau}_r$.

It is easy to see that the identity map $\text{id} : (X, \tau) \rightarrow (\mathbb{R}, \tilde{\tau})$ is a topological embedding. So, it suffices to show that $(\mathbb{R}, \tilde{\tau}) \in SC_p(\mathfrak{M}_0^{<\omega})$.

For this purpose we consider the separately continuous functions $L, R : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$L(x, y) := \begin{cases} 2xy/(x^2 + y^2) & \text{if } x, y < 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$R(x, y) := \begin{cases} 2xy/(x^2 + y^2) & \text{if } x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For every $a \in \mathbb{R}$, consider the shifted functions $L_a, R_a : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$L_a(x, y) := L(x - a, y - a), \quad R_a(x, y) := R(x - a, y - a), \quad \forall x, y \in \mathbb{R}.$$

Let $a \in \mathbb{R}$, $\varepsilon > 0$ and $(x, y) \in \tilde{X}_\ell \times \tilde{X}_\ell$. If $x < a$ and $y < a$, we can find $\delta > 0$ such that

$$(4.1) \quad |L_a(x, y) - L_b(x, y)| < \varepsilon, \quad \forall b \in (a - \delta, a + \delta) \in \tilde{\tau}_\leq.$$

If $x > a$ or $y > a$, we can find $\delta > 0$ such that

$$(4.2) \quad L_a(x, y) = L_b(x, y) = 0, \quad \forall b \in (a - \delta, a + \delta) \in \tilde{\tau}_\leq.$$

Assume that $x = y = a$. Then $a \in \tilde{X}_\ell$ and

$$(4.3) \quad L_a(x, y) = L_b(x, y) = 0, \quad \forall b \in (-\infty, a] \in \tilde{\tau}_\leq.$$

Since $SC_p(\tilde{X}_\ell \times \tilde{X}_\ell)$ carries the topology of pointwise convergence, (4.1)–(4.3) imply that the map

$$\mathcal{L} : (\mathbb{R}, \tilde{\tau}_\leq) \rightarrow SC_p(\tilde{X}_\ell \times \tilde{X}_\ell), \quad \mathcal{L} : a \mapsto L_a|_{\tilde{X}_\ell \times \tilde{X}_\ell},$$

is continuous. Note also that if $a \in \tilde{X}_\ell$ and $b > a$, then $L_b(a, a) = 1$ and (see (4.3))

$$(4.4) \quad \mathcal{L}((-\infty, a]) = \mathcal{L}(\mathbb{R}) \cap U,$$

where $U = \{f \in SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) : |f(a, a)| < 1/2\}$ is open in $SC_p(\tilde{X}_\ell \times \tilde{X}_\ell)$.

Analogously it can be shown that the function

$$\mathcal{R} : (\mathbb{R}, \tilde{\tau}_r) \rightarrow SC_p(\tilde{X}_r \times \tilde{X}_r), \quad \mathcal{R} : b \mapsto R_b|_{\tilde{X}_r \times \tilde{X}_r},$$

is continuous, and if $b \in \tilde{X}_r$ and $c < b$, then $R_c(b, b) = 1$ and

$$(4.5) \quad \mathcal{R}([b, +\infty)) = \mathcal{R}(\mathbb{R}) \cap V,$$

where $V = \{f \in SC_p(\tilde{X}_r \times \tilde{X}_r) : |f(b, b)| < 1/2\}$ is open in $SC_p(\tilde{X}_r \times \tilde{X}_r)$.

Set $T := \mathbb{R} \times SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) \times SC_p(\tilde{X}_r \times \tilde{X}_r) \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ and define

$$\mathcal{S} := (\text{id}, \mathcal{L}, \mathcal{R}) : (\mathbb{R}, \tilde{\tau}) \rightarrow T.$$

Clearly, the map \mathcal{S} is continuous and injective. If $(a, b) \in \tilde{\tau}_\leq$, then

$$(4.6) \quad \mathcal{S}((a, b)) = \mathcal{S}(\mathbb{R}) \cap [(a, b) \times SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) \times SC_p(\tilde{X}_r \times \tilde{X}_r)].$$

If $a \in \tilde{X}_\ell$ and $b \in \tilde{X}_r$, then (4.4) and (4.5) imply

$$(4.7) \quad \mathcal{S}((-\infty, a]) = \mathcal{S}(\mathbb{R}) \cap [\mathbb{R} \times U \times SC_p(\tilde{X}_r \times \tilde{X}_r)],$$

$$(4.8) \quad \mathcal{S}([b, \infty)) = \mathcal{S}(\mathbb{R}) \cap [\mathbb{R} \times SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) \times V].$$

The equalities (4.6)–(4.8) imply that \mathcal{S} is a topological embedding of the GO -space $(\mathbb{R}, \tilde{\tau})$ into $T \in \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$, which implies that the GO -space $(\mathbb{R}, \tilde{\tau})$ and its subspace (X, τ) both belong to $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$. ■

5. Proof of Theorem 1.5. Fix any space $X \in \mathbf{C}_p^{\mathfrak{M}_0, \mathfrak{T}}[\mathfrak{M}_0]$. By Theorem 1.3, the space X embeds into $SC_p(Z^{<\omega})$ for some separable metrizable Z .

Then $X \subset SC_p(Z^{<\omega}) \subset \mathbb{R}^{Z^{<\omega}}$ has weight

$$w(X) \leq w(SC_p(Z^{<\omega})) \leq w(\mathbb{R}^{Z^{<\omega}}) \leq |Z^{<\omega}| \leq \mathfrak{c}.$$

To see that $iw(X) \leq \mathfrak{c}$, choose a countable dense subset D of Z and observe that the restriction operator

$$R : SC_p(Z^{<\omega}) \rightarrow SC_p(D^{<\omega}) \subset \mathbb{R}^{D^{<\omega}}, \quad R : f \mapsto f|_{D^{<\omega}},$$

is continuous.

Let us show that R is also injective. Since R is a linear operator, it is enough to show that a function $f \in SC_p(X^n)$ equals zero if $f|_{D^n} = 0$. Since f is separately continuous and D is dense in X , we have

$$f|_{X \times D^{n-1}} = f|_{X^2 \times D^{n-2}} = \dots = f|_{X^n} = 0.$$

Therefore, $R : X \rightarrow \mathbb{R}^{D^{<\omega}}$ is continuous and injective, which implies that $iw(X) \leq w(\mathbb{R}^{D^{<\omega}}) \leq |D^{<\omega}| \leq \aleph_0$ and $|X| \leq |\mathbb{R}^{D^{<\omega}}| \leq |\mathbb{R}^\omega| = \mathfrak{c}$.

6. Proof of Theorem 1.8. For a linearly ordered set X , by a *cut* of L we understand a pair (A, B) consisting of two subsets A, B of X such that $X = A \cup B$ and $a < b$ for any $a \in A$ and $b \in B$. Each point $x \in X$ determines two cuts

$$x_- = ((\leftarrow, x), [x, \rightarrow)) \quad \text{and} \quad x_+ = ((\leftarrow, x], (x, \rightarrow))$$

of X . The set of all cuts of X will be denoted by \check{X} . The set \check{X} is linearly ordered by $(A, B) \leq (C, D)$ if for each $a \in A$ there is $c \in C$ such that $a \leq c$. It is easy to see that $X^\pm = \{x_-, x_+ : x \in X\}$ is order dense in the linearly ordered set \check{X} .

To each linearly ordered set L we can assign the *ordered group* $H(L)$ consisting of all functions $f : L \rightarrow \mathbb{Q}$ with finite support

$$\text{supp}(f) = \{x \in L : f(x) \neq 0\};$$

this group is called the *Hahn group* (see [5, 1.25], where it is denoted by $\mathfrak{F}_0(\mathbb{Q}, L)$). The order on $H(L)$ is defined by letting $f > 0$ iff $f(\min(\text{supp}(f))) > 0$. The Hahn group $H^2(L) := H(H(L))$ has the structure of an ordered field. For $f, g : H(L) \rightarrow \mathbb{Q}$ in $H^2(\mathbb{Q})$ their product in $H^2(L)$ is defined as the convolution $f * g : z \mapsto \sum_{x+y=z} f(x)g(y)$. By [5, Theorem 2.15], $H^2(L)$ is a real-closed ordered field of cardinality $\max\{|L|, \aleph_0\}$.

For any (order dense) subspace $D \subset L$ we can identify $H(D)$ with the (order dense) subgroup $\{f \in H(L) : \text{supp}(f) \subset D\}$ of $H(L)$ and the field $H^2(D)$ with the (order dense) subfield $\{f \in H^2(L) : \text{supp}(f) \subset H(D) \subset H(L)\}$ of $H^2(L)$.

The proof of Theorem 1.8 is based on the following recent deep result of Chernikov and Shelah [4, Corollary 2.11].

FACT 6.1 (Chernikov–Shelah). *For any infinite cardinal κ there are seven linearly ordered spaces L_0, \dots, L_6 such that*

$$|L_0| \leq \kappa, \quad |L_6| \geq 2^\kappa \quad \text{and} \quad |L_{k+1}| \leq |\check{L}_k|$$

for every $k \in \{0, 1, 2, 3, 4, 5\}$.

Specifically, this result will be applied in the proof of the following lemma.

LEMMA 6.2. *For every ordinal $\alpha < \omega_1$, each Tychonoff space of weight $\leq \beth_\alpha$ belongs to the class $\mathbf{C}_p[\mathfrak{M}_0]$.*

Proof. We use transfinite induction. For $\alpha = 0$ the statement is trivially true. Assume that for some countable ordinal α and all ordinals $\beta < \alpha$ we have proved that all Tychonoff spaces of weight $\leq \beth_\beta$ belong to $\mathbf{C}_p[\mathfrak{M}_0]$. If the ordinal α is limit, then $\beth_\alpha = \sup_{\beta < \alpha} \beth_\beta$. By the inductive assumption, the cardinals \beth_β , $\beta < \alpha$, endowed with the discrete topology belong to $\mathbf{C}_p[\mathfrak{M}_0]$. Since this class is closed under taking countable topological sums, the cardinal \beth_α belongs to $\mathbf{C}_p[\mathfrak{M}_0]$. Then the Tychonoff power $\mathbb{R}^{\beth_\alpha}$, being homeomorphic to $C_p(\beth_\alpha)$, belongs to $\mathbf{C}_p[\mathfrak{M}_0]$ too. Since each Tychonoff space of weight $\leq \beth_\alpha$ embeds into $\mathbb{R}^{\beth_\alpha} \in \mathbf{C}_p[\mathfrak{M}_0]$, the class $\mathbf{C}_p[\mathfrak{M}_0]$ contains all Tychonoff spaces of weight $\leq \beth_\alpha$. This completes the inductive step in the case where α is a limit ordinal.

Now assume that $\alpha = \beta + 1$ is a successor ordinal. Then $\beth_\alpha = 2^{\beth_\beta}$. By Fact 6.1, there are infinite linearly ordered sets L_0, \dots, L_6 such that $|L_0| \leq \beth_\beta$, $\beth_\alpha \leq |L_6|$ and $|L_{k+1}| \leq |\check{L}_k|$ for all $k \in \{0, \dots, 5\}$. By induction, for every $k \in \{0, \dots, 6\}$ we shall prove that each Tychonoff space of weight $\leq |L_k|$ belongs to $\mathbf{C}_p[\mathfrak{M}_0]$. For $k = 0$ this follows from the inequality $|L_0| \leq \beth_\beta$ and the inductive assumption. Assume that for some $k \in \{0, \dots, 6\}$ we have proved that every Tychonoff space of weight $\leq |L_k|$ belongs to $\mathbf{C}_p[\mathfrak{M}_0]$. Consider the linearly ordered space \check{L}_k , and observe that L_k^\pm is an order dense subset of \check{L}_k and

$$\{(\leftarrow, x) : x \in L_k^\pm\} \cup \{(x, \rightarrow) : x \in L_k^\pm\}$$

is a base of the order topology of \check{L}_k . So, the linearly ordered space \check{L}_k has topological weight $\leq |L_k|$. Now consider the Hahn field $H^2(\check{L}_k)$ over the linearly ordered set \check{L}_k and its order dense subfield $H^2(L_k^\pm)$. Taking into account that the weight of an ordered field is equal to its density, we deduce that the ordered field $F_k := H^2(\check{L}_k)$ has topological weight

$$w(F_k) \leq |H^2(L_k^\pm)| = |L_k^\pm| = |L_k|.$$

By the inductive assumption, the space F_k belongs to $\mathbf{C}_p[\mathfrak{M}_0]$ and so does the function space $C_p(F_k, C_p(F_k, F_k))$. The latter is canonically homeomorphic to the subspace $SC_p(F_k \times F_k, F_k)$ of $F_k^{F_k \times F_k}$ consisting of separately continuous functions.

By Lemma 4.2, the space $SC_p(F_k \times F_k, F_k) \in \mathbf{C}_p[\mathfrak{M}_0]$ contains a discrete subspace D of cardinality $|F_k|$. Observe that

$$|D| = |F_k| = |H^2(\check{L}_k)| = |\check{L}_k| \geq |L_{k+1}|.$$

Hence $\mathbf{C}_p[\mathfrak{M}_0]$ also contains $C_p(D)$, which is homeomorphic to \mathbb{R}^D . Since each Tychonoff space of cardinality $\leq |L_{k+1}| \leq |D|$ embeds into \mathbb{R}^D , the class $\mathbf{C}_p[\mathfrak{M}_0]$ contains all Tychonoff spaces of cardinality $\leq |L_{k+1}|$. This completes the inductive step. For $k = 6$ we get $|L_6| \geq \beth_\alpha$, which implies that $\mathbf{C}_p[\mathfrak{M}_0]$ contains all Tychonoff spaces of weight $\leq \beth_\alpha$. ■

Proof of Theorem 1.8. Denote by $\mathbf{Tych}(\beth_{\omega_1})$ the class of all Tychonoff spaces of weight strictly less than \beth_{ω_1} . Clearly, $\mathbf{Tych}(\beth_{\omega_1})$ is stable. Let $X, Y \in \mathbf{Tych}(\beth_{\omega_1})$. Then $\kappa = \max\{w(X), w(Y)\} < \beth_{\omega_1}$, $|X| \leq 2^{w(X)} \leq 2^\kappa$ and

$$w(C_p(X, Y)) \leq w(Y^X) \leq w(Y)^{|X|} \leq \kappa^{|X|} \leq 2^{\kappa \cdot |X|} \leq 2^{2^\kappa} < \beth_{\omega_1},$$

which implies $C_p(X, Y) \in \mathbf{Tych}(\beth_{\omega_1})$. Therefore, $\mathbf{Tych}(\beth_{\omega_1})$ is C_p -stable and hence $\mathbf{C}_p[\mathfrak{M}_0] \subset \mathbf{Tych}(\beth_{\omega_1})$. On the other hand, Lemma 6.2 guarantees that $\mathbf{Tych}(\beth_{\omega_1}) \subset \mathbf{C}_p[\mathfrak{M}_0]$, and hence $\mathbf{C}_p[\mathfrak{M}_0] = \mathbf{Tych}(\beth_{\omega_1})$.

Since each Tychonoff space X has weight $w(X) \leq 2^{|X|}$ and cardinality $|X| \leq 2^{w(X)}$, the inequalities $w(X) < \beth_{\omega_1}$ and $|X| < \beth_{\omega_1}$ are equivalent, which implies that $\mathbf{Tych}(\beth_{\omega_1})$ coincides with the class of all Tychonoff spaces of cardinality $< \beth_{\omega_1}$. ■

7. Open problems. By Theorem 1.4, $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ contains all cosmic spaces and all metrizable spaces of weight $\leq \mathfrak{c}$. The classes of cosmic spaces and of metrizable spaces are contained in the class of σ -spaces. It can be shown that each σ -space X of weight $\leq \mathfrak{c}$ admits a continuous injective map onto a metrizable space of weight $\leq \mathfrak{c}$ and hence has countable i -weight.

PROBLEM 7.1. *Does $\mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$ contain all σ -spaces of weight $\leq \mathfrak{c}$?*

It is easy to see that for every metrizable separable space X the space $SC_p(X^{<\omega})$ is canonically homeomorphic to the countable product $\prod_{n \in \mathbb{N}} SC_p(X^n)$. We can ask how different the function spaces $SC_p(X^n)$ are for various $n \in \mathbb{N}$.

PROBLEM 7.2. *Is it true that $SC_p([0, 1]^m)$ embeds into $SC_p([0, 1]^n)$ for no positive integers $n < m$?*

(The answer is affirmative if $n = 1$, as $SC_p([0, 1]) = C_p([0, 1])$ is cosmic while $SC_p([0, 1]^m)$ is not.)

The affirmative answer to the following problem would simplify the proof of Theorem 1.8. Observe that this problem has an affirmative answer under the Generalized Continuum Hypothesis.

PROBLEM 7.3. *Is it true that for every infinite cardinal κ there exists a Tychonoff space X of weight $w(X) \leq \kappa$ such that $SC_p(X \times X)$ contains a discrete subspace of cardinality 2^κ ?*

Using Fact 2.1 and (the proof of) Theorem 1.2 one can show that for the class \mathfrak{C} of all cosmic spaces we get $\mathbf{C}_p^{\mathfrak{C};\mathfrak{I}}[\mathfrak{M}_0] = \mathbf{C}_p^{\mathfrak{M}_0;\mathfrak{I}}[\mathfrak{M}_0] = \mathbf{SC}_p(\mathfrak{M}_0^{<\omega})$. It would be interesting to determine the extensions $\mathbf{C}_p^{\mathcal{X};\mathfrak{I}}[\mathfrak{M}_0]$ of \mathfrak{M}_0 for some other classes \mathcal{X} of topological spaces, in particular, for the class \mathfrak{L} of Lindelöf spaces.

PROBLEM 7.4. *Describe $\mathbf{C}_p^{\mathfrak{L};\mathfrak{I}}[\mathfrak{M}_0]$. Does it contain non-metrizable compacta?*

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