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ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE DELTOID CURVE

BY

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Abstract. We study a family of bivariate orthogonal polynomials associated to the deltoid curve. These polynomials arise when classifying bivariate diffusion operators that have discrete spectral decomposition given by orthogonal polynomials with respect to some compactly supported probability measure on the interior of the deltoid curve.

1. Introduction. Orthogonal polynomials in the interior of the deltoid curve, referred to below as the *deltoid domain* Ω, is one example of the eleven families of orthogonal polynomials on a compact domain in dimension 2 which are at the same time eigenvectors of an elliptic diffusion operators (see [2]). These families may be considered as the natural two-dimensional generalizations of the Jacobi polynomials. The deltoid case is one of the most intriguing, and has been put forward by Koornwinder [12, 13, 14] (see also [25, 16]).

The deltoid domain is a bounded set Ω in $\mathbb{R}^2$, the boundary of which is described by some degree 4 algebraic equation $P(X,Y) = 0$, so that $P(X,Y) > 0$ in Ω (see equation (3.1) below). On this domain one considers the family of probability measures $\mu_\alpha(dX,dY) = C_\alpha P^\alpha dXdY$, where C_α is a normalizing constant and $\alpha > -5/6$ is the basic condition to ensure the finiteness of the measure (see Proposition 4.5 below). For each such α there exists a family of polynomials which form an orthonormal basis for $\mathbb{L}^2(\mu_\alpha)$ and which are at the same time eigenvectors of a second order elliptic differential operator $\mathcal{L}^{(\alpha)}$ on Ω (which is a diffusion operator).

Bounded domains in $\mathbb{R}^2$ for which such structure exists are quite rare (there are eleven families up to affine transformations, see [2]) and it turns out that all the structure (operators, measures, orthogonal polynomials, etc.) may be entirely described from the polynomial $P(X,Y)$ which describes the boundary of the domain.

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In this deltoid case, the operator $\mathcal{L}^{(\alpha)}$ and the associated orthogonal polynomials have been studied by T. Koornwinder [12, 13], with a slight change of variables. In some respect, our work may be seen as an extension of those papers. In two special cases ($\alpha = -1/2$ and $\alpha = 1/2$), the operators have a natural geometric interpretation, respectively of the Laplace operator in $\mathbb{R}^2$ acting on functions which are invariant under the symmetries of a regular triangular lattice, and of the Casimir operator on SU(3), which we present in detail for the sake of completeness. They both relate to the root system A_2, which turns out to be deeply connected with this deltoid model; see [4, 5, 6, 15, 11, 22], and also [7, 8] for related topics on Tits triangle buildings of type $\tilde{A}_2$.

These special cases have been particularly investigated (see [3, 9], and [23, 24] for a spectral point of view). Those two cases are referred to below as the *geometric cases*. Their analysis may provide some insight into the general situation, which turns out to be more delicate.

In this paper, we concentrate on the eigenvectors of $\mathcal{L}^{(\alpha)}$, that is, the families of orthogonal polynomials. We derive a 3-term recurrence formula, quite unexpected in this context since such formulae are in general specific to dimension one and not to be expected in higher dimension. Those recurrence formulae take a particularly simple form in the two geometric cases. The geometric cases provide some simpler expressions for the polynomials, and allow specific representations of these families to be derived. In the particular case $\alpha = 1/2$, the use of Schur–Weyl duality leads to the construction of polynomials through the characters of the symmetric groups $\mathcal{S}_n$ and their associated Young diagrams. We then investigate generating functions, that is, explicit functions whose power series expansions provide the required sequence of polynomials. Generating functions are not easy to devise in general. We develop a technique related to diffusion operators to give some explicit form for them. We first provide in the general case a *partial* generating function (that is, a function which generates a subfamily of the polynomials). For a complete generating function, the two geometric cases ($\alpha = \pm 1/2$) are already known, and we propose a generic form for them in the case $\alpha = k + 1/2$, $k \in \mathbb{N}$. Our approach is based on the construction of harmonic functions in four variables, with respect to some specific elliptic operator related to the operator $\mathcal{L}^{(\alpha)}$. Unfortunately, this leads to a complicated form for which we do not have a simple general expression, and although we conjecture a general form of the generating function for those values of α, we have only been able to check it for the first values of k, with however a recursive method for a similar construction for any $k \in \mathbb{N}$.

The paper is organized as follows: Section 2 is a short presentation of the general setting of symmetric diffusion processes associated with orthogonal polynomials, mostly inspired from [1] and [2]. In Section 3, we give explicit formulae for the measure and the generator associated to the deltoid model, and introduce the complex variables in which the associated operator $\mathcal{L}^{(\alpha)}$ takes a much simpler form, leading to the explicit values for the eigenvalues of the operator. Section 4 is the presentation of the Euclidean case (that is, the case $\alpha = -1/2$), while Section 5 presents the SU(3) case, that is, $\alpha = 1/2$. Section 6 concentrates on recurrence formulae in the general case, and the technical proof of the main Lemma 6.5 is postponed to an appendix (Section 9). Section 7 provides the simpler representation of the eigenvectors in the geometric cases. In particular, we provide a representation formula in the SU(3) case which relies on a relation between the generator $\mathcal{L}^{(1/2)}$ and the generator of a standard Markov chain on the symmetric group, which is a special case of the Schur–Weyl duality formula (see (7.13)). This method is valid in the general case $\alpha \neq 1/2$, but takes an especially simple form for $\alpha = 1/2$, which allows for an explicit representation of the eigenvectors. Finally, in Section 8, we first provide a partial generating function in the general case, which relies on the study of the action of $\mathcal{L}^{(\alpha)}$ on the polynomial $P(T) = 1-3\bar{z}T+3zT^2-T^3$, where $z = X+iY$ is a complex generic variable in the deltoid domain. We then propose a complete generating function whenever $\alpha = k + 1/2$, $k \in \mathbb{N}$, with explicit forms for a few values of k.

2. Orthogonal polynomials and diffusion generators. Let Ω be an open bounded domain in $\mathbb{R}^d, d \geq 1$, with piecewise smooth boundary, and let μ a probability measure on $\bar{\Omega}$. Recall from [25, p. 32] that a family of polynomials $P_\tau : \mathbb{R}^d \to \mathbb{R}$ is *orthogonal* in $L^2(\Omega, \mu)$ if

$$\int P_\tau(x)P_{\tau'}(x)\,\mu(dx) = 0,$$

where $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{N}^d$ is a multi-index, whenever $|\tau| := \tau_1 + \cdots + \tau_d \neq \tau'_1 + \cdots + \tau'_d = |\tau'|$. In contrast to the real one variable setting, this family need not be unique in higher dimensions due to various orders one may choose when applying the Gram–Schmidt process to the canonical basis $(x_1^{\tau_1} \ldots x_d^{\tau_d})_{\tau\in\mathbb{N}^d}$ (see [25, bottom of p. 31]). However, in many situations, there are natural choices for this family of orthogonal polynomials. In particular, it may happen that they are also eigenvectors of some diffusion differential operator. This is the case for the classical families of orthogonal polynomials in dimension 1, Hermite, Laguerre and Jacobi polynomials (although only the last one corresponds to a bounded domain, see [20]).

On the other hand, when solving stochastic differential equations in probability theory, one is often led to consider second order differential operators on Ω which are symmetric in $\mathcal{L}^2(\mu)$, at least when one restricts attention to the set $\mathcal{C}_c^\infty(\Omega)$ of smooth functions compactly supported in Ω. When μ has a smooth positive density ρ on Ω, these operators may be represented as

$$\mathcal{L}f := \frac{1}{\rho}\sum_{k,j=1}^{d} \partial_k(g_{kj}\rho\partial_j f) = \sum_{k,j=1}^{d} g_{kj}\partial^2_{kj} f + \sum_{j=1}^{d} b_j\partial_j f \tag{2.1}$$

where $g = (g_{kj}(x))_{k,j=1}^d, x \in \Omega$, is a symmetric non-negative matrix depending smoothly on $x \in \Omega$ and

$$b_j = \frac{1}{\rho}\sum_{k=1}^{d} \partial_k(g_{kj}\rho), \quad j \in \{1, \dots, d\}.$$

The coefficients $b_j(x)$ are called the *drift terms* of the operator $\mathcal{L}$.

We call such operators *symmetric diffusion operators*. They are related to Markov diffusion processes (ξ_t) with values in Ω through the fact that for any smooth function f, the process $f(\xi_t) - \int_0^t \mathcal{L}f(\xi_s)\,ds$ is a (local) martingale. When the operator $\mathcal{L}$ is essentially self-adjoint, this entirely characterizes the law of the process (ξ_t) (at least as long as we only consider finite-dimensional marginals). The operator $\mathcal{L}$ is called the *infinitesimal generator* of the process (ξ_t).

Working with such diffusion operators, it is often convenient to introduce the so-called *carré du champ operator*

$$\Gamma(f, g) = \tfrac{1}{2}\big(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f\big),$$

and observe that $\mathcal{L}$ is entirely determined by Γ and μ through the integration by parts formula

$$\int_\Omega f\mathcal{L}g\,d\mu = \int_\Omega g\mathcal{L}f\,d\mu = -\int_\Omega \Gamma(f, g)\,d\mu,$$

valid at least when f and g are smooth and compactly supported in Ω. Moreover, from the representation (2.1), it is immediate that $b_i(x) = \mathcal{L}(x_i)$ and $g_{ij} = \Gamma(x_i, x_j)$.

More generally, the change of variable formula, valid for any smooth $\Phi : \mathbb{R}^k \to \mathbb{R}$, and any k-uple $f = (f_1, \dots, f_k)$ of smooth functions, reads

$$\mathcal{L}(\Phi(f)) = \sum_i \partial_i\Phi(f)\mathcal{L}f_i + \sum_{i,j} \partial^2_{ij}\Phi(f)\Gamma(f_i, f_j). \tag{2.2}$$

In particular, whenever for $i, j = 1, \dots, k$, there exist functions B_i and G_{ij} such that $\mathcal{L}f_i = B_i(f)$ and $\Gamma(f_i, f_j) = G_{ij}(f)$, one has

$$\mathcal{L}(\Phi(f)) = (\mathcal{L}_1\Phi)(f) \tag{2.3}$$

where $\mathcal{L}_1$ is the new diffusion operator acting on the image of Ω under the function f (which is not necessarily a local diffeomorphism), as

$$\mathcal{L}_1(\Phi) = \sum_{i,j} G_{ij}(x)\partial^2_{ij}\Phi + \sum_i B_i(x)\partial_i\Phi,$$

which is called the image of $\mathcal{L}$ under the function f.

In the probabilistic interpretation, if (ξ_t) is the stochastic process with generator $\mathcal{L}$, then $f(\xi_t)$ is again a diffusion Markov process with generator $\mathcal{L}_1$. In particular, the operator $\mathcal{L}_1$ is symmetric with respect to the image measure of ρ under the map f, which, following (2.1), may often be an efficient way to determine the image measure. When such a situation occurs, we shall say that $\mathcal{L}$ *projects onto* $\mathcal{L}_1$.

In what follows, we restrict for simplicity to the case where the matrix $g(x)$ is positive definite on Ω. It is then natural to raise the question of determining when such an $\mathcal{L}$ may be extended as a self-adjoint operator (see [26]) with spectral decomposition given by a family of orthogonal polynomials with respect to μ. In other words, one wants to determine for which choice of ρ and g there is a complete family of μ-orthogonal polynomials which are at the same time eigenvectors for $\mathcal{L}$. This will produce a natural choice for a basis of orthogonal polynomials.

It turns out that the general answer to this question is the following.

The functions $g_{ij}(x)$ are polynomials with degree at most two, and the boundary $\partial\Omega$ is included in the algebraic set $\{\det(g) = 0\}$. More precisely, if $\{P(x) = 0\}$ denotes the irreducible equation of the boundary $\partial\Omega$, then there exists a family of degree 1 polynomials $L_i(x)$ such that for any i, we have the algebraic equation

$$\sum_j g_{ij}\partial_j P = L_i P. \tag{2.4}$$

Moreover, the sets of admissible density measures ρ are entirely described by the algebraic structure of the boundary. In particular, when the determinant $\det(g)$ is irreducible, the only admissible density measures ρ are $C(\lambda)\det(g)^\lambda$ for any real λ such that $\det(g)^\lambda$ is $\mathcal{L}^1(\Omega, dx)$ (see [2]). Once the boundary $\partial\Omega$ is given through its irreducible equation, the coefficients $g_{ij}(x)$ are entirely determined from (2.4). It turns out that they are in general unique up to some scaling factor.

3. The deltoid model. In dimension 2, up to affine transformations, there are only eleven bounded sets Ω on which there exists a symmetric diffusion operator for which the associated eigenvectors are orthogonal polynomials with respect to a reversible measure (see [2]). One of the most intriguing ones is the interior of the deltoid curve, which is a degree 4 algebraic curve with equation

$$P(x) = (x_1^2 + x_2^2)^2 + 18(x_1^2 + x_2^2) - 8x_1^3 + 24x_1x_2^2 - 27 = 0. \tag{3.1}$$

The curve $P = 0$ also has the following parametric representation:

$$x_1(\theta) = 2\cos\theta + \cos 2\theta, \quad x_2(\theta) = 2\sin\theta - \sin 2\theta.$$

In complex notation, $z(\theta) = x_1(\theta) + ix_2(\theta) = 2e^{i\theta} + e^{-2i\theta}$.

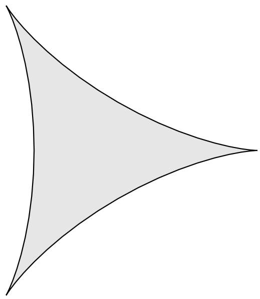


Fig. 1. The deltoid domain Ω

We shall denote by Ω the interior of the deltoid curve, by $\bar{\Omega}$ its closure, and by $\partial\Omega$ its boundary.

For this particular choice, the matrix $g_{ij}(x)$ is uniquely determined by equation (2.4) up to some scaling factor, and is given by

$$\begin{cases} g_{11}(x_1,x_2) = -(3x_1^2 - x_2^2 - 6x_1 - 9), \\ g_{12}(x_1,x_2) = -2x_2(2x_1+3), \\ g_{22}(x_1,x_2) = -(3x_2^2 - x_1^2 + 6x_1 - 9). \end{cases} \tag{3.2}$$

Hence we deduce that $\det(g) = -3P(x)$. Moreover, in this representation, for the measure $\mu(dx) = c(\alpha)|P(x)|^\alpha dx$, the drift terms in the equation read

$$b_1(x_1,x_2) = -2(6\alpha+5)x_1, \quad b_2(x_1,x_2) = -2(6\alpha+5)x_2. \tag{3.3}$$

The general operator $\mathcal{L}^{(\alpha)}$ on the interior of the deltoid curve for which a family of orthogonal polynomial is formed of eigenvectors of $\mathcal{L}^{(\alpha)}$ is therefore given by

$$\begin{aligned} \mathcal{L}^{(\alpha)} = {} & g_{11}(x_1,x_2)\partial_1^2 + g_{22}(x_1,x_2)\partial_2^2 + 2g_{12}(x_1,x_2)\partial_{1,2}^2 \\ & - 2(6\alpha+5)x_1\partial_1 - 2(6\alpha+5)x_2\partial_2 \end{aligned}$$

with associated measure $c(\alpha)\rho^\alpha dx$, with

$$\rho(x) = \tfrac{1}{3}\det(g) = P(x),$$

where P is given in (3.1).

As long as we only deal with polynomials, it turns out that it is easier to use complex variables. Indeed, let $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. Then the generator is entirely characterized by

$$\begin{cases} \Gamma(z,z) = -4z^2 + 12\bar{z}, \\ \Gamma(\bar{z},z) = -2z\bar{z} + 18, \\ \Gamma(\bar{z},\bar{z}) = -4\bar{z}^2 + 12z, \\ \mathcal{L}^{(\alpha)}z = -2(6\alpha+5)z, \quad \mathcal{L}^{(\alpha)}\bar{z} = -2(6\alpha+5)\bar{z}. \end{cases} \tag{3.4}$$

We can simplify the operator by setting $z = 3z_1, \bar z = 3\bar z_1$ and multiplying $\mathcal{L}^{(\alpha)}$ by $1/4$, which does not change the eigenvectors and multiplies the eigenvalues by $1/4$. This gives (renaming z_1 and $\bar z_1$ as z and $\bar z$)

$$
(3.5)\qquad \begin{cases} \Gamma(z,z) = \bar z - z^2, \\ \Gamma(\bar z, z) = \frac{1}{2}(1 - z\bar z), \\ \Gamma(\bar z,\bar z) = z - \bar z^2, \\ \mathcal{L}^{(\alpha)} z = -\frac{1}{2}(6\alpha+5)z, \quad \mathcal{L}^{(\alpha)}\bar z = -\frac{1}{2}(6\alpha+5)\bar z. \end{cases}
$$

In these coordinates, we may choose P to be

$$
(3.6)\quad P(z,\bar z) = 4\big((\Gamma(z,\bar z)^2 - \Gamma(z,z)\Gamma(\bar z,\bar z)\big) = 1 - 6z\bar z - 3z^2\bar z^2 + 4(z^3+\bar z^3),
$$

which is positive in Ω.

REMARK 3.1. It is worth observing that if for complex numbers z_1, z_2, z_3 such that $z_1z_2z_3 = 1$ and $|z_i| = 1$, we set $z = (z_1+z_2+z_3)/3$, then

$$
P(z,\bar z) = -\tfrac{1}{27}(z_1 - z_2)^2(z_2-z_3)^2(z_3-z_1)^2,
$$

which is a non-negative real valued function.

In particular, giving a particular role to the case $\alpha = -1/2$, one has

$$
(3.7)\qquad \mathcal{L}^{(\alpha)} = \mathcal{L}^{(-1/2)} - \tfrac{3}{2}(2\alpha+1)(z\partial_z + \bar z\partial_{\bar z}).
$$

This model has been studied in [12, 13], where the relationship with homogeneous spaces of rank 2 and the root system A_2 has been put forward. Observe that the case $\alpha = -1/2$ corresponds to the Laplace–Beltrami operator associated with the Riemannian metric g^{-1} associated with the inverse matrix of g.

Our aim here is to study the associated orthogonal polynomials together with the associated eigenvalues, and various representations for it. Indeed, this family belongs to the larger class of Hall polynomials associated with root systems (here the root system A_2) (see [19, 18]), and our aim here is to present some properties of these polynomials specific to this model.

4. $\mathcal{L}^{(-1/2)}$ as a projection of the Euclidean Laplacian. As already mentioned, the case $\alpha = -1/2$ corresponds to the Laplace–Beltrami operator associated to the inverse matrix g^{-1}. If one computes the associated curvature (here, in dimension 2, the scalar curvature is sufficient to characterize the metric), we may observe that it vanishes, and therefore it is not much surprising that the operator is the image, in the sense described in Section 2, of the ordinary Laplace operator in $\mathbb{R}^2$.

To describe this, consider the three third roots of unity in the complex plane, $(\mathbf{1}, \mathbf{j}, \bar{\mathbf{j}})$, that is, the points in $\mathbb{R}^2$ with coordinates $(1,0)$, $(-1/2, \sqrt{3}/2)$, $(-1/2, -\sqrt{3}/2)$, and, for two points $\mathbf{x}_i = (x_i, y_i) \in \mathbb{R}^2$, denote by $\mathbf{x}_1 \cdot \mathbf{x}_2$

$= x_1x_2 + y_1y_2$ their scalar product in $\mathbb{R}^2$. Then, consider the function $z : \mathbb{R}^2 \to \mathbb{C}$ given by

$$z(\mathbf{x}) = \frac{1}{3}(e^{i\mathbf{1}\cdot\mathbf{x}} + e^{i\mathbf{j}\cdot\mathbf{x}} + e^{i\bar{\mathbf{j}}\cdot\mathbf{x}}). \tag{4.1}$$

Our aim is to identify the above operator $\mathcal{L}^{(-1/2)}$ as the image of the Laplace operator Δ on $\mathbb{R}^2$ through the action of $z : \mathbb{R}^2 \to \bar{\Omega}$.

For this, denote by Δ the two-dimensional Laplace operator and by Γ_Δ its associated square field operator, that is, $\Gamma_\Delta(f, g) = \partial_x f \partial_x g + \partial_y f \partial_y g$.

Lemma 4.1.

$$\begin{cases} \Gamma_\Delta(z, z) = -z^2 + \bar{z}, \\ \Gamma_\Delta(\bar{z}, z) = \frac{1}{2}(1 - z\bar{z}), \\ \Gamma_\Delta(\bar{z}, \bar{z}) = -\bar{z}^2 + z, \\ \Delta z = -z, \quad \Delta\bar{z} = -\bar{z}. \end{cases}$$

Proof. For $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, let $e_{\mathbf{a}} : \mathbb{R}^2 \to \mathbb{C}$ be the function $e_{\mathbf{a}}(\mathbf{x}) = \mathbf{e}^{\mathbf{i}\mathbf{a}\cdot\mathbf{x}}$. Then

$$\Delta(e_{\mathbf{a}}) = -\|\mathbf{a}\|^2 e_{\mathbf{a}}, \qquad \Gamma(e_{\mathbf{a}}, e_{\mathbf{b}}) = -(\mathbf{a}\cdot\mathbf{b})e_{\mathbf{a}+\mathbf{b}}.$$

Hence, the stated formulae follow immediately. ∎

Remark 4.2. In connection with Remark 3.1, observe that, with the above notation, $z = (e_{\mathbf{1}} + e_{\mathbf{j}} + e_{\bar{\mathbf{j}}})/3 = (z_1 + z_2 + z_3)/3$, with $|z_i| = 1$ and $z_1z_2z_3 = 1$, the last formula being a consequence of $\mathbf{1} + \mathbf{j} + \bar{\mathbf{j}} = 0$.

We shall see (Proposition 4.3) that $z(\mathbb{R}^2) = \bar{\Omega}$. Then an immediate comparison with (3.5) shows that $\mathcal{L}^{(-1/2)}$ is the image of Δ under the function $z : \mathbb{R}^2 \to \bar{\Omega}$ (Proposition 4.4).

The function z is invariant under the symmetries with respect to the lines of the regular triangular lattice $\mathbb{L}_1$ whose fundamental domain is the regular triangle $\mathcal{A}$ with vertices $(0, 0)$, $(4\pi/3, 0)$, $(4\pi/3)e^{i\pi/3}$ (see Fig. 2). We shall say that a function having those invariances *has the symmetries of the lattice* $\mathbb{L}_1$. To see this, it is enough to observe that z is invariant by rotation through $2\pi/3$ and by symmetry with respect to the horizontal lines $\{y = 0\}$ and the line $\{y = 2\pi/\sqrt{3}\}$. As a consequence, z is uniquely determined by its restriction to $\mathcal{A}$.

Proposition 4.3. *The function z is a one-to-one map from $\bar{\mathcal{A}}$ onto $\bar{\Omega}$ and from $\partial\mathcal{A}$ onto $\partial\Omega$. In particular, it maps the whole plane onto $\bar{\Omega}$.*

Proof. Recall that $\bar{\Omega}$ is the set of points in $\mathbb{R}^2$ such that $P(z, \bar{z}) \geq 0$, where P is defined in (3.6). Now, for any smooth function $f : \mathbb{R}^2 \to \mathbb{C}$, one has $\Gamma_\Delta(f, \bar{f})^2 \geq \Gamma_\Delta(f, f)\Gamma_\Delta(\bar{f}, \bar{f})$. Applied to $f = z$, this shows that z maps $\mathbb{R}^2$ into $\bar{\Omega}$.

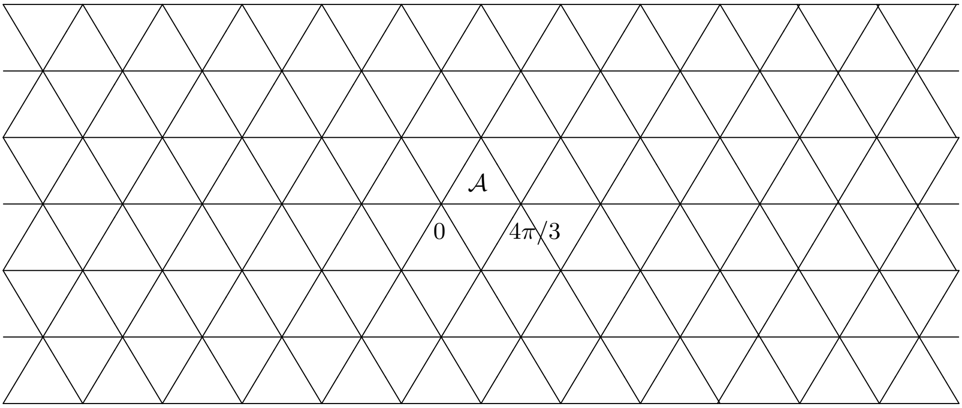


Fig. 2. The lattice $\mathbb{L}_1$ of regular triangles

Recall also the parametric equation of $\partial\Omega$ in complex notation,

$$z_1(\theta) = 2e^{i\theta} + e^{-2i\theta},$$

so that $z_1(\theta) = z(-2\theta, 0)$, where θ runs over any interval of length 2π. Then the invariance of z under rotations of angles $\pm 2\pi/3$ shows that the images of the intervals

$$[-4\pi/3, 0], \quad [4\pi/3, 8\pi/3]$$

coincide with the images of the oblique edges of $\mathcal{A}$ (the cusps of $\partial\Omega$ are the images of $\{\theta = 0, 4\pi/3, 8\pi/3\}$). Thus z maps ∂A onto $\partial\Omega$ and it is easy to check from the complex parametrization of $\partial\Omega$ that z is one-to-one there. But then $z(\mathcal{A}) = \bar{\Omega}$, since otherwise $\bar{\Omega}$ would not be simply connected, which leads to a contradiction.

Now, we shall use the following parametrization of $\bar{\Omega}$:

$$z(x_1, x_2) = e^{ix_1} + 2e^{-ix_1/2}\cos\left(\frac{\sqrt{3}}{2}x_2\right), \qquad x = (x_1, x_2) \in \mathcal{A}.$$

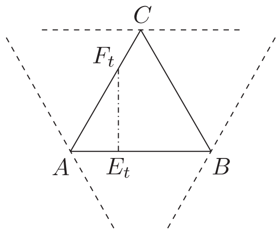
For fixed $x_1 \in [0, 2\pi/3]$, the image by z of the vertical segments

$$[(x_1, 0), (x_1, \sqrt{3}x_1)] \in \mathcal{A}$$

is the line segment $I(x_1) = [A(x_1), B(x_1)]$ where

$$A(x_1) = (\cos(x_1) + 2\cos(x_1/2), \sin(x_1) - 2\sin(x_1/2)),$$
$$B(x_1) = (2\cos(x_1) + \cos(2x_1), 2\sin(x_1) - \sin(2x_1)).$$

Thus, the coordinates of $A(x_1)$ are decreasing as functions of x_1, while those of $B(x_1)$ are decreasing and increasing respectively. Equivalently, $A(x_1)$ runs over the half of the lowest branch of $\partial\Omega$ starting from $(3, 0)$, while $B(x_1)$ runs over the whole highest one since clearly $B(x_1) = A(-2x_1)$. Indeed, two line segments $I(x_1), I(x_1'), 0 \le x_1 \ne x_1' \le 2\pi/3$, never intersect. A similar reasoning applies when $x_1 \in [2\pi/3, 4\pi/3]$ and the segment $[(x_1, -\sqrt{3}\,x_1 + 4\pi/\sqrt{3}), A(x_1)]$ runs over the remaining half of the lowest branch while $B(x_1)$ runs over the whole third one. As a matter of fact, z is one-to-one from $\bar{\mathcal{A}}$ onto $\bar{\Omega}$. Finally, $\bar{\mathcal{A}}$ is a fundamental domain for the action of the affine group $\mathcal{D}_3$ on $\mathbb{R}^2$ so that every $x \in \mathbb{R}^2$ is conjugate to a unique element of $\mathcal{A}$. The proposition is proved. ∎



We can now identify the operator $\mathcal{L}^{(-1/2)}$ as an image of the 2-dimensional Laplace operator acting on functions which are invariant under the symmetries in the lines of the triangular lattice.

Proposition 4.4. *A measurable function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ has the symmetries of the lattice $\mathbb{L}_1$ if and only if it can be written as $f = g(z)$, where $g : \Omega \to \mathbb{R}$ is a measurable function.*

Moreover, when $f \in \mathcal{C}^2$, then we may choose $g \in \mathcal{C}^2$, in which case

$$\Delta(g(z)) = \mathcal{L}^{(-1/2)}(g)(z). \tag{4.2}$$

In other words, $\mathcal{L}^{(-1/2)}$ is nothing else than the 2-dimensional Laplace operator acting on functions having the symmetries of $\mathbb{L}_1$.

Proof. If we denote by z^{-1} the inverse map $\Omega \to A$ of the restriction of z to A, then we just set $g = f \circ z^{-1}$. ∎

Using Remark 3.1 and the above diffeomorphism between the deltoid and the triangle, one obtains

Proposition 4.5. *The function P^α on the deltoid is integrable with respect to the Lebesgue measure if and only if $\alpha > -5/6$.*

Throughout, we use $\lambda = \frac{1}{2}(6\alpha + 5)$.

Proof. By the change of variables formula we have

$$\int_\Omega P^\alpha dx_1 dx_2 = \int_{\mathcal{A}} P^{\alpha+1/2}\, dx_1\, dx_2$$
$$= C \int_{\mathcal{A}} |(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^{2\alpha+1}\, dx_1\, dx_2$$

where $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}, z_3 = e^{i\theta_3}$ and $\theta_1 = x_1$, $\theta_2 = m - x_1/2 + \sqrt{3}\,x_2/2$, $\theta_3 = -(\theta_1 + \theta_2) = -x_1/2 - \sqrt{3}\,x_2/2$.

A rapid inspection of the integrability of this function on the triangle shows that, near the boundary and outside the corners of the triangle, the local integrability condition is $\alpha > -1$, while at the corner of the triangles, there is a more restrictive condition. Indeed, for the integrability of the measure near $(0,0)$, we need $(z_1-z_2)(z_2-z_3)(z_3-z_1)) \simeq -i\frac{3}{4}\sqrt{3}\,x_2(x_2^2-3x_1^2)$, and if we set $x_2=\sqrt{3}\,tx_1$, $t\in[0,1]$, we have $P^{\alpha+1/2} \simeq \big(-\frac{27}{4}t^2x_1^6(1-t^2)^2\big)^{\alpha+1/2}$, which is integrable for the measure $tdtdx_1$ if and only if $\alpha > -5/6$. ∎

5. $\mathcal{L}^{(1/2)}$ as a projection of the Casimir operator on SU(3)**.** Let $\mathcal{G}$ be a compact semisimple Lie linear group with Lie algebra $\mathcal{L}$, seen as the tangent space at Id for G, with Lie bracket $[A,B]$ (see [10]). On $\mathcal{L}$, the Killing form is a scalar product defined by $\langle A,B\rangle = -\operatorname{trace}(AB)$. On the other hand, to any $A\in\mathcal{L}$ is associated a vector field X_A on $\mathcal{G}$ defined as $X_A(f)(g) = \partial_t|_{t=0}f(ge^{tA})$. Given an orthonormal basis $(A_1,\dots,A_d)$ in $\mathcal{L}$ with respect to the Killing form, the *Casimir operator* is defined as $\Delta_G = \sum_i X_{A_i}^2$. It is also the Laplace–Beltrami operator on $\mathcal{G}$ when $\mathcal{G}$ inherits the Riemannian structure from the Killing form in $\mathcal{L}$. $\Delta_{\mathcal{G}}$ is a second order differential operator in the sense that it satisfies the change of variable formula (2.2). We shall denote by $\Gamma_{\mathcal{G}}$ the corresponding carré du champ operator.

The Casimir operator commutes with the Lie group action. More precisely, if, for $g\in\mathcal{G}$ and for any function $f:\mathcal{G}\to\mathbb{R}$, one defines the right action $R_g(f)(k)=f(kg)$, then $\Delta_{\mathcal{G}}R_g = R_g\Delta_{\mathcal{G}}$, and the same holds true for the left action $L_g(f)(k) = f(gk)$.

In order to entirely determine the action of $\Delta_{\mathcal{G}}$ on functions of $\mathcal{G}$, it is enough to compute $\Delta_{\mathcal{G}}(f_i)$ and $\Gamma_{\mathcal{G}}(f_i,f_j)$ for a set of functions which generates all functions on $\mathcal{G}$ (say as σ-algebras). Once again, it could be helpful to consider complex valued functions. On SU(n), if one represents g as a matrix (z_{ij}) with complex entries, we shall consider the coordinates $g\mapsto z_{ij}$ and $g\mapsto \bar z_{ij}$ as generating functions.

When performing the above computations in SU(n), one ends up with the following.

Proposition 5.1. *The action of the Casimir operator of* SU(n) *on the entries* (z_{ij}) *of the matrix* $g\in\mathrm{SU}(n)$ *is given by*

$$\begin{cases} \Delta_{\mathrm{SU}(n)}(z_{kl}) = -2\frac{(n-1)(n+1)}{n}z_{kl},\\ \Delta_{\mathrm{SU}(n)}(\bar z_{kl}) = -2\frac{(n-1)(n+1)}{n}\bar z_{kl},\\ \Gamma_{\mathrm{SU}(n)}(z_{ij},z_{kl}) = -2z_{il}z_{kj} + \frac{2}{n}z_{ij}z_{kl},\\ \Gamma_{\mathrm{SU}(n)}(\bar z_{ij},\bar z_{kl}) = -2\bar z_{il}\bar z_{kj} + \frac{2}{n}\bar z_{ij}\bar z_{kl},\\ \Gamma_{\mathrm{SU}(n)}(z_{ij},\bar z_{kl}) = 2\big(\delta_{ik}\delta_{jl} - \frac{1}{n}z_{ij}\bar z_{kl}\big). \end{cases}$$

We shall not give the proof of these formulae, which is straightforward from the definition, although a bit tedious.

Since we shall use this later on, we shall compute the action of $\Delta_{\mathrm{SU}(n)}$ on special functions, namely the traces of the powers. For any $p \in \mathbb{Z}$, consider the functions $\mathrm{SU}(n) \to \mathbb{C}$, $T_p(g) = \mathrm{trace}(g^p)$. For $p \geq 1$, one has

$$T_p(g) = \sum_{i_1,\dots,i_p=1}^{n} z_{i_1i_2} z_{i_2i_3} \cdots z_{i_pi_1}; \tag{5.1}$$

the same formula holds for $p \leq -1$ upon replacing z_{ij} by $\bar{z}_{ij}$ (and of course $T_{-p} = \overline{T}_p$).

Corollary 5.2. *The action of the Casimir operator on the functions T_p, $p \geq 1$, is given by*

$$\Delta_{\mathrm{SU}(n)} T_p = -p\bigg(2\Big(\frac{n^2-p}{n}\Big)T_p + \sum_{i=1}^{p-1} T_i T_{p-i}\bigg), \tag{5.2}$$

with the conjugate formula for $p \leq -1$, while, for any $p, q \in \mathbb{Z}$,

$$\Gamma_{\mathrm{SU}(n)}(T_p, T_q) = 2|pq|\bigg(\frac{T_pT_q}{n} - T_{p+q}\bigg). \tag{5.3}$$

Proof. From the change of variable formula, one has, for any m-uple of functions $(f_1, \dots, f_m)$ and any diffusion generator $\mathcal{L}$,

$$\mathcal{L}(f_1 \cdots f_m) = f_1 \cdots f_m \bigg(\sum_{i=1}^{m} \frac{\mathcal{L}f_i}{f_i} + \sum_{i,j=1}^{m} \frac{\Gamma(f_i, f_j)}{f_i f_j} - \sum_{i=1}^{m} \frac{\Gamma(f_i, f_i)}{f_i^2}\bigg), \tag{5.4}$$

and, for any m-uple $(f_1, \dots, f_m)$ and any k-uple $(g_1, \dots, g_k)$,

$$\Gamma(f_1 \cdots f_m, g_1 \cdots g_k) = f_1 \cdots f_m g_1 \cdots g_k \bigg(\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{\Gamma(f_i, g_j)}{f_i g_j}\bigg).$$

It remains to apply these formulae to the explicit expression (5.1) of T_p. ∎

In particular, if we set $Z = T_1$, $\bar{Z} = T_{-1}$, then

$$\Delta_{\mathrm{SU}(n)} Z = -2\frac{n^2-1}{n} Z, \qquad \Delta_{\mathrm{SU}(n)} \bar{Z} = -2\frac{n^2-1}{n} \bar{Z}, \tag{5.5}$$

and

$$\Gamma_{\mathrm{SU}(n)}(Z, Z) = 2(Z^2/n - T_2), \qquad \Gamma_{\mathrm{SU}(n)}(\bar{Z}, \bar{Z}) = 2(\bar{Z}^2/n - \overline{T}_2), \tag{5.6}$$

$$\Gamma_{\mathrm{SU}(n)}(Z, \bar{Z})) = 2(3 - Z\bar{Z}/n). \tag{5.7}$$

Now, consider more precisely the case $n = 3$. For any matrix in $\mathrm{SU}(3)$, if (μ_1, μ_2, μ_3) denote its eigenvalues, then $T_p = \mu_1^p + \mu_2^p + \mu_3^p$. The μ_i are complex numbers with $|\mu_i| = 1$ and $\mu_1\mu_2\mu_3 = 1$. With $Z = \mu_1 + \mu_2 + \mu_3$, they are a solution of the equation

$$X^3 - ZX^2 + \bar{Z}X - 1 = 0,$$

and multiplying this by X^p and summing over the three values μ_1, μ_2, μ_3, one gets, for any $p \in \mathbb{Z}$,

$$T_{p+3} - ZT_{p+2} + \bar{Z}T_{p+1} - T_p = 0, \tag{5.8}$$

which for $p = -1$ gives $T_2 = Z^2 - 2\bar{Z}$, and similarly $\overline{T}_2 = \bar{Z}^2 - 2Z$. Inserting these values in (5.5) and (5.6) leads to

$$\Delta_{\mathrm{SU}(3)} Z = -\tfrac{16}{3} Z, \qquad \Delta_{\mathrm{SU}(3)}(\bar{Z}) = -\tfrac{16}{3}\bar{Z}$$

and

$$\Gamma_{\mathrm{SU}(3)}(Z, Z) = \tfrac{4}{3}(3\bar{Z} - Z^2), \qquad \Gamma_{\mathrm{SU}(3)}(\bar{Z}, \bar{Z}) = \tfrac{4}{3}(3Z - \bar{Z}^2),$$
$$\Gamma_{\mathrm{SU}(3)}(Z, \bar{Z}) = \tfrac{2}{3}(Z\bar{Z} - 9).$$

Setting $z = Z/3$, on observes that $\frac{3}{4}\Delta_{\mathrm{SU}(3)}$ acting on functions of $(z, \bar{z})$ is nothing other than $\mathcal{L}^{(1/2)}$. Observe also that the functions on SU(3) which depend only on $(z, \bar{z})$ are exactly those which depend only on the spectrum of the matrix $g \in \mathrm{SU}(3)$, that is, the functions which are invariant under $g \mapsto h^{-1}gh$ for any $h \in \mathrm{SU}(3)$. Indeed, as long as polynomials are concerned, those functions are exactly functions depending only on the traces T_p, $p \in \mathbb{Z}$, and formula (5.8) shows that these functions are again polynomials in the variables $(z, \bar{z})$.

To summarize, we have

PROPOSITION 5.3. *The operator* $\frac{4}{3}\mathcal{L}^{(1/2)}$ *is the image of the Casimir operator on* SU(3) *acting on spectral functions, through the map* $\mathrm{SU}(3) \to \mathbb{C}$ *defined by* $z(g) = \frac{1}{3}\,\mathrm{trace}(g)$.

6. Eigenvalues and eigenvectors. We proceed now to the determination of the eigenvalues of $\mathcal{L}^{(\alpha)}$, and give a recurrence formula for the corresponding eigenvectors. To simplify the notation, we shall set $\lambda = \frac{1}{2}(6\alpha + 5)$, so that $\lambda > 0$ will appear in each case as the minimal nonzero eigenvalue of $-\mathcal{L}^{(\alpha)}$.

In dimension 1, it is well known (and easy to check) that for any probability measure μ for which the polynomials are dense in $\mathbb{L}^2(\mu)$, the unique (up to sign) associated sequence of orthogonal polynomials satisfies a 3-term recurrence formula (see [21]), usually written in the form

$$xP_n = a_n P_{n+1} + b_n P_n + a_{n-1} P_{n-1}.$$

This is not the case in dimension 2, since one gets in general a recurrence formula involving at each step n an increasing number of terms. Indeed, if, for each degree n, one denotes by $\mathcal{P}_n$ the space of polynomials of total degree at most n and by $\mathcal{H}_n$ the space of polynomials in $\mathcal{P}_n$ orthogonal to $\mathcal{P}_{n-1}$, then for any $P \in \mathcal{H}_n$,

$$x_1 P = Q_{n+1} + Q_n + Q_{n-1}, \qquad x_2 P = R_{n+1} + R_n + R_{n-1},$$

where Q_i and R_i belong to $\mathcal{H}_i$. But the space $\mathcal{H}_i$ has dimension $i+1$, and in general one should not expect any simple recurrence formula.

However, looking more precisely at the form of the operators $\mathcal{L}^{(\alpha)}$ in the variables $(z, \bar{z})$, one should expect for the sequence of eigenvectors of $\mathcal{L}^{(\alpha)}$ a 6-term recurrence formula. It comes as a surprise that indeed one is able to get a 3-term recurrence formula, as in dimension 1.

We start by investigating the eigenvalues. Recall first that we are looking for polynomials $P^{(\alpha)}_{p,q}$ such that

$$\mathcal{L}^{(\alpha)}(P^{(\alpha)}_{p,q}) = -\lambda_{p,q} P^{(\alpha)}_{p,q}$$

where $(p,q) \in \mathbb{N}^2$ is a bi-index whose weight $p+q$ is the degree of $P^{(\alpha)}_{p,q}$.

Proposition 6.1. *The eigenvalues of* $\mathcal{L}^{(\alpha)}$ *are*

$$\lambda_{p,q} = (\lambda - 1)(p+q) + p^2 + q^2 + pq,$$

where $\lambda = \frac{1}{2}(6\alpha + 5)$.

Proof. The complex representation easily leads to the eigenvalues. Indeed, if $\mathcal{P}_n$ denotes the space of polynomials (now in the variables $(z, \bar{z})$) with total degree at most n, one may write any $P \in \mathcal{P}_n$ as

$$P = \sum_{p=0}^{n} a_{p,q} z^p \bar{z}^{n-p} + Q = P_n + Q,$$

where $Q \in \mathcal{P}_{n-1}$.

Now, looking at the action of $\mathcal{L}^{(\alpha)}$ on the highest degree term P_n of P, one sees that if $\mathcal{L}^{(\alpha)} P = -\mu P$, then the highest degree term $\hat{P}_n$ of $\mathcal{L}^{(\alpha)} P_n$ is equal to $-\mu \hat{P}_n$. It remains to find the action of $\mathcal{L}^{(\alpha)}$ on those highest terms. Fortunately, in coordinates $(z, \bar{z})$, this action is diagonal (which is not the case in coordinates (x_1, x_2)).

Indeed, the change of variable formula (2.2) gives

$$\begin{aligned}\mathcal{L}^{(\alpha)}(z^p \bar{z}^q) = {} & p z^{p-1} \bar{z}^q \mathcal{L}^{(\alpha)} z + q z^{q-1} z^p \mathcal{L}^{(\alpha)} \bar{z} \\ & + p(p-1) z^{p-2} z^q \Gamma(z,z) + q(q-1) \bar{z}^{q-2} z^p \Gamma(\bar{z}, \bar{z}) \\ & + 2pq z^{p-1} \bar{z}^{q-1} \Gamma(z, \bar{z}),\end{aligned}$$

whose highest term is $-\lambda_{p,q} z^p \bar{z}^q$ with $\lambda_{p,q} = (\lambda - 1)(p+q) + p^2 + q^2 + pq$, where $\lambda = \frac{1}{2}(6\alpha + 5)$. ■

Remark 6.2. When $\alpha \notin \mathbb{Q}$, the eigenspaces associated to the eigenvalues $\lambda_{p,q}$ are at most two-dimensional (and exactly two-dimensional when $p \neq q$). Indeed, writing $\sigma = p+q$ and $\pi = pq$, with similar notation σ', π' for (p', q'), we have $\lambda_{p,q} = (\lambda - 1)\sigma + \sigma^2 - \pi$, and therefore if $\lambda_{p,q} = \lambda_{p',q'}$, then either $\sigma = \sigma'$ and then $\pi = \pi'$, or $\lambda = 1 - \sigma - \sigma' + (\pi - \pi')/(\sigma - \sigma')$, whence $\lambda \in \mathbb{Q}$.

We shall see moreover that for any (p, q) there exists exactly one polynomial $P^{(\alpha)}_{p,q}(z, \bar{z}) = z^p \bar{z}^q +$ lower degree terms, which is an eigenvector of $\mathcal{L}^{(\alpha)}$. Indeed, the following result shows that, for any p, q, there exists at least one polynomial $P^{(\alpha)}_{p,q}(z^p \bar{z}^q)$ with unique highest degree term $z^p \bar{z}^q$ which is an eigenvector for $\mathcal{L}^{(\alpha)}$.

THEOREM 6.3. *Recall that* $\lambda = \frac{1}{2}(6\alpha + 5) > 0$. *Let* $P^{(\alpha)}_{p,q}(z, \bar{z})$ *be the family of polynomials with unique highest term* $z^p \bar{z}^q$ *defined by induction from*

$$P^{(\alpha)}_{0,0} = 1, \qquad P^{(\alpha)}_{0,1} = \bar{z}, \qquad P^{(\alpha)}_{1,0} = z,$$

$$\begin{cases} P^{(\alpha)}_{p+1,q} = zP^{(\alpha)}_{p,q} + a_1(\lambda, p)P^{(\alpha)}_{p-1,q+1} + a_2(\lambda, p, q)P^{(\alpha)}_{p,q-1}, \\ P^{(\alpha)}_{p,q+1} = \bar{z}P^{(\alpha)}_{p,q} + a_1(\lambda, q)P^{(\alpha)}_{p+1,q-1} + a_2(\lambda, q, p)P^{(\alpha)}_{p-1,q}, \end{cases} \tag{6.1}$$

with

$$a_1(\lambda, p) = -\frac{p(3p + 2\lambda - 5)}{(\lambda + 3p - 1)(\lambda + 3p - 4)},$$
$$a_2(\lambda, p, q) = -\frac{N_{p,q}}{D_{p,q}},$$

where

$$N_{p,q} = q(3q + 2\lambda - 5)(\lambda + 3(p + q) - 1)(\lambda + p + q - 2),$$
$$D_{p,q} = (\lambda + 3q - 1)(2\lambda + 3(p + q) - 5)(2\lambda + 3(p + q) - 2)(\lambda + 3q - 4).$$

Then, for the operator $\mathcal{L}^{(\alpha)}$ *determined from* (3.5), *we have*

$$L^{(\alpha)} P^{(\alpha)}_{p,q} = -\lambda_{p,q} P^{(\alpha)}_{p,q} \quad \textit{where} \quad \lambda_{p,q} = (\lambda - 1)(p + q) + p^2 + q^2 + pq.$$

REMARK 6.4. The only possible values of λ for which the denominators vanish in the above formulae are $\lambda = 1$ and $\lambda = 4$, which correspond to $\alpha = \pm 1/2$ and to the values $(p, q) \in \{(0, 0), (1, 0), (0, 1)\}$. In those situations, we have to replace $a_1(\lambda, p)$ and $a_2(\lambda, p, q)$ by

$$a_1(\lambda, p) = \lim_{\epsilon \to 0} a_1(\lambda + \epsilon, p) \quad \text{and} \quad a_2(\lambda, p, q) = \lim_{\epsilon \to 0} a_2(\lambda + \epsilon, p, q).$$

Moreover, in both cases,

$$a_1(1, p) = a_1(4, p) = -1/3, \qquad a_2(1, p, q) = a_2(4, p, q) = -1/9,$$

for every (p, q) except for $(p, q) \in \{(0, 0), (1, 0), (0, 1)\}$. We have indeed

$$a_1(1, 1) = -2/3, \qquad a_2(1, 0, 1) = -1/3,$$
$$a_1(4, 1) = -1/3, \qquad a_2(4, 0, 1) = -1/9,$$
$$a_1(\lambda, 0) = a_2(\lambda, p, 0) = 0,$$

In the case $\alpha = 1/2$ the recurrence formulae simplify for every $p, q \geq 0$ to

$$P^{(1/2)}_{p+1,q} = zP^{(1/2)}_{p,q} - \tfrac{1}{3}P^{(1/2)}_{p-1,q+1} - \tfrac{1}{9}P^{(1/2)}_{p,q-1}, \tag{6.2}$$

But in the other case $\alpha = -1/2$, the recurrence formulae are the same except for $(p, q) \in \{(1, 0), (0, 1)\}$ corresponding to

$$P^{(-1/2)}_{1,1} = z\bar{z} - \tfrac{1}{3}, \quad P^{(-1/2)}_{2,0} = z^2 - \tfrac{2}{3}\bar{z}, \quad P^{(-1/2)}_{0,2} = \bar{z}^2 - \tfrac{2}{3}z.$$

Therefore, for $\alpha = -1/2, 1/2$, the recurrence formulae for the polynomials are the same, except for the first two coefficients (which explains that they do not give the same family of polynomials).

It is worth observing that since $a_1(\lambda, 0) = a_2(\lambda, p, 0) = 0$, formula (6.1) makes sense for $p = 0$ and $q = 0$, and defines completely the family $P^{(\alpha)}_{p,q}$ for any $(p, q) \in \mathbb{N}^2$. One observes that $P^{(\alpha)}_{p,q}(z, \bar{z}) = z^p\bar{z}^q +$ lower degree terms, with real coefficients. It is also easily checked that $P^{(\alpha)}_{q,p} = \overline{P^{(\alpha)}_{p,q}}$.

The proof of Theorem 6.3 is rather technical, and relies on the following

Lemma 6.5. *For the family of polynomials defined in* (6.1) *and the Γ operator defined in* (3.5), *we have*

$$\Gamma(z, P^{(\alpha)}_{p,q}) = \alpha_0(p, q)P^{(\alpha)}_{p+1,q} + \alpha_1(p, q)P^{(\alpha)}_{p-1,q+1} + \alpha_2(p, q)P^{(\alpha)}_{p,q-1},$$
$$\Gamma(\bar{z}, P^{(\alpha)}_{p,q}) = \alpha_0(q, p)P^{(\alpha)}_{p,q+1} + \alpha_1(q, p)P^{(\alpha)}_{p+1,q-1} + \alpha_2(q, p)P^{(\alpha)}_{p-1,q},$$

with

$$\alpha_0(p, q) = -\tfrac{1}{2}(q + 2p),$$
$$\alpha_1(p, q) = \frac{1}{2}\,\frac{p(2\lambda + 3q - 5)(\lambda + p - q - 1)}{(\lambda + 3p - 1)(\lambda + 3p - 4)},$$
$$\alpha_2(p, q) = \frac{1}{2}\,\frac{N^1_{p,q}}{D^1_{p,q}},$$

where

$$N^1_{p,q} = q(3q + 2\lambda - 5)(\lambda + 3(p + q) - 1)(\lambda + p + q - 2)(2\lambda + p + 2q - 2),$$
$$D^1_{p,q} = (\lambda + 3q - 1)(2\lambda + 3(p + q) - 5)(2\lambda + 3(p + q) - 2)(\lambda + 3q - 4).$$

It is worth observing that although the definition of Γ does not involve the parameter α (or equivalently λ), the recurrence formula defining $P^{(\alpha)}_{p,q}$ does, and Lemma 6.5 is valid for any α. However, it is not clear from the proof for which family of recurrence formulae on $P^{(\alpha)}_{p,q}$ the 3-terms recurrence formulae for $\Gamma(z, P^{(\alpha)}_{p,q})$ and $\Gamma(\bar{z}, P^{(\alpha)}_{p,q})$ are still valid.

Since the proof of Lemma 6.5 is quite technical, we postpone it to the Appendix (Section 9).

Proof of Theorem 6.3. Recall that $\lambda = \frac{1}{2}(6\alpha + 5) > 0$. Assume by induction that $\mathcal{L}^{(\alpha)} P_{p_1,q_1} = -\lambda_{p_1,q_1} P_{p_1,q_1}$ when $p_1 + q_1 \le p + q$, where $\lambda_{p,q} = (\lambda - 1)(p + q) + p^2 + q^2 + pq$. As before, we simply write the change of variable formula, and using Lemma 6.5, we get

$$\begin{aligned}\mathcal{L}^{(\alpha)} P_{p+1,q} &= \mathcal{L}^{(\alpha)}(zP_{p,q}) + a_1(\lambda, p)\mathcal{L}^{(\alpha)} P_{p-1,q+1} + a_2(\lambda, p, q)\mathcal{L}^{(\alpha)} P_{p,q-1} \\ &= z\mathcal{L}^{(\alpha)} P_{p,q} + P_{p,q}\mathcal{L}^{(\alpha)} z + 2\Gamma(z, P_{p,q}) \\ &\quad - a_1(\lambda, p)\lambda_{p-1,q+1} P_{p-1,q+1} - a_2(\lambda, p, q)\lambda_{p,q-1} P_{p,q-1} \\ &= -(\lambda_{p,q} + \lambda) zP_{p,q} + 2\alpha_0(p, q)P_{p+1,q} \\ &\quad + (2\alpha_1(p, q) - a_1(\lambda, p))P_{p-1,q+1} \\ &\quad + (2\alpha_2(p, q) - a_2(\lambda, p, q)\lambda_{p,q-1})P_{p,q-1}.\end{aligned}$$

But

$$zP_{p,q} = P_{p+1,q} - a_1(\lambda, p)P_{p-1,q+1} - a_2(\lambda, p, q)P_{p,q-1},$$

so that

$$\mathcal{L}^{(\alpha)} P_{p+1,q} = B_1(p, q)P_{p+1,q} + B_2(p, q)P_{p-1,q+1} + B_3(p, q)P_{p,q-1},$$

where

$$\begin{aligned}B_1(p, q) &= -(\lambda_{p,q} + \lambda) + 2\alpha_0(p, q), \\ B_2(p, q) &= 2\alpha_1(p, q) + (\lambda_{p,q} - \lambda_{p-1,q+1} + \lambda)a_1(\lambda, p), \\ B_3(p, q) &= 2\alpha_2(p, q) + (\lambda_{p,q} - \lambda_{p,q-1} + \lambda)a_2(\lambda, p, q).\end{aligned}$$

Everything boils down to the following formulae, which are straightforward to check:

$$B_1(p, q) = -\lambda_{p+1,q}, \qquad B_2(p, q) = B_3(p, q) = 0.$$

The same proof applies for $\mathcal{L}^{(\alpha)} P_{p,q+1} = -\lambda_{p,q+1} P_{p,q+1}$. The conclusion follows. ∎

Remark 6.6. From the recurrence formula, it is easily checked that

$$\begin{aligned}P_{p,q} &= z^p\bar{z}^q + A_{p,q} z^{p+1}\bar{z}^{q-2} + B_{p,q} z^{p-2}\bar{z}^{q+1} \\ &\quad + C_{p,q} z^{p-1}\bar{z}^{q-1} + D_{p,q} z^{p-4}\bar{z}^{q+2} + F_{p,q} z^{p+2}\bar{z}^{q-4} + R,\end{aligned}$$

where $\deg(R) \le p + q - 3$. This general form may be easily deduced from the form of the operator, and should produce a 6-term recurrence formula. The fact that the recurrence formula contains only three terms (as in dimension 1) is indeed quite mysterious.

Corollary 6.7. *For any p, q, there exists a unique polynomial $P^{(\alpha)}_{p,q}$ with unique highest degree term $z^p\bar{z}^q$, which is an eigenvector of $\mathcal{L}^{(\alpha)}$ with eigenvalue $-\lambda_{p,q} = -((\lambda-1)(p+q)+p^2+q^2+pq)$. As already mentioned, the eigenspace associated to $\lambda_{p,q}$ has dimension 2 when $p \neq q$ and dimension 1 for $p = q$ whenever $\lambda \notin \mathbb{Q}$.*

Proof. From the fact that the operator is symmetric in the space of polynomials with total degree less than $p+q$, we know that there are exactly $p+q+1$ linearly independent eigenvectors for this operator with total degree $p+q$. Since Theorem 6.3 provides the exact number of such eigenvectors, we know that we have described a complete set of them. ∎

7. Other representations of eigenpolynomials. In this section, we come back to the two different representations for $\mathcal{L}^{(-1/2)}$ and $\mathcal{L}^{(1/2)}$ which provide new representations for the eigenvectors $P^{(\alpha)}_{p,q}$ in those specific cases. These new representations will allow us to get in those cases linearization formulae for the product together with generating functions.

Although the case $\alpha = -1/2$ is quite easy, since it comes from a Euclidean Laplace operator, it gives rise to another family of recurrence formulae. On the other hand, the case $\alpha = 1/2$, which comes from the Casimir operator on SU(3), leads to new representations of the eigenvectors $P^{(\alpha)}_{p,q}$ related to the irreducible representations of the symmetric group. Finally, comparing the two cases allows us to generalize the SU(3) formulae to the general situation, although this does not provide any new simple form for the eigenvectors.

7.1. Case $\alpha = -1/2$. In this case, one may see quite easily that all the polynomials $P^{(-1/2)}_{p,q}$ have simple expressions in terms of $P^{(-1/2)}_{p,0}$ and $P^{(-1/2)}_{0,q}$. We have

Proposition 7.1.

(1) *For any $p \geq q \geq 1$,*

$$P^{(-1/2)}_{p,q} = P^{(-1/2)}_{p,0} P^{(-1/2)}_{0,q} - 3^{-2q} P^{(-1/2)}_{p-q,0}.$$

(2) *For any $q \geq p \geq 1$,*

$$P^{(-1/2)}_{p,q} = P^{(-1/2)}_{p,0} P^{(-1/2)}_{0,q} - 3^{-2p} P^{(-1/2)}_{0,p-q}.$$

Proof. With the representation (3.5) of the operator $\mathcal{L}^{(-1/2)}$, one may represent the function z as a function $\mathbb{R}^2 \to \mathbb{C}$ through equation (4.1). As in Section 5, we may then write $z = (z_1 + z_2 + z_3)/3$, where z_i are three complex numbers satisfying $|z_i| = 1$ and $z_1 z_2 z_3 = 1$.

Then, for any $p \in \mathbb{Z}$, setting $T_p = z_1^p + z_2^p + z_3^p$, one has, by (5.8),

$$T_{p+2} - 3zT_{p+1} + 3\bar{z}T_p - T_{p-1} = 0, \tag{7.1}$$

with $T_1 = 3z$ and $T_{-1} = 3\bar{z}$, $T_0 = 3$. One may observe first that this formula is unchanged if we replace p by $-p$ and z by $\bar{z}$. Setting $T_p = 3^{|p|} Q_p$, one gets

$$Q_{p+1} = zQ_p - \frac{1}{3}\bar{z}Q_{p-1} + \frac{1}{3^3}Q_{p-2}. \tag{7.2}$$

From this, it is clear that Q_p is a polynomial of degree less than p in $(z,\bar z)$, of the form $Q_p = z^p +$ lower degree terms. Now, if we replace z_1, z_2, z_3 by $e_{\mathbf{1}}, e_{\mathbf{j}}, e_{\bar{\mathbf{j}}}$ we see that Q_p is an eigenvector for the Laplace operator Δ in $\mathbb{R}^2$, with eigenvalue p^2. Therefore,

$$(7.3)\qquad \forall p\ge 0,\quad Q_p = P_{p,0}^{(-1/2)},\quad Q_{-p} = \overline{Q}_p = P_{0,p}^{(-1/2)}.$$

Comparing (7.2) with the recurrence formulae for $P_{p,q}^{(\alpha)}$, the first line in (6.1) gives in this case ($\lambda = 1$)

$$P_{p+1,0}^{(-1/2)} = zP_{p,0}^{(-1/2)} - \tfrac{1}{3}P_{p-1,1}^{(-1/2)},$$

which leads to

$$P_{p-1,1}^{(-1/2)} = \bar z P_{p-1,0}^{(-1/2)} - \tfrac{1}{3^2}P_{p-2,0}^{(-1/2)},$$

which is the second line in (6.1).

On the other hand, coming back to the representation $T_p = z_1^p + z_2^p + z_3^p$, one sees that, for any $(p,q)\in\mathbb{Z}^2$,

$$T_pT_q - T_{p+q} = \sum_{i\neq j} z_i^p z_j^q = \sum_{i\neq j} z_i^{p-q} z_j^{-q},$$

from which we get, for any $(p,q)\in\mathbb{Z}^2$,

$$(7.4)\qquad T_pT_q - T_{p+q} = T_{p-q}T_{-q} - T_{p-2q} = T_{q-p}T_{-p} - T_{q-2p}.$$

When $(z_1,z_2,z_3) = (e_{\mathbf{1}}, e_{\mathbf{j}}, e_{\bar{\mathbf{j}}})$, this becomes a sum of terms of the form $e_{\beta(p,q)}$, where $|\beta_{p,q}|^2 = p^2+q^2-pq$. Therefore, for the $\mathcal{L}^{(-1/2)}$ operator, writing $T_pT_{-q} - T_{p-q}$ as a polynomial in $(z,\bar z)$, we see that this is an eigenvector associated with the eigenvalue p^2+q^2+pq. Looking at the highest degree term, and translating this in terms of the polynomials $Q_p = 3^{-|p|}T_p$, we obtain

$$(7.5)\qquad \forall p,q\ge 1,\quad P_{p,q}^{(-1/2)} = Q_pQ_{-q} - 3^{-2\min(p,q)}Q_{p-q},$$

giving a representation of $P_{(p,q)}^{(-1/2)}$ in terms of the polynomials $P_{p,0}^{(-1/2)}$ and $P_{0,q}^{(-1/2)}$ which is not easy to obtain directly from the recurrence formula (6.1).

When $p,q\ge 0$, $Q_pQ_q - Q_{p+q}$ is also an eigenvector for the Laplace operator associated with the eigenvalue p^2+q^2-pq. Indeed, using (7.4) which is valid for any $(p,q)\in\mathbb{Z}^2$, and comparing with (6.1), we end up, for $p\ge q\ge 0$, with

$$(7.6)\qquad P_{p-q,q}^{(-1/2)} = Q_pQ_q - Q_{p+q}.$$

The conclusions of Proposition 7.1 are immediate consequences of these formulae. ∎

Observe that equations (7.5) and (7.6) may be seen as multiplication formulae, that is, expressions of the products $P_{p,0}^{(-1/2)}P_{q,0}^{(-1/2)}$, $P_{p,0}^{(-1/2)}P_{0,q}^{(-1/2)}$ and $P_{0,p}^{(-1/2)}P_{0,q}^{(-1/2)}$ as linear combinations of the polynomials $P_{r,s}^{(-1/2)}$.

7.2. Case $\alpha = 1/2$**.** We now turn to the inspection of the family $P^{(1/2)}_{p,q}$. As in the case $\alpha = -1/2$, we may build the various polynomials $P^{(1/2)}_{p,q}$ from the polynomials $P^{(1/2)}_{p,0}$ and $P^{(1/2)}_{0,q}$, but the construction is more subtle, and relies on combinatorial expressions.

We know that $\frac{4}{3}\mathcal{L}^{(1/2)}$ may be represented as the action of the Casimir operator on SU(3) on spectral functions. Once again, one may represent $z = (z_1 + z_2 + z_3)/3$ with $|z_i| = 1$ and $z_1 z_2 z_3 = 1$. The complex variables z_i are now the eigenvalues of the matrix $g \in \mathrm{SU}(3)$. Now, the function $T_p = z_1^p + z_2^p + z_3^p$ is also $T_p = \mathrm{trace}(g^p)$, but may be expressed through the same polynomials in $(z, \bar{z})$ as in the previous section.

To formulate the main result of this section, we need to introduce some notation.

For any $n \in \mathbb{N}$, $n \geq 1$, let Π_n be the set of sequences of integers $p_1 \geq \cdots \geq p_k$, $p_i \geq 1$, such that $p_1 + \cdots + p_k = n$. An element $\pi \in \Pi_n$ is called a *Young diagram*, and k is its *length*. To any $\pi \in \Pi_n$ corresponds a conjugacy class in the group $\mathcal{S}_n$ of permutations of n elements. Any $\sigma \in \mathcal{S}_n$ may be decomposed into cycles with decreasing lengths $p_1 \geq \cdots \geq p_k$ and any two permutations are conjugate to each other if they have the same cycle decomposition, so that conjugacy classes are in one-to-one correspondence with elements of Π_n. We shall denote by $\pi(\sigma) \in \Pi_n$ the cycle decomposition of $\sigma \in \mathcal{S}_n$.

For $\sigma \in \mathcal{S}_n$, we denote by $|\sigma|$ the size of its conjugacy class.

A *class function* on $\mathcal{S}_n$ is a function $\mathcal{S}_n \to \mathbb{R}$ which is constant on conjugacy classes.

The characters of the group $\mathcal{S}_n$ are class functions, and are also in one-to-one correspondence with Young diagrams. For any character χ and any permutation σ, one may compute the value $\chi(\sigma)$, which is in fact a function of $\pi(\sigma)$, and is given through a character table.

Proposition 7.2. *Let* $n \geq 1$*. For any* $\sigma \in \mathcal{S}_n$ *with* $\pi(\sigma) = (p_1 \geq \cdots \geq p_k) \in \Pi_n$ *let* $S_\sigma = T_{p_1} \cdots T_{p_k}$*. Let* ξ *be a character on* $\mathcal{S}_n$*. Then* $\sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) T_\sigma$ *is an eigenvector of* $\mathcal{L}^{(1/2)}$*, and a polynomial in* $(z, \bar{z})$ *of degree* $d \leq n$*.*

Remark 7.3. This result is similar to the results of the previous section, in that it allows one to construct $P^{(1/2)}_{p,q}$ as polynomials in $P^{(-1/2)}_{(p,0)}$ and $P^{(-1/2)}_{(0,p)}$, which are used as basic blocks.

Proof of Proposition 7.2. Most of the computations are valid for any $\alpha > -5/6$. Indeed, in the computation below, one may see that the finite-dimensional linear vector space spanned by the polynomials T_σ, $\sigma \in \Pi_n$, is always stable under the action of $\mathcal{L}^{(\alpha)}$. But the special case $\alpha = 1/2$ provides a simpler form which allows for explicit diagonalization. This result may be seen as a particular interpretation of a general fact known as the Schur–Weyl duality (see [17] for example).

Let us start with some basic computations. Using (5.2) and (5.3), one sees that, for $p \geq 1$,

$$\mathcal{L}^{(1/2)}(T_p) = -\frac{p}{4}\Big(2(9-p)T_p + 3\sum_{i=1}^{p-1} T_i T_{p-i}\Big), \tag{7.7}$$

while for $p, q \geq 0$,

$$\Gamma(T_p, T_q) = \frac{pq}{2}\big(T_p T_q - 3T_{p+q}\big), \tag{7.8}$$

with similar formulae for $p, q \leq 0$.

If we remember that $\mathcal{L}^{(1/2)} = \mathcal{L}^{(-1/2)} - 3(z\partial_z + \bar{z}\partial_{\bar{z}})$, we get, for $p \geq 0$, the formula

$$(z\partial_z + \bar{z}\partial_{\bar{z}})T_p = -\frac{p}{2}(p-3)T_p + \frac{p}{4}\sum_{i=1}^{p-1} T_i T_{p-i}. \tag{7.9}$$

With the help of (3.7), for $p \geq 1$ and for a general parameter α, we end up with

$$\begin{aligned} \mathcal{L}^{(\alpha)} T_p = &-\frac{p}{4}\big(p(1-6\alpha) + 9(2\alpha+1)\big)T_p \\ &- \frac{3p}{8}(2\alpha+1)\sum_{i=1}^{p-1} T_i T_{p-i}. \end{aligned} \tag{7.10}$$

We first perform a slight change in the normalization of the variables T_p, setting $T_p = c\hat{T}_p$ with $c = \sqrt{2/(2\alpha+1)}$, in order to reduce (7.10) and (7.8) to

$$\mathcal{L}^{(\alpha)}\hat{T}_p = -\mu_{p,\alpha}\hat{T}_p - \frac{1}{c}\,\frac{3p}{4}\sum_{i=1}^{p-1} \hat{T}_i\hat{T}_{p-i}, \tag{7.11}$$

$$\Gamma(\hat{T}_p, \hat{T}_q) = \frac{pq}{2}\left(\hat{T}_p\hat{T}_q - \frac{3}{c}\hat{T}_{p+q}\right), \tag{7.12}$$

with $\mu_{p,\alpha} = \frac{p}{4}(p(1-6\alpha) + 9(2\alpha+1))$. We change S_σ into $\hat{S}_\sigma$ accordingly. We now look at the action of $\mathcal{L}^{(\alpha)}$ on $\hat{S}_\sigma$ and compare it with the action of the operator $\mathcal{T}(f)(\sigma) = \sum_{\tau\in\mathcal{T}_n} f(\sigma\tau)$, where $\mathcal{T}_n$ is the set of transpositions.

It is worth observing that if $\tau = (ij)$ is a transposition, the cycle decomposition of $\sigma\tau$ splits one cycle into two subcycles when i and j belong to the same cycle and glues together two cycles when i and j belong to two different cycles.

Therefore, for a permutation σ with $\pi(\sigma) = (p_1, \ldots, p_k)$, through an easy combinatorial argument, one gets

$$\sum_{\tau \in \mathcal{T}_n} \hat{S}_{\tau\sigma} = \frac{1}{2} \sum_{i=1}^{k} p_i \frac{\hat{S}_\sigma}{\hat{S}_{p_i}} \sum_{j=1}^{p_i - 1} \hat{S}_j \hat{S}_{p_i - j} + \sum_{i,j=1}^{k} p_i p_j \frac{\hat{S}_\sigma}{\hat{S}_{p_i} \hat{S}_{p_j}} \hat{S}_{p_i + p_j}.$$

Now, through the change of variable formula, and (7.11) and (7.8), with the help of (5.4), we get

$$\begin{aligned} \mathcal{L}^{(\alpha)} \hat{S}_\sigma = \hat{S}_\sigma \Bigg(- \sum_{i=1}^{k} \mu_{p_i,\alpha} + \sum_{i \neq j} \frac{p_i p_j}{2} \\ - \frac{3}{2c} \Bigg(\sum_{i=1}^{k} \frac{p_i}{2\hat{S}_{p_i}} \sum_{j=1}^{p_i - 1} \hat{S}_j \hat{S}_{p_i - j} + \sum_{i \neq j} p_i p_j \frac{\hat{S}_{p_i + p_j}}{\hat{S}_{pi} \hat{S}_{p_j}} \Bigg) \Bigg) \\ = \hat{S}_\sigma \Bigg(-\mu_{\sigma,\alpha} - \frac{3}{2c} \Bigg(\sum_{i=1}^{k} \frac{p_i}{2\hat{S}_{p_i}} \sum_{j=1}^{p_i - 1} \hat{S}_j \hat{S}_{p_i - j} + \sum_{i \neq j} p_i p_j \frac{\hat{S}_{p_i + p_j}}{\hat{S}_{pi} \hat{S}_{p_j}} \Bigg) \Bigg) \end{aligned}$$

where, for $\pi(\sigma) = (p_1, \ldots, p_k)$,

$$\mu_{\sigma,\alpha} = \sum_{i=1}^{k} \mu_{p_i,\alpha} - \sum_{i \neq j} \frac{p_i p_j}{2} = \frac{3}{4}(1 - 2\alpha) \sum_{i=1}^{k} p_i^2 + \frac{9}{4}(2\alpha + 1)n - \frac{n^2}{2}.$$

Finally,

$$\mathcal{L}^{(\alpha)} \hat{S}_\sigma = -\mu_{\alpha,\sigma} \hat{S}_\sigma - \frac{3}{2c} \sum_{\tau \in \mathcal{T}_n} \hat{S}_{\tau\sigma} = -\mu_{\alpha,\sigma} \hat{S}_\sigma - \frac{3}{2c} \mathcal{T}(\hat{S}_\sigma). \tag{7.13}$$

Observe that for $\alpha = 1/2$ (and only in this case), $\mu_{\sigma,\alpha}$ depends on n only, and therefore finding eigenvectors for $\mathcal{L}^{(1/2)}$ amounts to finding eigenvectors for the linear operator $S_\sigma \mapsto \mathcal{T}(S_\sigma)$. But the latter corresponds to the operator $\sum_{\tau \in \mathcal{T}_n} \tau$ in the group algebra of the group $\mathcal{S}_n$, which commutes with every group element. It is therefore diagonal on any irreducible representation. Turning back to our setting, we conclude that for any character χ of $\mathcal{S}_n$, $\sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) \hat{S}_\sigma$ is an eigenvector for $\mathcal{L}^{(1/2)}$. ∎

Remark 7.4. Since all our expressions are constant on conjugacy classes, one may have written S_π for $\pi \in \Pi_n$ instead of S_σ, and $\xi(\pi)$ for the constant value of $\xi(\sigma)$ on the class $\pi(\sigma)$. Then the expression for the eigenvectors becomes $\sum_{\pi \in \Pi_n} |\pi| \xi(\pi) S_\pi$, where $|\pi|$ is the size of the conjugacy class associated to π. Recall that, for $\pi = (p_1, \ldots, p_k)$,

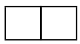
$$|\pi| = \frac{n!}{\prod_{j=1}^{n} k_j!} \prod_{j=1}^{k} \frac{1}{p_j},$$

where k_j is the number of cycles of length j in π.

It turns out that the expression $\sum_{\sigma\in\mathcal{S}_n}\chi(\sigma)T_\sigma$ takes a simple form in terms of the dual diagram associated to χ. As a function of the matrix $M \in \mathrm{SU}(3)$, and following [17, formula (20)], this is nothing else, up to some constant, than the Schur function $s_\lambda(\mu_1,\mu_2,\mu_3)$, where λ is the Young diagram associated with the character χ, and μ_1,μ_2,μ_3 are the eigenvalues of the matrix. It vanishes on all diagrams of length $k \geq 4$. Thanks to Jacobi–Trudi's identities (see [18, formula I.3.5]), this may be expressed through the dual diagram λ' of length p as $\det(e_{\lambda'_i-i+j})_{1\leq i,j\leq p}$ where in this particular context $e_0 = 1$, $e_1 = 3z$, $e_2 = 3\bar{z}$, $e_3 = 1$, and $e_i = 0$ when $i < 0$ or $i \geq 4$.

As an example, below we list the eigenvectors given by this construction for $n = 2, 3, 4$.

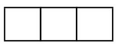
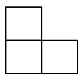
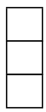
The group $\mathcal{S}_2$ has two conjugacy classes χ_1,χ_2, corresponding to the partitions $(2,0)$ and $(1,1)$,

$\chi_1 =$ , $\chi_2 =$ ,

with corresponding eigenvectors

$$Q_1(z,\bar{z}) = 3z^2 - \bar{z}, \qquad Q_2(z,\bar{z}) = \bar{z}.$$

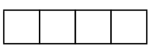
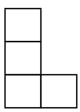
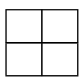
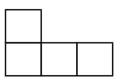
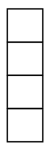
For $\mathcal{S}_3$, we have three conjugacy classes χ_1,χ_2,χ_3 corresponding to the partitions $(3,0,0), (2,1,0), (1,1,1)$,

$\chi_1 =$ , $\chi_2 =$ , $\chi_3 =$ ,

and corresponding eigenvectors (up to some constant)

$$Q_1(z,\bar{z}) = 27z^3 - 15z\bar{z} + 1, \qquad Q_2(z,\bar{z}) = 9z\bar{z} - 1, \qquad Q_3(z,\bar{z}) = 1.$$

For $\mathcal{S}_4$, we have five conjugacy classes, with Young diagrams χ_1,χ_2,χ_3, χ_4,χ_5, corresponding to $(4,0,0,0), (2,1,1,0), (2,2,0,0), (3,1,0,0), (1,1,1,1)$,

$\chi_1 =$ , $\chi_2 =$ , $\chi_3 =$ , $\chi_4 =$ , $\chi_5 =$ ,

with corresponding eigenvectors

$$Q_1(z,\bar{z}) = 27z^4 - 27\bar{z}z^2 + 3\bar{z}^2 + 2z,\ Q_2(z,\bar{z}) = z,\ Q_3(z,\bar{z}) = 3\bar{z}^2 - z,$$
$$Q_4(z,\bar{z}) = 9\bar{z}z^2 - 3\bar{z}^2 - z, Q_5(z,\bar{z}) = 0.$$

Unfortunately, the correspondence between Young diagrams of length $l \leq 3$ and eigenvectors is not one-to-one. For example, $(2,2)$ provides (up to some constant) the same polynomial as $(2,0)$ and $(3,1,1)$, namely$P_{2,0}$.

8. Generating functions. In this section, we first provide a partial generating function in the general case for the family $P^{(\alpha)}_{0,n}$ or equivalently $P^{(\alpha)}_{n,0}$, which leads as in the previous section to some simple representation of the polynomials $P^{(\alpha)}_{n,m}$ as linear combinations of $P^{(\alpha)}_{p,0}P^{(\alpha)}_{0,q}$. We then provide a general generating function for some specific values of α, including the two geometric cases.

8.1. Partial generating functions in the general case. In this section, we propose an alternative representation of the eigenvectors in the general case, together with a partial generating function. A generating function is an explicit function $G(X, Y, z, \bar{z})$ such that the asymptotic expansion of G around $X = Y = 0$ reads

$$G_\alpha(X, Y, z, \bar{z}) = \sum_{p,q} X^p Y^q a_{p,q} P^{(\alpha)}_{p,q}(z, \bar{z}),$$

where we assume that the series is convergent for small values of X and Y.

Such a generating function is not unique in general. For example, applying $X\partial_X$ to some generating function G provides a new one, with $a_{p,q}$ replaced by $pa_{p,q}$. The same being true for $Y\partial_Y$, we may construct in this way as many generating functions as we wish, from a first given one.

We first provide a partial generating function for the general value of α.

THEOREM 8.1. *Let* $P(X) = 1 - 3\bar{z}X + 3zX^2 - X^3$ *and* $\beta = -(1+2\alpha)/2$. *Then*

$$P(X)^\beta = \sum_n c_n P^{(\alpha)}_{0,n} X^n, \quad \textit{where} \quad c_n = (-3)^n \frac{\beta(\beta-1)\dots(\beta-n+1)}{n!}$$

(*we replace* P^β *by* $\log P$ *when* $\alpha = -1/2$).

Thus, $P(X)^\beta$ is a generating function for the family $P^{(\alpha)}_{0,n}$. Similarly, $\bar{P}(X)^\beta$ is a generating function for the family $P^{(\alpha)}_{n,0}$, where $\bar{P}(X) = X^3 - 3\bar{z}X^2 + 3zX - 1$.

We start with a few results which will be useful later.

PROPOSITION 8.2. *Let*

$$P(X) = 1 - 3\bar{z}X + 3zX^2 - X^3. \tag{8.1}$$

Then, still with $\lambda = \frac{1}{2}(6\alpha + 5)$,

$$\mathcal{L}^{(\alpha)}(P(X)) = -\lambda XP' + \frac{\lambda}{2}X^2P'',$$

$$\Gamma(P(X), P(Y)) = \frac{XY}{2}\Big(P'(X)P'(Y) + 3\frac{P'(X)P(Y) - P(X)P'(Y)}{X - Y}\Big),$$

from which

$$\Gamma(P(X), P(X)) = \frac{X^2}{2}(3PP'' - 2P'^2).$$

Also, with $\bar P(Y) = 1 - 3zY + 3\bar z Y^2 - Y^3 = -Y^3 P(1/Y)$,

$$\Gamma(P(X), \bar P(Y)) = \frac{XY}{2(XY-1)}\big(3XP'\bar P + 3Y\bar P' P - 9P\bar P - (XY-1)P'\bar P'\big).$$

Proof. The proof boils down to a simple verification, using the linearity of $\mathcal{L}$ and the bilinearity of Γ. The formula for $\Gamma(P(X), P(X))$ may be obtained directly from $\Gamma(P(X), P(X)) = \lim_{Y\to X} \Gamma(P(X), P(Y))$. ■

LEMMA 8.3. *Let* $Q = P^\beta$ *with* $\beta = (1-\lambda)/3 = -(1+2\alpha)/2$. *Then, for* $\alpha \neq -1/2$,

$$\mathcal{L}^{(\alpha)}(Q(X)) = -\lambda X Q'(X) - X^2 Q''(X), \tag{8.2}$$

$$\Gamma(Q(X), Q(Y)) = \frac{XY}{2}\bigg(Q'(X)Q'(Y) + 3\beta \frac{Q'(X)Q(Y) - Q(X)Q'(Y)}{X-Y}\bigg). \tag{8.3}$$

$$\Gamma(Q(X), \bar Q(Y)) = 3\beta^2 \frac{XY}{2(XY-1)} Q(X)\bar Q(Y)(XS + Y\bar S - 3) - \frac{XY}{2} Q'(X)\bar Q'(Y), \tag{8.4}$$

where $S = \frac{P'}{P}(X)$, $\bar S = \frac{\bar P'}{\bar P}(Y)$. (*For* $\alpha = -1/2$, *one should replace* $Q = P^\beta$ *by* $Q = \log P$).

Proof. Let us look first at $\mathcal{L}^{(\alpha)}(Q)$. With (2.2), we have

$$\mathcal{L}^{(\alpha)}(Q(X)) = \beta P(X)^{\beta-1}\mathcal{L}^{(\alpha)}(P(X)) + \beta(\beta-1)P(X)^{\beta-2}\Gamma(P(X), P(X)), \tag{8.5}$$

and from Proposition 8.2 we obtain

$$\mathcal{L}^{(\alpha)}(Q(X)) = -\lambda X Q'(X) + X^2\big[(\lambda/2 + \tfrac{3}{2}(\beta-1))\beta P''(X)P(X)^{\beta-1} - \beta(\beta-1)P'^2(X)P^{\beta-2}(X)\big].$$

For $\beta = (1-\lambda)/3$, we get $\lambda/2 + \frac{3}{2}(\beta-1) = -1$, giving the announced result.

Turning now to (8.3), we write

$$\begin{aligned}\Gamma(Q(X), Q(Y)) &= \beta^2 P^{\beta-1}(X)P^{\beta-1}(Y)\Gamma(P(X), P(Y)) \\ &= \frac{XY}{2}\bigg(\beta^2 P'(X)P^{\beta-1}(X)P'(Y)P^{\beta-1}(Y) \\ &\quad + \frac{3\beta}{X-Y}\big(\beta P^\beta(Y)P'(X)P^{\beta-1}(X) - \beta P^\beta(X)P'(Y)P^{\beta-1}(Y)\big)\bigg)\end{aligned}$$

It remains to write $Q' = \beta P' P^{\beta-1}$ to obtain (8.3).

Formula (8.4) is obtained in the same way. ■

Proof of Theorem 8.1. From (8.2), setting $D_X = X\partial_X$, one sees that

$$\mathcal{L}^{(\alpha)}(Q) + (\lambda - 1)D_X Q + D_X^2 Q = 0.$$

On the other hand,

$$Q(X) = \sum_n X^n R_n(z, \bar{z}),$$

where R_n is a polynomial in $(z, \bar{z})$ with highest order term $c_n \bar{z}^n$.

Plugging this in the asymptotic expansion (which is allowed as soon as X is small enough, z and $\bar{z}$ being both bounded), one sees that

$$\sum_n X^n \big(\mathcal{L}^{(\alpha)} R_n + ((\lambda - 1)n + n^2)R_n\big) = 0,$$

which in turn shows that R_n is an eigenvector of $\mathcal{L}^{(\alpha)}$, with eigenvalue $-(\lambda - 1)n - n^2$. This implies that $R_n = c_n P^{(\alpha)}_{0,n}$. This is the announced result. ∎

From this partial generating function, one may deduce a description of the polynomials $P^{(\alpha)}_{p,q}$ in terms of $P^{(\alpha)}_{0,r}$ and $P^{(\alpha)}_{r,0}$, similar to the explicit form that we deduced in Section 7.

For this, let us introduce the following operator in four variables $(X, Y, z, \bar{z})$:

$$\mathcal{L}_0 = \mathcal{L}^{(\alpha)} + D_X^2 + D_Y^2 + D_X D_Y + (\lambda - 1)(D_X + D_Y). \tag{8.6}$$

Then, with the notation of Lemma 8.3, we have

LEMMA 8.4.

$$\mathcal{L}_0(Q(X)\bar{Q}(Y)) = \frac{3\beta^2 XY}{XY - 1} Q(X)\bar{Q}(Y)(XS + Y\bar{S} - 3),$$

where as before $S = \frac{P'}{P}(X)$ *and* $\bar{S} = \frac{\bar{P}'}{\bar{P}}(Y)$.

Proof. The proof is straightforward using (8.2) and (8.4). ∎

This proposition leads us to a new representation of $P^{(\alpha)}_{n,m}$.

PROPOSITION 8.5. *There exist constants* $d(m, n, p, \alpha)$ *such that*

$$P^{(\alpha)}_{m,n} = \sum_{p=0}^{\min(m,n)} d_{m,n,p,\alpha} P^{(\alpha)}_{m-p,0} P^{(\alpha)}_{0,n-p}.$$

Proof. Writing for simplicity

$$Q(X) = \sum_{n \geq 0} R_n X^n, \qquad \overline{Q}(Y) = \sum_{m \geq 0} \bar{R}_m Y^m,$$

where we recall that R_m is proportional to $P^{(\alpha)}_{0,n}$, and $\bar{R}_n$ to $P^{(\alpha)}_{n,0}$, we deduce from Lemma 8.4 that

$$\mathcal{L}^{(\alpha)}(R_n\bar{R}_m - R_{n-1}\bar{R}_{m-1}) = -\lambda_{n,m}R_n\bar{R}_m - \delta_{n,m}R_{n-1}\bar{R}_{m-1}$$

where

$$\lambda_{n,m} = (\lambda-1)(n+m) + n^2 + m^2 + nm,$$
$$\delta_{n,m} = (1-\lambda)(\lambda+n+m-3) - \lambda_{n-1,m-1}.$$

Finally, by easy induction, one sees that $\mathcal{L}^{(\alpha)}(R_n\bar{R}_m)$ is a linear combination of $R_{n-p}\bar{R}_{m-p}$, $0 \le p \le \min(n,m)$. Hence $P^{(\alpha)}_{m,n}$ may be written as

$$P^{(\alpha)}_{m,n} = \sum_{p=0}^{\min(m,n)} d_{m,n,p,\alpha} P^{(\alpha)}_{m-p,0} P^{(\alpha)}_{0,n-p}. \ \blacksquare$$

Remark 8.6. Unfortunately, the expression for the constants $d_{m,n,p,\alpha}$ does not seem to have any simple form, from which one could deduce elementary expressions as the one described in Section 7 for $\alpha = -1/2$.

8.2. General generating functions in particular cases. We now turn to the description of a generating function (in two variables X and Y) for some specific values of α, namely $\alpha = k + 1/2$, $k \in \mathbb{N}$.

Following the method of Theorem 8.1, we are looking for a function $G(X, Y, z, \bar{z})$ which may be expanded for X and Y close to 0 in a power series

$$G(X, Y, z, \bar{z}) = \sum_{p,q} X^p Y^q A_{p,q}(z, \bar{z}),$$

where $A_{p,q}$ is a polynomial in $(z, \bar{z})$ with highest term $\bar{z}^p z^q$ and satisfying the differential identity

$$\mathcal{L}_0(G) = \big(\mathcal{L}^{(\alpha)} + D_X^2 + D_Y^2 + D_X D_Y + (\lambda-1)(D_X + D_Y)\big)G = 0,$$

where $D_X = X\partial_X$ and $D_Y = Y\partial_Y$.

We make the following

Conjecture. *Let $\alpha = k+1/2$, $k \in \mathbb{N}$, $D = X\partial_X + Y\partial_Y$ and $U(X,Y) = \big(P(X)\bar{P}(Y)\big)^{-k-1}$. For each $k \in \mathbb{N}$ there exists a family of polynomials $R_{k,i}(x)$, $i = 0, \dots, k$, in one variable, of degree $k-i$ (with $R_{k,k} = 1$), such that*

$$G(X, Y, z, \bar{z}) = (XY-1)^{k+1} \sum_{i=0}^{k} R_{k,i}\Big(\frac{XY}{XY-1}\Big) D^i U \tag{8.7}$$

satisfies

$$\mathcal{L}_0(G) = 0,$$

where $\mathcal{L}_0$ is defined in (8.6).

Thus, the asymptotic expansion $G(X,Y,z,\bar{z}) = \sum_{p,q} X^p Y^q A_{p,q}(z,\bar{z})$ is such that $A_{p,q}$ is a polynomial with unique highest degree term $c_{p,q}\bar{z}^p z^q$, proportional to $P_{q,p}(z,\bar{z})$.

Indeed, to compare with the previous section, we set $\beta = -(1+2\alpha)/2 = -k-1$. Then, with $U = P(X)^\beta \bar{P}(Y)^\beta$, we shall describe under which conditions on the polynomials $R_{k,i}$ the function defined in (8.7) satisfies $\mathcal{L}_0(G) = 0$. We shall see that this requires strong constraints on the coefficients of $R_{k,i}$. We have not been able to show that in the general case there is a nontrivial solution for this problem, but we can produce solutions for small values of k. (Indeed, with the help of computer algebra, we tested this form up to $k = 10$, without finding any regular structure in this family of polynomials, which could have induced a general form). Unfortunately, this method or a similar one does not seem to produce any interesting result for other values of α.

Let us start with some notation. For a function $f(x)$ of the real variable x, we write $d(f) = xf'(x)$. Let $g(x) = x/(x-1)$, and observe that $d(g) = g(1-g)$. Moreover, observe that there exist polynomials Q_i of degree $i-1$ such that $d^i(g) = Q_i(g)d(g)$ for $i \geq 1$. Those polynomials satisfy the recursive equation

$$Q_1(x) = 1, \qquad Q_{i+1}(x) = (1-2x)Q_i(x) + x(1-x)Q_i'(x).$$

We shall use the polynomials Q_i in the following.

Proposition 8.7. *Given k, suppose that the family of polynomials $R_{k,i}$ satisfy $R_{k,k} = 1$, $R_{k,-1} = 0$ and, for $i = k, \ldots, 0$,*

$$\begin{aligned} R'_{k,i-1} &= k(k+1)R_{k,i} - R'_{k,i}(2k+3+2kx) + R''_{k,i}x(x-1) \\ &\quad + (k+1)\sum_{j=i}^{k} \binom{j}{i-1} 2^{j+1-i} Q_{j+1-i} R_{k,j} \\ &\quad + 3(k+1)^2 \sum_{j=i+1}^{k} \binom{j}{i} 2^{j-i} Q_{j-i} R_{k,j} \end{aligned}$$

where by convention we have set $\binom{j}{-1} = 0$. Then the associated function G defined in (8.7) satisfies the equation $\mathcal{L}_0(G) = 0$.

Remark 8.8. One may see this system of equations as a recursive definition for the polynomials $R_{k,i}$, starting from $R_{k,k}$ down to $R_{k,0}$. At each step, the polynomials are defined up to some additive constant, so that in the end we have a choice of k parameters. Thus, the last equation $R_{k,-1} = 0$ is a compatibility condition. Experimentally (up to $k = 10$ at least), it turns out that this problem has a nontrivial solution. However, we were unable to prove this in the general case.

Proof of Proposition 8.7. We omit the index k in $R_{k,i}$ to simplify the notation. To stick with the notation of the previous section, we set $\beta = -(k+1)$. From (8.2) and (8.4), we have

$$\mathcal{L}^{(\alpha)}(U) = -(\lambda-1)DU - (D_X^2 + D_Y^2 + D_X D_Y)U + 3\beta g(XY)(DU - 3\beta U).$$

In other words,

$$\mathcal{L}_0(U) = 3\beta g(XY)(DU - 3\beta U).$$

Now, observe that $[\mathcal{L}_0, D] = 0$; using

$$D^k(AB) = \sum_{i=0}^{k} \binom{k}{i} D^i(A) D^{k-i} B,$$

we obtain, with $p = XY$,

$$\mathcal{L}_0 D^j U = 3\beta \sum_{i=0}^{j} \binom{j}{i} 2^{j-i} d^{j-i}(g)(p)(D^{i+1}U - 3\beta D^i U).$$

Now, using $D(f(p)) = 2d(f)(p)$, we have

$$\begin{aligned}\mathcal{L}_0(r(p)D^jU) &= r(p)\mathcal{L}_0 D^j U + 3d(r)(p)D^{j+1}U \\ &\quad + 3D^jU\big(d^2(r) - 2\beta d(r)\big)(p).\end{aligned} \tag{8.8}$$

Set $h(x) = (x-1)^{-\beta}$. We apply (8.8) with $r(x) = h(x)R_j(g)(x)$.

We have $dh = -\beta g h$. Then

$$d(hR_j(g)) = -\beta h g R_j(g) + hR_j'(g)dg = h\big(-\beta g R_j(g) + R_j'(g)dg\big).$$

$$\begin{aligned}d^2(hR_j(g)) - 2\beta d(hR_j(g)) &= h\big(\beta R_j(g)(\beta g(g+2) - dg) \\ &\quad + R_i'(g)dg\big((1 - 2\beta - 2g(1+\beta)\big) + R_j''(g)(dg)^2\big).\end{aligned}$$

Still with $p = XY$,

$$\begin{aligned}\mathcal{L}_0(h(p)R_j(g(p))D^jU) = h(p)\big[&R_j(g(p))\mathcal{L}_0 D^j U \\ &+ 3D^{j+1}U(R_j'(g(p)dg(p) - \beta g R_j(g(p)) \\ &+ 3D^jU\big(\beta R_j(g(p))(\beta g(g+2) - dg)(p) \\ &+ R_j'(g(p))dg(p)(1 - 2\beta - 2(1+\beta)g(p)) \\ &+ R_j''(g(p))(dg)^2(p)\big)\big].\end{aligned}$$

From this, we get

$$\mathcal{L}_0\left(h(XY)\sum_j R_j\left(\frac{XY}{XY-1}\right)D^jU\right) = 3h(XY)\sum_{i=0}^{k} T_i(XY)D^iU,$$

where

$$T_i = \beta \sum_{j=i-1}^{k} \binom{j}{i-1} 2^{j+1-i} d^{j+1-i}(g) R_j(g) - 3\beta^2 \sum_{j=i}^{k} \binom{j}{i} 2^{j-i} d^{j-i}(g) R_j(g)$$
$$+ R'_{i-1}(g)dg - \beta g R_{i-1}(g) + \beta(\beta g^2 + 2\beta g - dg)R_i(g)$$
$$+ R'_i(g)dg(1 - 2\beta - 2(1+\beta)g) + R''_i(g)(dg)^2,$$

while, for T_0, we set $R_{-1} = 0$ and $\binom{j}{-1} = 0$.

Looking more carefully at T_i for $i \geq 1$, one sees that the term $j = i-1$ in the first sum cancels with the second term of the second line, and that the term $j = i$ in the first sum added to the last term of the second line factorizes as $-\beta(\beta+1)dgR_i(g)$. On the other hand, since $d^i g = Q_i(g)dg$ for $i \geq 1$, we see that T_i factorizes as $T_i = dgS_i$ with

$$S_i(g) = \beta \sum_{j=i}^{k} \binom{j}{i-1} 2^{j+1-i} Q_{j+1-i}(g) R_j(g)$$
$$- 3\beta^2 \sum_{j=i+1}^{k} \binom{j}{i} 2^{j-i} Q_{j-i}(g) R_j(g)$$
$$+ R'_{i-1}(g) - \beta(\beta+1)R_i(g)$$
$$+ R'_i(g)((1 - 2\beta - 2(1+\beta)g) + R''_i(g)g(1-g).$$

Thus, the equations of Proposition 8.7 imposed on $R_{k,i}$ $(= R_i)$ are exactly chosen such that the polynomials S_i vanish, in which case

$$\mathcal{L}_0\bigg((XY-1)^{-\beta} \sum_j R_j\Big(\frac{XY}{XY-1}\Big) D^j U\bigg) = 0. \ ■$$

Remark 8.9. In the case $k = 0$, we obtain

$$G(X,Y) = \frac{1 - XY}{(1 - 3XZ + 3X^2\bar{Z} - X^3)(1 - 3Y\bar{Z} + 3Y^2 Z - Y^3)},$$

and this form has been proposed in [9], with however a completely different approach, based on representations of SU(3). The case $k = -1$, formally excluded from our computations, and corresponding to $\alpha = -1/2$, suggests that U should be replaced by $\log(P(X)) + \log(\bar{P}(Y))$. However, a direct approach using the explicit expression of Section 7.1 provides directly the following generating function:

$$G(X,Y,z,\bar{z})) = \Big(3 - X\frac{\bar{P}'}{\bar{P}}(X)\Big)\Big(3 - Y\frac{P'}{P}(Y)\Big)$$
$$+ \frac{1}{1-XY}\Big(X\frac{\bar{P}'}{\bar{P}}(X) + Y\frac{P'}{P}(Y) - 3\Big).$$

Beyond the case $k = 0$, which corresponds to $\alpha = 1/2$ and the SU(3) case, we provide for example the polynomials $R_{k,i}, i = 0, \dots, k-1$, for the first values of $k = 1, 2, 3$:

$$k = 1: \quad R_{1,0}(X) = 3(1 + 2X);$$
$$k = 2: \quad R_{2,1}(X) = 9(2X + 1), \qquad R_{2,0}(X) = 2(30X^2 + 51X + 10);$$
$$k = 3: \quad R_{3,2}(X) = 18(2X + 1), \qquad R_{3,1}(X) = 348X^2 + 516X + 107,$$
$$R_{3,0}(X) = 6(140X^3 + 486X^2 + 308X + 35).$$

9. Appendix: proof of Lemma 6.5. In this section we provide a detailed proof of Lemma 6.5. For this, we remove the parameter α from the formulae, since it will not change throughout the computations. Lemma 6.5 is proved by induction, from

$$\begin{aligned}\Gamma(z, P_{p+1,q}) &= \Gamma(z, zP_{p,q}) + a_1(\lambda, p)\Gamma(z, P_{p-1,q+1}) \\ &\quad + a_2(\lambda, p, q)\Gamma(z, P_{p,q-1}) \\ &= \Gamma(z)P_{p,q} + z\Gamma(z, P_{p,q}) + a_1(\lambda, p)\Gamma(z, P_{p-1,q+1}) \\ &\quad + a_2(\lambda, p, q)\Gamma(z, P_{p,q-1}),\end{aligned}$$

and finally

$$\begin{aligned}\Gamma(z, P_{p+1,q}) &= (\bar{z} - z^2)P_{p,q} + z\Gamma(z, P_{p,q}) \\ &\quad + a_1(\lambda, p)\Gamma(z, P_{p-1,q+1}) + a_2(\lambda, p, q)\Gamma(z, P_{p,q-1}).\end{aligned}$$

And by the definition (6.3) we have

$$\begin{aligned}\bar{z}P_{p,q} &= P_{p,q+1} - a_1(\lambda, q)P_{p+1,q-1} - a_2(\lambda, q, p)P_{p-1,q}, \\ zP_{p,q} &= P_{p+1,q} - a_1(\lambda, p)P_{p-1,q+1} - a_2(\lambda, p, q)P_{p,q-1},\end{aligned}$$

so that

$$\begin{aligned}z^2P_{p,q} &= z(zP_{p,q}) \\ &= zP_{p+1,q} - a_1(\lambda, p)zP_{p-1,q+1} - a_2(\lambda, p, q)zP_{p,q-1}.\end{aligned} \tag{9.1}$$

Furthermore,

$$\begin{aligned}zP_{p+1,q} &= P_{p+2,q} - a_1(\lambda, p+1)P_{p,q+1} - a_2(\lambda, p+1, q)P_{p+1,q-1}, \\ zP_{p-1,q+1} &= P_{p,q+1} - a_1(\lambda, p-1)P_{p-2,q+2} - a_2(\lambda, p-1, q+1)P_{p-1,q}, \\ zP_{p,q-1} &= P_{p+1,q-1} - a_1(\lambda, p)P_{p-1,q} - a_2(\lambda, p, q-1)P_{p,q-2},\end{aligned}$$

which gives

$$\begin{aligned} z^2 P_{p,q} = P_{p+2,q} &- \big(a_1(\lambda,p+1)+a_1(\lambda,p)\big)P_{p,q+1}\\ &- \big(a_2(\lambda,p+1,q)+a_2(\lambda,p,q)\big)P_{p+1,q-1}\\ &+ a_1(\lambda,p)\big(a_2(\lambda,p-1,q+1)+a_2(\lambda,p,q)\big)P_{p-1,q}\\ &+ a_2(\lambda,p,q)a_2(\lambda,p,q-1)P_{p,q-2}\\ &+ a_1(\lambda,p)a_1(\lambda,p-1)P_{p-2,q+2}. \end{aligned}$$

On the other hand, from the induction hypothesis we have

$$\begin{aligned} z\Gamma(z,P_{p,q}) &+ a_1(\lambda,p)\Gamma(z,P_{p-1,q+1}) + a_2(\lambda,p,q)\Gamma(z,P_{p,q-1})\\ &= \alpha_0(p,q)P_{p+2,q} + \gamma_1(p,q)P_{p,q+1} + \gamma_2(p,q)P_{p+1,q-1}\\ &\quad + \gamma_3(p,q)P_{p-1,q} + \gamma_4(p,q)P_{p-2,q-2} + \gamma_5(p,q)P_{p,q-2}, \end{aligned}$$

with

$$\begin{aligned} \gamma_1(p,q) &= \alpha_1(p,q) - \alpha_0(p,q)a_1(\lambda,p+1) + a_1(\lambda,p)\alpha_0(p-1,q+1),\\ \gamma_2(p,q) &= \alpha_2(p,q) - \alpha_0(p,q)a_2(\lambda,p+1,q) + a_2(\lambda,p,q)\alpha_0(p,q-1),\\ \gamma_3(p,q) &= a_1(\lambda,p)\alpha_2(p-1,q+1) + \alpha_1(\lambda,p)a_2(\lambda,p,q)\\ &\quad - \alpha_1(p,q)a_2(\lambda,p-1,q+1) - \alpha_2(p,q)a_1(\lambda,p),\\ \gamma_4(p,q) &= a_1(\lambda,p)\alpha_1(p-1,q+1) - a_1(\lambda,p-1)\alpha_1(p,q),\\ \gamma_5(p,q) &= a_2(\lambda,p,q)\alpha_2(p,q-1) - a_2(\lambda,p,q-1)\alpha_2(p,q). \end{aligned}$$

Substituting everything in (9.1), we get

$$\begin{aligned} \Gamma(z,P_{p+1,q}) = (\alpha_0(p,q)-1)P_{p+2,q} &+ A_1(p,q)P_{p,q+1}\\ &+ A_2(p,q)P_{p+1,q-1} + A_3(p,q)P_{p-1,q}\\ &+ A_4(p,q)P_{p-2,q+2} + A_5(p,q)P_{p,q-2}, \end{aligned}$$

where

$$\begin{aligned} A_1(p,q) &= 1 + \alpha_1(p,q) - \alpha_0(p,q)a_1(\lambda,p+1)\\ &\quad + a_1(\lambda,p)\alpha_0(p-1,q+1) + a_1(\lambda,p+1) + a_1(\lambda,p),\\ A_2(p,q) &= -a_1(\lambda,q) + a_2(\lambda,p+1,q) + a_2(\lambda,p,q) + \alpha_2(p,q)\\ &\quad - \alpha_0(p,q)a_2(\lambda,p+1,q) + a_2(\lambda,p,q)\alpha_0(p,q-1),\\ A_3(p,q) &= -a_2(\lambda,q,p) - a_1(\lambda,p)a_2(\lambda,p-1,q+1) - a_2(\lambda,p,q)a_1(\lambda,p)\\ &\quad + a_1(\lambda,p)\alpha_2(p-1,q+1) + \alpha_1(p,q-1)a_2(\lambda,p,q)\\ &\quad - \alpha_1(p,q)a_2(\lambda,p-1,q+1) - \alpha_2(p,q)a_1(\lambda,p),\\ A_4(p,q) &= a_1(\lambda,p)\alpha_1(p-1,q+1) - a_1(\lambda,p-1)\alpha_1(p,q)\\ &\quad - a_1(\lambda,p)a_1(\lambda,p-1),\\ A_5(p,q) &= a_2(\lambda,p,q)\alpha_2(p,q-1) - a_2(\lambda,p,q-1)\alpha_2(p,q)\\ &\quad - a_2(\lambda,p,q)a_2(\lambda,p,q-1). \end{aligned}$$

A simple calculation shows that

$$\begin{aligned}&1+\alpha_1(p,q)=\alpha_1(p+1,q),\\&A_1(p,q)=\alpha_1(p+1,q),\qquad A_2(p,q)=\alpha_2(p+1,q),\\&A_3(p,q)=A_4(p,q)=A_5(p,q)=0,\end{aligned}$$

which concludes the induction formula for $\Gamma(z, P_{p+1,q})$. The same method leads to the result for $\Gamma(\bar{z}, P_{p+1,q})$, and exchanging p and q in the above amounts to exchanging z and $\bar{z}$.

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