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LIMIT SETS IN NORMED LINEAR SPACES

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Abstract. The sets of all limit points of series with terms tending to 0 in normed linear spaces are characterized. An immediate conclusion is that a normed linear space $(X, \|\cdot\|)$ is infinite-dimensional if and only if there exists a series $\sum x_n$ of terms of X with $x_n \to 0$ whose set of limit points contains exactly two different points of X. The last assertion could be extended to an arbitrary (greater than 1) finite number of points.

Introduction. Let (x_n) be a sequence in a topological space X. A point $x \in X$ is a *limit point* of (x_n) if every neighbourhood of x contains infinitely many terms of this sequence. The set of all limit points of (x_n) is called its *limit set*.

In the following we discuss normed linear spaces, real or complex, equipped with topologies induced by their norms. A series in a normed linear space can be considered as the sequence of its partial sums, and limit points of the series are limit points of this sequence.

Every separable subspace Y of a space X is clearly the limit set of a sequence in X. The situation changes if we restrict our consideration to sequences (x_n) satisfying $\|x_{n+1} - x_n\| \to 0$ or, equivalently, to series $\sum x_n$ of terms of X with $x_n \to 0$. It is known that for $X = \mathbb{R}$ the limit sets of such series are exactly closed subintervals of $[-\infty, \infty]$, where $[-\infty, \infty]$ stands for the two-point compactification of $\mathbb{R}$. Likewise in an arbitrary metric space, under some additional assumptions, for instance compactness of the sequence, the limit set must be connected (see [2], [14]). Recall that a sequence is said to be *compact* provided that each of its subsequences has a convergent subsequence. On the other hand, if the dimension of a normed linear space X is finite and greater than 1, then any pair of lines or even closed halflines in X can be the limit set of some series $\sum x_n$ of terms of X with $x_n \to 0$; in other words the limit sets of such series can be disconnected (see e.g. [20]). At the same time, in $\mathbb{R}^2 \cup \{\infty\}$ (or, equivalently, $\mathbb{C} \cup \{\infty\}$) with the topol-

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ogy of the one-point compactification of $\mathbb{R}^2$, the set of limit points of every series $\sum x_n$ with $x_n \in \mathbb{R}^2$, $x_n \to 0$, is connected (see [4, Theorem 11]). The properties of complex series and their transformations were deeply investigated by Jasek [4–6] whose works are not commonly known although they definitely deserve remembrance and propagation. Moreover, [4] provides a broad historical overview of the subject.

In the present paper we investigate the topological structure of the limit sets of series $\sum x_n$ with $x_n \to 0$ in normed linear spaces, paying special attention to the infinite-dimensional case.

1. Sequences, series and continuous images of a halfline. The following theorem states an equivalence between limit sets of sequences, series and some special subsets of closures of continuous images of a halfline.

THEOREM 1.1. *Let $(X, \|\cdot\|)$ be a normed linear space and let Y be a nonempty subset of X. The following conditions are equivalent:*

(1) *There exists a sequence $(x_n) \subset X$ whose set of limit points is Y and such that*
$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$
(2) *There exists a series $\sum x_n$ of terms of X with $x_n \to 0$ whose set of limit points is Y.*
(3) *There is a continuous mapping $\gamma : [0,\infty) \to X$ such that*
$$Y = \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty))).$$

Proof. The equivalence (1)⇔(2) is obvious.

Observe that

$$z \in \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty))) \Leftrightarrow \forall\epsilon > 0\ \forall n \in \mathbb{N}\ \exists s \in [n,\infty)\ \ \|\gamma(s) - z\| < \epsilon. \tag{1.1}$$

To show (1)⇒(3), define $\gamma : [0,\infty) \to X$ by setting

$$\gamma(n + t) = (1 - t)x_n + tx_{n+1}$$

for each $n \in \mathbb{N} \cup \{0\}$ and $t \in [0, 1]$. Clearly, $Y \subset \mathrm{cl}(\gamma([n,\infty)))$ for every $n \in \mathbb{N} \cup \{0\}$. To show the converse inclusion, fix $\epsilon > 0$. By (1.1), if $z \in \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty)))$ then for every $n \in \mathbb{N}$ there exists $s_n \geq n$ such that $\|\gamma(s_n) - z\| < \epsilon$. Denote by k_n and t_n the integer part and the fractional part of s_n, respectively. We thus obtain two sequences $(k_n) \in \mathbb{N} \cup \{0\}$ and $(t_n) \in [0, 1]$ satisfying $\|\gamma(k_n + t_n) - z\| < \epsilon$ and $k_n \geq n$ for each $n \in \mathbb{N} \cup \{0\}$. Since

$$\|x_{k_n} - z\| \leq \|\gamma(k_n) - \gamma(k_n + t_n)\| + \|\gamma(k_n + t_n) - z\| < t_n\|x_{k_n} - x_{k_n+1}\| + \epsilon$$

and, by the assumption,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

we deduce that z is a limit point of the sequence (x_n). Therefore $Y = \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty)))$.

To prove (3)⇒(1), define a sequence (x_n) inductively. Set $x_0 = \gamma(0)$. Next pick finitely many numbers from every interval $[n, n+1]$ and take their values under γ as the consecutive elements of the sequence. Suppose that the x_i have been defined for all $i \in \{0, 1, \ldots, m\}$ and that $x_m = \gamma(n)$ for some $n \in \mathbb{N}$. Since $\gamma|_{[n,n+1]}$ is uniformly continuous, there is a positive integer k such that if $|t-t'| < 1/k$ then $\|\gamma(t)-\gamma(t')\| < 1/n$ for any $t, t' \in [n, n+1]$. Set $x_{m+i} = \gamma(n+i/k)$ for all $i \in \{1, \ldots, k\}$. It is easily seen that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$ and that the limit set of (x_n) is contained in Y. On the other hand, for each $s \in [n, n+1]$ one can find $k_n \in \mathbb{N}$ such that $x_{k_n} \in [n, n+1]$ and $\|\gamma(s)-x_{k_n}\| < 1/n$. Applying (1.1) again, we conclude that Y is a subset of the limit set of (x_n). ■

A consequence of Theorem 1.1 is the following classical result that can also be derived from the Riemann Rearrangement Theorem (see [18] or [16]) or its generalizations (see [19, Section 4]).

COROLLARY 1.2. *Let $(x_n) \subset \mathbb{R}$ be a sequence converging to* 0. *Then the limit set of $\sum x_n$ is a closed connected subset of $[-\infty, \infty]$. Moreover, every closed connected subset of $[-\infty, \infty]$ is the limit set of such a series.*

2. Infinite-dimensional spaces

THEOREM 2.1. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed linear space and let Y be a nonempty subset of X. Then the conditions* (1)–(3) *of Theorem* 1.1 *are equivalent to*

(4) *Y is a closed and separable subset of X.*

Proof. (3)⇒(4). The set Y is closed, as the intersection of closed sets. Since $[0,\infty)$ is separable and γ is continuous, we see that $\gamma([0,\infty))$, $\mathrm{cl}(\gamma([0,\infty)))$ and $Y \subset (\mathrm{cl}\,\gamma([0,\infty)))$ are separable as well.

(4)⇒(3). Let A be a countable dense subset of Y. Consider an infinite sequence (y_n) whose range is equal to A and such that each element of A occurs infinitely many times in the sequence. By the Riesz lemma (see [3, Chapter I]) there exists a sequence $(z_n) \subset X$ of norm 1 vectors satisfying

$$\forall n \in \mathbb{N} \quad \mathrm{dist}(z_{n+1}, Z_n) > 0.5 \tag{2.1}$$

where Z_n denotes the linear subspace of X spanned by $\{y_1, z_1, y_2, z_2, \ldots, y_n, z_n, y_{n+1}\}$.

Set $y_0 = y_1$ and consider the mapping $\gamma : [0,\infty) \to X$ which is linear on every segment $[n, n+1]$, $n \in \mathbb{N} \cup \{0\}$, and $\gamma(2k) = y_k$, $\gamma(2k+1) = z_{k+1}$ for each $k \in \mathbb{N} \cup \{0\}$.

It is obvious that $Y \subset \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty)))$.

Assume that $x \in \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty)))$. By (1.1) the point x is the limit of a sequence $x_n = (1-\lambda_n)y_n + \lambda_n z_{\alpha(n)}$ for n running through some infinite set $N \subset \mathbb{N}$, $\alpha(n) \in \{n, n+1\}$, $\lambda_n \in (0,1]$. Consider the case of $\alpha(n) = n$ for all $n \in N$; the proof for $\alpha(n) = n+1$ is analogous. For $n, m \in N$ and $n > m$ we have

$$\|x_n - x_m\| = \lambda_n \left\| z_n - \left(\frac{\lambda_n - 1}{\lambda_n} \cdot y_n + \frac{1-\lambda_m}{\lambda_n} \cdot y_m + \frac{\lambda_m}{\lambda_n} \cdot z_m \right) \right\|.$$

Note that $\frac{\lambda_n - 1}{\lambda_n} \cdot y_n + \frac{1-\lambda_m}{\lambda_n} \cdot y_m + \frac{\lambda_m}{\lambda_n} \cdot z_m \in Z_{n-1}$. By (2.1) we obtain

$$\|x_n - x_m\| \geq \lambda_n \cdot 0.5,$$

so (λ_n) tends to zero. Since $\|(1-\lambda_n)y_n\| \leq \|x_n\| + \lambda_n\|z_n\|$ and the sequences $(\|x_n\|)$, $(\|z_n\|)$ are bounded, it follows that $(\|y_n\|)$ is bounded as well (all the sequences for $n \in N$). Finally,

$$\|x_n - y_n\| = \|(1-\lambda_n)y_n + \lambda_n z_n - y_n\| \leq \lambda_n(\|y_n\| + \|z_n\|) \to 0,$$

so $x \in Y$. ∎

COROLLARY 2.2. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed linear space and let $Y \neq \emptyset$ be a closed countable (finite or infinite) subset of X. Then there exists a series $\sum x_n$ with terms in X such that $x_n \to 0$ and whose limit set is equal to Y.*

Interesting examples can be found in [8], [15], [20], [22], [11] (see also [9], [7], [16], [18]). Among them there are series all of whose rearrangements (i.e. permutations of the terms) have discrete limit sets [20] or even exactly two limit points [8]. Series $\sum x_n$ in incomplete normed linear spaces have interesting properties concerning convergence of $\sum \|x_n\|^p$ for $p > 0$. By [17, Theorem 1], if X is a normed linear space, (x_n) is a Cauchy sequence in X, and $(\alpha_n) \subset (0,\infty)$ is an arbitrary sequence converging to 0 and satisfying $\|x_{n+1} - x_n\| < \alpha_n$, $n \in \mathbb{N}$, then there exists a sequence $(y_n) \subset X$ such that $\|x_n - y_n\| \to 0$ and $\|y_{n+1} - y_n\| = \alpha_n$ for all $n \in \mathbb{N}$. In this situation the sequence (y_n), which is formally a series $y_1 + \sum_{n=1}^{\infty}(y_{n+1} - y_n)$, is either convergent or has no limit points; anyway the sequences (x_n) and (y_n) have the same limit in the completion $\tilde{X}$ of the space X. In particular, if X is incomplete then for any divergent Cauchy sequence (x_n) and every $p > 0$ one can choose sequences $(y_n), (\tilde{y}_n)$ in X that both converge in $\tilde{X}$ to the limit of (x_n) but $(\|y_{n+1} - y_n\|) \in l^p \setminus \bigcup_{0<q<p} l^q$ and $(\|\tilde{y}_{n+1} - \tilde{y}_n\|) \in \bigcap_{q>p} l^q \setminus l^p$ (see [21, Proposition 1.2]).

3. Finite-dimensional spaces. It is well known that every n-dimensional linear space over $\mathbb{R}$ or $\mathbb{C}$ is homeomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ (and thus to $\mathbb{R}^{2n}$), respectively. The one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ is homeomorphic to the sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

In the following, by $\dim X$ we mean the topological dimension of X (e.g. the covering dimension); in particular $\dim \mathbb{C}^n = \dim \mathbb{R}^{2n} = 2n$.

A subset $A \subset X$ is a *retract* of a space X provided that there is a continuous function $r : X \to A$, called a *retraction*, such that $r|_A = \mathrm{id}_A$. A metric space X is called an *absolute neighbourhood retract* (ANR) if for every metric space Y containing X as a closed subspace there is a neighbourhood U of X in Y such that X is a retract of U. Every normed linear space is an ANR. It is easily seen that if Z is an ANR and X is a retract of an open subset of Z then X is an ANR too. In particular the sphere $S^n \subset \mathbb{R}^{n+1}$ is an ANR for every $n \in \mathbb{N}$. We use this fact in the proof of the theorem below. Recall that a *continuum* is a compact connected metric space.

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a finite-dimensional normed linear space and let Y be a nonempty subset of X. If $\dim X \geq 2$ then the conditions* (1)–(3) *of Theorem* 1.1 *are equivalent to each of the following:*

(5) *either Y is a continuum, or $Y \cup \{\infty\}$ is a continuum in the one-point compactification $X \cup \{\infty\}$ of X,*

(6) *either Y is a continuum, or Y is closed in X and every component of Y is unbounded.*

Proof. The equivalence (5)⇔(6) is easily seen.

(3)⇒(5). Take $\gamma^* : [0,\infty) \to X \cup \{\infty\}$ defined by $\gamma^*(t) = \gamma(t)$ for $t \in [0,\infty)$. Then $\mathrm{cl}(\gamma^*([n,\infty)))$ is a compact connected subset of the compact space $X \cup \{\infty\}$ for every $n \in \mathbb{N}$. Since the intersection of a descending sequence of continua is a continuum (see [10], §47, II, Theorem 5), the set

$$Y^* = \bigcap_{n=0}^{\infty} \mathrm{cl}\big(\gamma^*([n,\infty))\big)$$

is a subcontinuum of Y^*. Therefore $Y = Y^* \setminus \{\infty\}$ satisfies (5).

(5)⇒(3). If Y is a continuum, set $Y^* = Y$, otherwise set $Y^* = Y \cup \{\infty\}$. Thus Y^* is a continuum. By [1] there exists a compactification C of the halfline $[0,\infty)$ having Y^* as the remainder. Let $\gamma_0 : [0,\infty) \to C$ be an embedding. Consider the space $Z = X \cup \{\infty\} \cup C$. Let d be an admissible metric on Z. Note that for every $\delta > 0$ one can find a number $n \in \mathbb{N}$ such that $\gamma_0([n,\infty)) \subset B(Y^*,\delta) = \{x \in Z : d(x, Y^*) < \delta\}$.

Since $X \cup \{\infty\}$ is an ANR, there is a retraction $r : X \cup \{\infty\} \cup C \to X \cup \{\infty\}$. Note that Y^* and $\gamma_0([0,\infty))$ are disjoint, $\dim X \geq 2$, so r can be done in such a way that $r(\gamma_0([0,\infty))) \subset X$. Finally define $\gamma(t) = r(\gamma_0(t))$ for each $t \in [0,\infty)$. It is easily seen that $Y^* \subset \bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty)))$. To show the converse implication, take $z \notin Y^*$ and set $\lambda = d(z, Y^*)$. Since Y^* is compact, $\lambda > 0$. The retraction r is uniformly continuous, so there is a number $\delta > 0$ such that $d(r(x), r(y)) < \lambda/2$ whenever $d(x,y) < \delta$. Recall that $\gamma_0([n,\infty)) \subset B(Y^*, \delta)$ for some $n \in \mathbb{N}$. Therefore

$$\gamma([n,\infty)) = r(\gamma_0([n,\infty))) \subset r(B(Y^*,\delta)) \subset B(Y^*, \lambda/2) \cap (X \cup \{\infty\}).$$

This yields

$$d(z, \gamma([n,\infty))) \geq d(z, Y^*) - d(\gamma([n,\infty)), Y^*) > \lambda - \lambda/2,$$

so $z \notin \mathrm{cl}(\gamma([n,\infty)))$. Thus γ satisfies (3). ∎

Remark 3.2. Theorem 3.1 was established in 1993 by W. J. Charatonik. A few days before submitting the present paper we found that parts of Theorem 3.1 have been independently proven by Nash-Williams and White. Namely, (3.1) coincides with [12, Proposition 3.1] and (6) can be derived from [13, Corollaries 4.11–4.12]. The results obtained by Nash-Williams and White are stronger: depending on the properties of a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, subsets of $\mathbb{R}^n \cup \{\infty\}$, $n \geq 2$, are described that can be obtained as the limit sets of $\sum a_{\sigma(n)}$ for some series $\sum a_n$ converging to 0. Nevertheless, the methods used in [12] and [13] are different and much more complicated than the ones employed in the present paper.

Corollary 3.3. *Let Y be a subset of a finite-dimensional normed linear space X and let Y have at least two elements. If Y is the limit set of a series $\sum x_n$ with terms in X such that $x_n \to 0$ then Y is a perfect set.*

Combining Theorem 3.1 and Corollary 2.2 leads to the following observation.

Corollary 3.4. *If $(X, \|\cdot\|)$ is a normed linear space and* $\dim X \geq 2$ *then every union Y of lines or closed halflines that is closed in X is the limit set of a series $\sum x_n$ in X with $x_n \to 0$. In particular, the union of finitely many lines or closed halflines is the limit set of such a series.*

Remark 3.5. We are convinced, but we cannot provide a proof, that the series $\sum x_n$ as in Corollary 3.4 always admits a rearrangement whose limit set is $X_0 \cup \{\infty\}$ where X_0 stands for the closed linear subspace of X spanned by Y.

The following theorem is a consequence of Theorem 3.1 and Corollaries 1.2 and 2.2.

Theorem 3.6. *Let $(X, \|\cdot\|)$ be a normed linear space. Then X is infinite-dimensional if and only if there exists a series $\sum x_n$ in X with* $\lim_{n\to\infty} x_n = 0$ *and whose limit set is finite of cardinality at least* 2.

According to Theorem 1.1, the above statement is equivalent to the following one:

THEOREM 3.7. *Let $(X, \|\cdot\|)$ be a normed linear space. Then X is infinite-dimensional if and only if there is a continuous mapping $\gamma : [0,\infty) \to X$ such that the set $\bigcap_{n=0}^{\infty} \mathrm{cl}(\gamma([n,\infty)))$ consists of exactly two points.*

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REFERENCES

[1] J. M. Aarts and P. van Emde Boas, *Continua as remainders in compact extensions*, Nieuw Arch. Wisk. 15 (1967), 34–37.

[2] H. G. Barone, *Limit points of sequences and their transforms by methods of summability*, Duke Math. J. 5 (1939), 740–752.

[3] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.

[4] B. Jasek, *Transformations of complex series*, Colloq. Math. 9 (1962), 265–275.

[5] B. Jasek, *Complex series and connected sets*, Dissertationes Math. 52 (1966).

[6] B. Jasek, *On a certain transformation of complex series*, Ann. Polon. Math. 22 (1970), 277–289.

[7] M. I. Kadets and V. M. Kadets, *Series in Banach Spaces. Conditional and Unconditional Convergence*, Birkhäuser, Basel, 1997.

[8] M. I. Kadets and K. Woźniakowski, *On series whose permutations have only two sums*, Bull. Polish Acad. Sci. Math. 37 (1989), 15–21.

[9] V. M. Kadets, *How many points can the range of sums of a series in a Banach space contain?*, Teor. Funktsiĭ Funktsional. Anal. Prilozh. 54 (1990), 54–57 (in Russian).

[10] K. Kuratowski, *Topology*, Vol. II, PWN, Warszawa, 1968.

[11] S. Levental, V. Mandrekar and S. A. Chobanyan, *Towards Nikishin's theorem on the almost sure convergence of rearrangements of functional series*, Funct. Anal. Appl. 45 (2011), 33–45.

[12] C. St. J. A. Nash-Williams and D. J. White, *Rearrangement of vector series. I*, Math. Proc. Cambridge Philos. Soc. 130 (2001), 89–109.

[13] C. St. J. A. Nash-Williams and D. J. White, *Rearrangement of vector series. II*, Math. Proc. Cambridge Philos. Soc. 130 (2001), 111–134.

[14] R. Niechajewicz, *Sets of limit points of compact sequences in metric spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 25 (1977), 251–253.

[15] O. S. Osipov, *The integral analog of a series with a two-point sum range*, Sibirsk. Math. Zh. 50 (2009), 1348–1355 (in Russian).

[16] F. Prus-Wiśniowski, *Two refinements of the Riemann Deragement Theorem*, in: Real Functions, Density Topology and Related Topics, M. Filipczak and E. Wagner-Bojakowska (eds.), Łódź Univ. Press, Łódź, 2011, 165–172.

[17] R. Wituła, *Divergent vector sequences y_n with $\Delta y_n \to 0$*, Colloq. Math. 107 (2007), 263–266.

[18] R. Wituła, *The Riemann Derangement Theorem and divergent permutations*, Tatra Mt. Math. Publ. 52 (2012), 75–82.

[19] R. Wituła, *The family $\mathfrak{F}$ of permutations of* $\mathbb{N}$, Math. Slovaca 65 (2015), 1457–1474.

[20] R. Wituła, E. Hetmaniok and K. Kaczmarek, *On series whose rearrangements possess discrete sets of limit points*, J. Appl. Anal. 20 (2014), 93–96.

[21] R. Wituła and D. Słota, *On some new subfamilies of classical spaces of absolutely p-summable sequences*, Tatra Mt. Math. Publ. 49 (2011), 27–48.
[22] J. O. Wojtaszczyk, *A series whose sum range is an arbitrary finite set*, Studia Math. 171 (2005), 261–281.

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