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On-line Packing Cubes into n Unit Cubes

by

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Summary. If $n \geq 3$ and $d \in \{3,4\}$ or if $n \geq 1$ and $d \geq 5$, then any sequence of d-dimensional cubes of edge lengths not greater than 1 whose total volume does not exceed $(n+1) \cdot 2^{-d}$ can be on-line packed into n unit d-dimensional cubes.

1. Introduction. For $i = 1, 2, \dots$ let $Q_i = \lambda_i I$, where $\lambda_i \in (0,1]$ and $I = [0,1]^d$. We say that the cubes $Q_1, Q_2, \dots$ *can be packed* (in parallel way) into a domain $D \subset \mathbb{R}^d$ if there are $\sigma_i \in \mathbb{R}^d$ such that $\bigcup(\sigma_i + Q_i) \subseteq D$ and $\sigma_i + Q_i$ have pairwise disjoint interiors. By an *on-line* packing we mean a packing in which the members of a sequence of cubes Q_i are revealed one by one. First we only know λ_1 but we do not know $\lambda_2, \lambda_3, \dots$. We choose the appropriate σ_1 and pack Q_1. For $i > 1$, we learn λ_{i+1} only when $\sigma_1, \dots, \sigma_i$ have been defined, i.e., we do not know what Q_{i+1} is before we assign a position of Q_i, which cannot be changed afterwards. Surveys of results concerning packings and on-line packings are given in [1], [5] and [9].

Januszewski [7] proved that any sequence of squares of side lengths not greater than 1 whose total area does not exceed $\frac{1}{4}(n+1)$ can be on-line packed into n pairwise disjoint squares of sides of length 1 provided $n \geq 3$. Note that it is an open question whether this holds for $n = 2$ and $n = 1$. For $n = 1$, the following upper bounds of total area of squares of side lengths not greater than 1 which can be on-line packed into the unit square were found: 5/16 [8], 1/3 [6], 11/32 [4], 3/8 [2] and 2/5 [3].

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We consider the problem of on-line packing of d-dimensional cubes into n unit d-dimensional cubes. Let $I_j = \tau_j + [0,1]^d$, where $\tau_j \in \mathbb{R}^d$ for $j = 1, \ldots, n$ be pairwise disjoint cubes. Moreover, let $J_n = I_1 \cup \cdots \cup I_n$.

Observe that $n+1$ cubes $(1/2+\epsilon)\cdot I$ (of total volume greater than $(n+1)\cdot 2^{-d}$) cannot be packed into J_n for any $\epsilon > 0$. The reason is that the interior of any cube $(1/2+\epsilon)\cdot I$ packed into a unit cube I_k contains the center of I_k.

The aim of this paper is to show that if either $n \geq 3$ and $d \in \{3,4\}$ or if $n \geq 1$ and $d \geq 5$, then any sequence of d-dimensional cubes of edge lengths not greater than 1 whose total volume does not exceed $(n+1)\cdot 2^{-d}$ can be on-line packed into n unit d-dimensional cubes.

2. Containers. In the main packing method some small cubes Q_i will first be packed into special cubes P_i, and then $P_i \supset Q_i$ will be packed into J_n by the method described in this section.

Let $l \in \{2,3\}$. For each positive integer p, by an (l,p)-*cube* we mean the cube $\frac{2}{l\cdot 2^p} I$. Let w be a positive integer and let A be the union of $(l,1)$-cubes $A_1, \ldots, A_w$ with pairwise disjoint interiors. We call these cubes $(l,1)$-*containers*. For each positive integer q any (l,q)-container can be dissected into 2^d congruent $(l,q+1)$-cubes also called $(l,q+1)$-*containers*. Let us number all $(l,2)$-containers contained in A_k (for $k = 1, \ldots, w$) with integers from $(k-1)\cdot 2^d + 1$ to $k \cdot 2^d$. Furthermore, for each q, all $(l,q+1)$-containers contained in an (l,q)-container whose number is m are numbered with the integers $(m-1)\cdot 2^d + 1, \ldots, m \cdot 2^d$.

We present a method of the on-line packing of sequences of (l,p_i)-cubes into A.

We just pack every (l,p_i)-cube of the sequence in the congruent (l,p_i)-container of A with the smallest possible number. By an *empty* (l,p_i)-*container* we mean an (l,p_i)-container whose interior has an empty intersection with all cubes packed before. We stop the packing process if a successive (l,p_i)-cube in the sequence cannot be packed, i.e., if no empty (l,p_i)-container of A exists. We call this approach the *method of the first fitting container*.

The following proposition says that the above method is extremely efficient. The volume of A is denoted by $|A|$.

Proposition 2.1. *Every sequence of (l,p_i)-cubes whose total volume is smaller than or equal to $|A|$ can be on-line packed in A by the method of the first fitting container.*

Proof. Assume that the total volume of the (l,p_i)-cubes in the sequence is not greater than $|A|$ and that the packing procedure stops when we wish to pack an (l,r)-cube. Clearly the volume of this cube is $(2/l)^d \cdot 2^{-dr}$. Since every (l, p_i)-cube has been packed in the first fitting container, we conclude that there is no empty (l, u)-container for any $u < r$. Moreover, there are at most $2^d - 1$ empty containers of every size $(l, r+1), (l, r+2), \ldots$ at this time. Since a finite number of (l, p_i)-cubes have been packed, the number of those empty (l, p_i)-containers is finite. Thus the sum of the volumes of the empty (l, p_i)-containers is smaller than

$$(2^d - 1)(2/l)^d(2^{-d(r+1)} + 2^{-d(r+2)} + \cdots) = (2/l)^d \cdot 2^{-dr}.$$

Consequently, the total volume of the (l, p_i)-cubes packed up to now is greater than $|A| - (2/l)^d \cdot 2^{-dr}$. Since we have just obtained an (l, r)-cube of volume $(2/l)^d \cdot 2^{-dr}$, the total volume of the (l, p_i)-cubes in the sequence is greater than $|A|$, which is a contradiction. ∎

3. Packing algorithm. Let $d \geq 3$ and let (Q_i) be a sequence of cubes $Q_i = q_i I$, where $q_i \in (0, 1]$. We consider the following types of cubes:

- Q_i is *very big* if $q_i > 2/3$;
- Q_i is *big* if $1/2 < q_i \leq 2/3$;
- other cubes are *small*; a small cube Q_i is
  - 2-*small* if $q_i \in \bigcup_{j=1}^{\infty}(2/3 \cdot 2^{-j},\ 2^{-j}]$;
  - 3-*small* if $q_i \in \bigcup_{j=1}^{\infty}(2^{-1-j},\ 2/3 \cdot 2^{-j}]$.

A unit cube $I_k \subset J_n$ is said to be *empty* if no cube has been packed into it; a 2-*cube* if a 2-small cube has been packed into it; a 3-*cube* if a 3-small cube has been packed into it; a *v-cube* if a very big cube has been packed into it; and a *b-cube* if a big cube has been packed into it and no other cube has been packed into it. However, if a 2-small cube has been packed into a b-cube I_k, then I_k is no longer a b-cube: it becomes a 2-cube. Moreover, if a 3-small cube has been packed into a b-cube I_k, then I_k is no longer a b-cube: it becomes a 3-cube.

In each of the unit cubes $I_k \subset J_n$ we select one of the vertices and denote it by v_k. Let F_k be the cube of edge length $3/4$ such that F_k is contained in a 2-cube I_k and one of the vertices of F_k is a vertex v_k of I_k. We partition any 2-cube I_k into 4^d $(2, 2)$-containers. We order them so that the $(2, 2)$-containers contained in $I_k \setminus F_k$ precede those contained in F_k. Let G_k be the cube of edge length $2/3$ such that G_k is contained in a 3-cube I_k and one of the vertices of G_k is v_k. We partition any 3-cube I_k into 3^d $(3, 1)$-containers. We order them so that the $(3, 1)$-containers contained in $I_k \setminus G_k$ precede those contained in G_k.

Packing very big cubes. If Q_i is very big, then we find the greatest $k \in \{1, \ldots, n\}$ such that I_k is empty and pack Q_i into I_k. Now I_k is a v-cube. No other cube will be packed into this v-cube.

Packing big cubes. A big cube Q_i will be packed into $I_k \subset J_n$ so that one vertex of $\sigma_i + Q_i$ is v_k. If Q_1 is big, then we pack it into I_1. Now I_1 is a b-cube. Assume that $i > 1$ and Q_i is big. If there is a 3-cube into which Q_i can be packed, then we pack Q_i into that cube. Now any $(3, 1)$-container contained in G is non-empty and I_k is still a 3-cube. Otherwise, if there is an empty unit cube of J_n, then we find the smallest $k \in \{1, \dots, n\}$ such that I_k is empty and we pack Q_i into it; now I_k is a b-cube. If there is no empty unit cube I_k and if there is a 2-cube I_k into which Q_i can be packed, then we pack Q_i into it. Now any $(2, 2)$-container contained in F_k is non-empty and I_k is still a 2-cube.

Packing 2-small cubes. If Q_1 is 2-small, then we pack it into I_1. If $1/3 < q_1 \le 1/2$, then we pack Q_1 so that one vertex of $\sigma_1 + Q_1$ is a vertex $v \ne v_1$ of I_1 and so that $\sigma_1 + Q_1$ has a non-empty intersection with the empty $(2, 2)$-container with the smallest possible number (i.e., with number 1 when we pack Q_1). If there is no vertex $v \ne v_1$ of I_1 at which Q_1 can be packed, then we pack this cube at the vertex v_1. The packed cube $\sigma_1 + Q_1$ is contained in the union of 2^d $(2, 2)$-containers. Now these containers are non-empty. If $q_1 \in \bigcup_{j=2}^{\infty}(2/3 \cdot 2^{-j}, 2^{-j}]$, then we find the smallest $(2, p)$-container P_1 containing Q_1 and we pack P_1, and hence also $Q_1 \subset P_1$, into I_1 by the method of the first fitting container. Clearly, I_1 is now a 2-cube. Assume that $i > 1$ and Q_i is 2-small. If there is a 2-cube into which Q_i can be packed, then we pack Q_i in the same way as Q_1. Otherwise, if there is an empty unit cube of J_n, then we find the smallest $k \in \{1, \dots, n\}$ such that I_k is empty and pack Q_i into I_k in the same way as we packed Q_1. Now I_k is a 2-cube. If there is no empty unit cube in J_n and if there is a b-cube I_k into which Q_i can be packed, then we pack Q_i into it. Now I_k is a 2-cube and any $(2, 2)$-container contained in F_k is non-empty.

Packing 3-small cubes. If Q_1 is 3-small, then we find the smallest $(3, p)$-container R_1 containing Q_1 and we pack R_1, and hence also $Q_1 \subset R_1$, into I_1 by the method of the first fitting container. Clearly, I_1 is now a 3-cube. Assume that $i > 1$ and Q_i is 3-small. If there is a 3-cube into which the smallest $(3, p)$-container R_i containing Q_i can be packed, then we pack R_i (together with Q_i) into this 3-cube by the method of the first fitting container. Otherwise, we find the smallest $k \in \{1, \dots, n\}$ such that I_k is either empty or a b-cube. We pack R_i together with Q_i into I_k by the method of the first fitting container. Now I_k is a 3-cube.

4. Efficiency of the packing algorithm

LEMMA 4.1. *Assume that there is no big cube in a sequence. Denote by* n_2 *the number of* 2*-cubes in* J_n*. If a sequence of* 2*-small cubes cannot be* *on-line packed into* 2*-cubes by the method described in Section* 3*, then the total volume of the cubes exceeds* $n_2 \cdot (2/3)^d$.

Proof. Let (Q_i) be a sequence of 2-small cubes as in the statement. Denote by Q_z the first cube from the sequence which cannot be packed into 2-cubes.

For every Q_i we find the smallest $(2, p_i)$-cube P_i containing Q_i. Since Q_z cannot be packed into 2-cubes, we deduce by Proposition 2.1 that

$$\sum_{i=1}^{z} |P_i| > n_2.$$

Moreover

$$|Q_i| = q_i^d > \left(\frac{2}{3 \cdot 2^{p_i}}\right)^d = \left(\frac{2}{3}\right)^d \cdot \left(\frac{1}{2^{p_i}}\right)^d = \left(\frac{2}{3}\right)^d |P_i|.$$

Thus

$$\sum_{i=1}^{z} |Q_i| > \left(\frac{2}{3}\right)^d \cdot \sum_{i=1}^{z} |P_i| > n_2 \left(\frac{2}{3}\right)^d. \ \blacksquare$$

LEMMA 4.2. *Denote by* n_2 *the number of* 2*-cubes in* J_n*. If a sequence of* 2*-small cubes and big cubes cannot be on-line packed into* 2*-cubes by the method described in Section* 3*, then the total volume of the cubes exceeds* $(n_2 + 1) \cdot 2^{-d}$.

Proof. Let (Q_i) be a sequence of 2-small cubes and big cubes as in the statement. Denote by Q_z the first cube from the sequence which cannot be packed into 2-cubes.

If a big cube is packed into a 2-cube, then the total volume of the cubes packed into this 2-cube is greater than $(1/2)^d$. Denote by m_b the number of big cubes packed into 2-cubes.

CASE 1: Q_z *is big.* Obviously, $q_z^d > (1/2)^d$.

SUBCASE 1a: $m_b = 0$. By Lemma 4.1 the total volume of the cubes packed into 2-cubes is greater than $(n_2 - 1)(2/3)^d$. It is easy to verify that $(2/3)^d > 2(1/2)^d$ for $d \geq 3$. If $n_2 > 1$, then

$$\sum_{i=1}^{z} |Q_i| > (n_2 - 1)\left(\frac{2}{3}\right)^d + q_z^d > n_2\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d = (n_2 + 1)\left(\frac{1}{2}\right)^d.$$

If $n_2 = 1$ and if there is a 2-small cube Q_w such that $q_w + q_z > 1$, then $q_w^d + q_z^d > (1 - q_z)^d + q_z^d$. Set $\varphi(q) = (1 - q)^d + q^d$. The function $\varphi(q)$ has a global minimum at $q_0 = 1/2$. Thus the total volume of the cubes packed into a 2-cube is greater than

$$\varphi(q_z) - q_z^d > \varphi(q_0) - q_z^d = 2\left(\frac{1}{2}\right)^d - q_z^d.$$

If $n_2 = 1$ and if there is no 2-small cube Q_w such that $q_w + q_z > 1$, then the total volume of the cubes packed into the 2-cube is greater than $(1 - (2/3)^d)(2/3)^d$. For $d \geq 3$ we have $(1 - (2/3)^d)(2/3)^d > (1/2)^d$. Moreover $(1/2)^d > 2(1/2)^d - q_z^d$. Consequently, if $n_2 = 1$, then

$$\sum_{i=1}^{z} |Q_i| > 2\left(\frac{1}{2}\right)^d - q_z^d + q_z^d = (n_2 + 1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 1b: $m_b \geq 1$. Denote by l the smallest number such that a big cube is packed into a 2-cube I_l. Denote by Q_w the first 2-small cube packed into I_l. Note that Q_w could not be packed into $n_2 - m_b$ 2-cubes into which no big cube is packed. By Lemma 4.1 the total volume of the cubes packed into those 2-cubes into which no big cube is packed plus the volume of Q_w is greater than $(n_2 - m_b)(2/3)^d$. Consequently,

$$\begin{aligned}\sum_{i=1}^{z} |Q_i| &\geq (n_2 - m_b)\left(\frac{2}{3}\right)^d + m_b\left(\frac{1}{2}\right)^d + q_z^d \\ &> (n_2 - m_b)\left(\frac{1}{2}\right)^d + m_b\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d = (n_2 + 1)\left(\frac{1}{2}\right)^d.\end{aligned}$$

CASE 2: Q_z is 2-*small*. Obviously, $q_z^d \leq (1/2)^d$.

SUBCASE 2a: $m_b = 0$. By Lemma 4.1 we get

$$\sum_{i=1}^{z} |Q_i| > n_2\left(\frac{2}{3}\right)^d > (n_2 + 1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 2b: $m_b \geq 1$. Denote by l the greatest number such that a big cube is packed into I_l. Furthermore, denote by Q_w the big cube packed into I_l. If $q_w + q_z > 1$, then $q_w^d + q_z^d > (1 - q_z)^d + q_z^d \geq 2(1/2)^d$. This implies that the total volume of the cubes packed into I_l is greater than $2(1/2)^d - q_z^d$. If $q_w + q_z < 1$, then the total volume of the cubes packed into I_l is greater than

$$\left(1 - \left(\frac{3}{4}\right)^d\right)\left(\frac{2}{3}\right)^d + \left(\frac{1}{2}\right)^d - q_z^d > 2\left(\frac{1}{2}\right)^d - q_z^d.$$

The total volume of the cubes packed into $m_b - 1$ other 2-cubes into which big cubes are packed is greater than or equal to $(m_b - 1)(1/2)^d$. The total volume of the cubes packed into those 2-cubes into which no big cube is packed is greater than or equal to

$$(n_2 - m_b)\left(\left(\frac{2}{3}\right)^d - q_z^d\right) \geq (n_2 - m_b)\left(\left(\frac{2}{3}\right)^d - \left(\frac{1}{2}\right)^d\right) \geq (n_2 - m_b)\left(\frac{1}{2}\right)^d.$$

Consequently,

$$\sum_{i=1}^{z}|Q_i| > (n_2 - m_b)\left(\frac{1}{2}\right)^d + (m_b - 1)\left(\frac{1}{2}\right)^d + 2\left(\frac{1}{2}\right)^d - q_z^d + q_z^d$$
$$= (n_2 + 1)\left(\frac{1}{2}\right)^d. \blacksquare$$

LEMMA 4.3. *Denote by n_3 the number of 3-cubes in J_n. If a sequence (Q_i) of cubes containing both 3-small cubes and big cubes cannot be on-line packed into 3-cubes by the method described in Section 3, then the total volume of the cubes exceeds $n_3 \cdot (3/4)^d$.*

Proof. Let (Q_i) be a sequence of cubes $q_i I$, where $q_i \in \bigcup_{j=1}^{\infty}(2^{-1-j}, 2/3 \cdot 2^{-j}]$. Assume that they cannot be packed into 3-cubes by the method presented in Section 3. Denote by Q_z the first cube from the sequence which cannot be packed into 3-cubes. Furthermore, denote by l_b the number of big cubes packed into 3-cubes.

CASE 1: $l_b = 0$ *and* Q_z *is* 3-*small.* For every Q_i we find the smallest $(3, p_i)$-container R_i containing Q_i. Since Q_z cannot be packed into 3-cubes, we deduce by Proposition 2.1 that $\sum_{i=1}^{z}|R_i| > n_3$. Moreover

$$|Q_i| = q_i^d > \left(\frac{1}{2 \cdot 2^{p_i}}\right)^d = \left(\frac{3}{4}\right)^d \cdot \left(\frac{2}{3 \cdot 2^{p_i}}\right)^d = \left(\frac{3}{4}\right)^d |R_i|.$$

Thus

$$\sum_{i=1}^{z}|Q_i| > \left(\frac{3}{4}\right)^d \cdot \sum_{i=1}^{z}|R_i| > n_3\left(\frac{3}{4}\right)^d.$$

CASE 2: $l_b = 0$ *and* Q_z *is big.* The total volume of the cubes packed into 3-cubes is greater than

$$(n_3 - 1)\left(\frac{3}{4}\right)^d + \left(1 - \left(\frac{2}{3}\right)^d\right)\left(\frac{3}{4}\right)^d = n_3\left(\frac{3}{4}\right)^d - \left(\frac{1}{2}\right)^d.$$

Consequently,

$$\sum_{i=1}^{z}|Q_i| > n_3\left(\frac{3}{4}\right)^d - \left(\frac{1}{2}\right)^d + q_z^d > n_3\left(\frac{3}{4}\right)^d.$$

CASE 3: $l_b \geq 1$. The total volume of the cubes packed into 3-cubes is greater than

$$(n_3 - l_b)\left(\frac{3}{4}\right)^d + l_b\left(1 - \left(\frac{2}{3}\right)^d\right)\left(\frac{3}{4}\right)^d + l_b\left(\frac{1}{2}\right)^d - q_z^d = n_3\left(\frac{3}{4}\right)^d - q_z^d.$$

Consequently,

$$\sum_{i=1}^{z} |Q_i| > n_3\left(\frac{3}{4}\right)^d - q_z^d + q_z^d = n_3\left(\frac{3}{4}\right)^d. \qquad \blacksquare$$

THEOREM 4.4. *If $n \geq 3$, then any sequence of d-dimensional cubes of edge lengths not greater than* 1 *whose total volume does not exceed* $(n+1)\cdot 2^{-d}$ *can be on-line packed into J_n.*

Proof. Let $n \geq 3$ and let (Q_i) be a sequence of d-dimensional cubes as in the statement. We pack the cubes by the method described in Section 3. Contrary to the statement, suppose that it is impossible to pack $Q_1, Q_2, \ldots$ into J_n by this method. Let Q_z be the cube which stops the packing process and let

$$\zeta = \sum_{i=1}^{z} |Q_i|.$$

We show that this leads to the false inequality

$$\zeta > (n+1)\cdot 2^{-d}.$$

Denote by n_2, n_3, n_b, n_v the number of 2-, 3-, b- and v-cubes respectively. Obviously $n_2 + n_3 + n_b + n_v = n$. We consider four cases.

CASE 1: *Q_z is big* $(1/2 < q_z \leq 2/3)$.

SUBCASE 1a: $n_3 \geq 1$ *and* $n_2 = 0$. By Lemma 4.3 we get

$$\zeta > n_3\left(\frac{3}{4}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d > (n_3+1)\left(\frac{1}{2}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d$$
$$\geq (n+1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 1b: $n_2 \geq 1$ *and* $n_3 = 0$. By Lemma 4.2 we get

$$\zeta > (n_2+1)\left(\frac{1}{2}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d \geq (n+1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 1c: $n_3 \geq 1$ *and* $n_2 \geq 1$. The total volume of the cubes packed into 3-cubes is greater than

$$n_3\left(\frac{3}{4}\right)^d - q_z^d > (n_3+1)\left(\frac{2}{3}\right)^d - \left(\frac{2}{3}\right)^d = n_3\left(\frac{2}{3}\right)^d.$$

The total volume of the cubes packed into 2-cubes is greater than $(n_2+1)(1/2)^d$

$-\, q_z^d$. Thus

$$\zeta > n_3\left(\frac{2}{3}\right)^d + (n_2+1)\left(\frac{1}{2}\right)^d - q_z^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d$$
$$\geq (n+1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 1d: $n_3 = 0$ *and* $n_2 = 0$. Obviously $q_z^d > (1/2)^d$. We get

$$\zeta > n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d > (n_b+n_v)\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d = (n+1)\left(\frac{1}{2}\right)^d.$$

CASE 2: Q_z *is very big* ($q_z > 2/3$). Obviously $q_z^d > (2/3)^d > 2(1/2)^d$. Note that if a very big cube Q_z cannot be packed into J_n, then it is possible that both one unit 2-cube and one unit 3-cube are almost empty (as in Fig. 1).

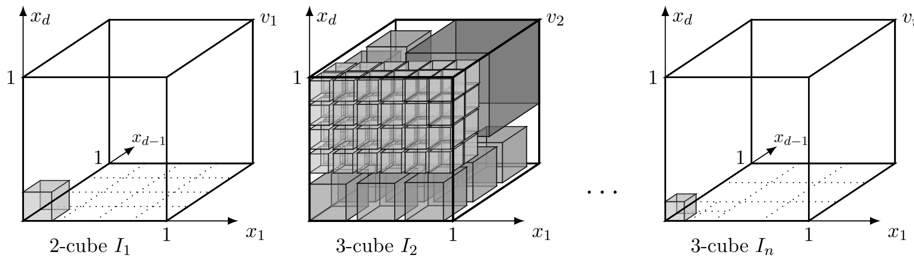

Fig. 1. There is no empty cube into which a very big cube Q_z could be packed.

If $n_3 \geq 1$, then

$$\zeta > [(n_2-1)+1]\left(\frac{1}{2}\right)^d + (n_3-1)\left(\frac{3}{4}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d$$
$$> (n-1)\left(\frac{1}{2}\right)^d + 2\left(\frac{1}{2}\right)^d = (n+1)\left(\frac{1}{2}\right)^d.$$

If $n_3 = 0$, then

$$\zeta > [(n_2-1)+1]\left(\frac{1}{2}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d > n\left(\frac{1}{2}\right)^d + 2\left(\frac{1}{2}\right)^d$$
$$> (n+1)\left(\frac{1}{2}\right)^d.$$

CASE 3: Q_z *is* 2*-small.* Assume that $n_3 \geq 1$. Denote by l the greatest number such that a 3-small cube is packed into I_l. If a big cube cannot be packed into I_l, then either a big cube is packed into I_l and the total volume of the cubes packed into I_l is greater than $(1/2)^d$, or no big cube is packed into I_l and the total volume of the cubes packed into I_l is greater than

$(1-(2/3)^d)(3/4)^d > (1/2)^d$. This implies that if $n_b \geq 1$ or if a big cube is packed into a 2-cube, then, by the description of packing of big cubes, the total volume of the cubes packed into 3-cubes is greater than

$$(n_3 - 1)\left(\frac{3}{4}\right)^d + \left(\frac{1}{2}\right)^d \geq n_3\left(\frac{1}{2}\right)^d.$$

SUBCASE 3a: $n_2 \geq 1$ *and no big cube is packed into* 2*-cubes.* The total volume of the cubes packed into 2-cubes is greater than $n_2(2/3)^d - q_z^d$. If $n_b = 0$, then it is possible that one unit 3-cube is almost empty. Consequently,

$$\zeta > n_2\left(\frac{2}{3}\right)^d - q_z^d + (n_3 - 1)\left(\frac{3}{4}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d \geq (n-1)\left(\frac{2}{3}\right)^d$$
$$> (n+1)\left(\frac{1}{2}\right)^d.$$

If $n_b \geq 1$, then

$$\zeta > n_2\left(\frac{2}{3}\right)^d - q_z^d + n_3\left(\frac{1}{2}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d$$
$$> (n_2 + 1)\left(\frac{1}{2}\right)^d + (n_3 + n_b + n_v)\left(\frac{1}{2}\right)^d = (n+1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 3b: *a big cube is packed into a* 2*-cube.* By Lemma 4.2 we get

$$\zeta > (n_2 + 1)\left(\frac{1}{2}\right)^d + n_3\left(\frac{1}{2}\right)^d + n_b\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d \geq (n+1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 3c: $n_2 = 0$. If $n_b \geq 1$ (see Fig. 2, where $n_b = n$), then the total volume of the cubes packed into b-cubes is greater then

$$(n_b - 1)\left(\frac{1}{2}\right)^d + 2\left(\frac{1}{2}\right)^d - q_z^d = (n_b + 1)\left(\frac{1}{2}\right)^d - q_z^d.$$

Hence

$$\zeta > (n_b + 1)\left(\frac{1}{2}\right)^d - q_z^d + n_3\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d + q_z^d \geq (n+1)\left(\frac{1}{2}\right)^d.$$

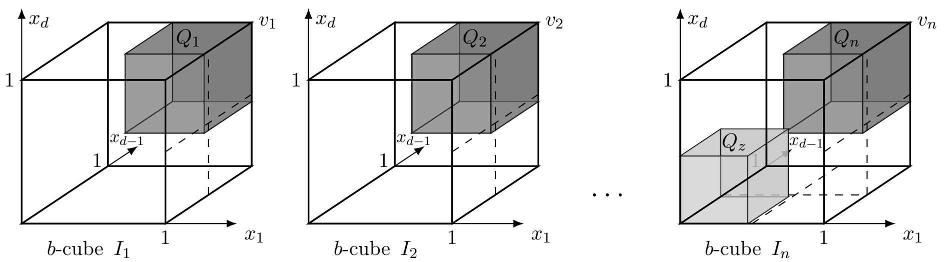

Fig. 2. Q_z is 2-small and $n_b = n$.

If $n_b = 0$, then

$$\zeta > (n_3 - 1)\left(\frac{3}{4}\right)^d + n_v\left(\frac{2}{3}\right)^d \geq (n-1)\left(\frac{2}{3}\right)^d > (n+1)\left(\frac{1}{2}\right)^d.$$

Case 4: *Q_z is 3-small.* This implies that $n_b = 0$.

Subcase 4a: $n_3 \geq 1$. The total volume of the cubes packed into 3-cubes is greater than $n_3(3/4)^d - q_z^d$. It is easy to verify that if $n_3 \geq 1$, then $n_3(3/4)^d > (n_3+2)(1/2)^d$ for $d \geq 3$. If $n_2 \geq 1$, then it is possible that one unit 2-cube is almost empty. Thus

$$\zeta > (n_2 - 1)\left(\frac{1}{2}\right)^d + (n_3+2)\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d \geq (n+1)\left(\frac{1}{2}\right)^d.$$

If $n_2 = 0$, then

$$\zeta > (n_3+2)\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d > (n+1)\left(\frac{1}{2}\right)^d.$$

Subcase 4b: $n_3 = 0$. If no big cube is packed into 2-cubes, then

$$\zeta > (n_2 - 1)\left(\frac{2}{3}\right)^d + n_v\left(\frac{2}{3}\right)^d = (n-1)\left(\frac{2}{3}\right)^d > (n+1)\left(\frac{1}{2}\right)^d.$$

If a big cube is packed into 2-cubes and $n_v \geq 1$, then

$$\zeta > [(n_2 - 1) + 1]\left(\frac{1}{2}\right)^d + n_v\left(\frac{2}{3}\right)^d > (n_2 + n_v + 1)\left(\frac{1}{2}\right)^d = (n+1)\left(\frac{1}{2}\right)^d.$$

If a big cube is packed into 2-cubes and $n_v = 0$ ($n_2 = n$), then both a big cube and a 2-small cube are packed into I_n. By Lemma 4.2 we get

$$\zeta > [(n_2 - 1) + 1]\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d = (n+1)\left(\frac{1}{2}\right)^d. \quad \blacksquare$$

5. Packing algorithm for $d \geq 5$. Let $d \geq 5$ and let (Q_i) be a sequence of cubes $q_i I$, where $q_i \in (0,1]$. We consider the following types of cubes:

- Q_i is *f-small* if $q_i \leq 1 - \frac{1}{4}\sqrt[d]{2}$;
- Q_i is *f-big* if $q_i > 1 - \frac{1}{4}\sqrt[d]{2}$.

A unit cube $I_k \subset J_n$ is said to be *empty* if no cube has been packed into it. A unit cube $I_k \subset J_n$ is said to be an *s-cube* if an f-small cube has been packed into it. A unit cube $I_k \subset J_n$ is said to be an *l-cube* if an f-big cube has been packed into it.

We pack f-small cubes by the method described in [8]. If Q_1 is f-small, then we pack it into I_1. Clearly, I_1 is now an s-cube. Assume that $i > 1$ and Q_i is f-small. If there is an s-cube into which Q_i can be packed, then we pack it into this s-cube. Otherwise, we find the smallest $k \in \{1, \ldots, n\}$ such that I_k is empty. We pack Q_i into I_k by the method described in [8]. Now I_k is an s-cube.

If Q_i is f-big, then we find the greatest $k \in \{1, \ldots, n\}$ such that I_k is empty and pack Q_i into I_k. Now I_k is an l-cube.

6. Efficiency of the packing algorithm for $d \geq 5$

LEMMA 6.1 (see [8]). *If $d \geq 5$, then every sequence of d-dimensional cubes of total volume at most $2\left(\frac{1}{2}\right)^d$ can be on-line packed into the unit cube I.*

LEMMA 6.2. *Denote by n_s the number of s-cubes in J_n. If $d \geq 5$ and if a sequence of f-small cubes cannot be on-line packed into s-cubes by the method described in Section 5, then the total volume of the cubes exceeds $(n_s + 1) \cdot 2^{-d}$.*

Proof. Let (Q_i) be a sequence of f-small cubes as in the statement. Denote by Q_z the first cube from the sequence which cannot be packed into s-cubes.

CASE 1: $n_s = 1$. By Lemma 6.1 we get

$$\sum_{i=1}^{z} |Q_i| > 2\left(\frac{1}{2}\right)^d = (n_s + 1)\left(\frac{1}{2}\right)^d.$$

CASE 2: $n_s \geq 2$ *and* $q_z \leq 1/2$. Obviously $q_z^d \leq \left(\frac{1}{2}\right)^d$. We get

$$\sum_{i=1}^{z} |Q_i| > n_s\left(2\left(\frac{1}{2}\right)^d - q_z^d\right) + q_z^d = 2n_s\left(\frac{1}{2}\right)^d - (n_s - 1)q_z^d$$
$$\geq (n_s + 1)\left(\frac{1}{2}\right)^d.$$

CASE 3: $n_s \geq 2$ *and* $q_z > 1/2$. Note that the total volume of the cubes packed into any two s-cubes is greater than $2(1/2)^d$.

SUBCASE 3a: n_s *is even.* We get

$$\sum_{i=1}^{z} |Q_i| > \frac{n_s}{2} \cdot 2\left(\frac{1}{2}\right)^d + q_z^d > n_s\left(\frac{1}{2}\right)^d + \left(\frac{1}{2}\right)^d = (n_s + 1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 3b: n_s *is odd.* The total volume of the cubes packed into an s-cube I_j with the greatest number j is greater than $2(1/2)^d - q_z^d$. The total volume of the cubes packed into $n_s - 1$ other s-cubes is greater than

$$\frac{n_s - 1}{2} \cdot 2\left(\frac{1}{2}\right)^d = (n_s - 1)\left(\frac{1}{2}\right)^d.$$

Consequently,

$$\sum_{i=1}^{z}|Q_i| > (n_s-1)\left(\frac{1}{2}\right)^d + 2\left(\frac{1}{2}\right)^d - q_z^d + q_z^d = (n_s+1)\left(\frac{1}{2}\right)^d. \ \blacksquare$$

THEOREM 6.3. *If $n \geq 1$ and $d \geq 5$, then any sequence of d-dimensional cubes of edge lengths not greater than 1 whose total volume does not exceed $(n+1)\cdot 2^{-d}$ can be on-line packed into J_n.*

Proof. Let $n \geq 1$ and let (Q_i) be a sequence of d-dimensional cubes as in the statement. We pack the cubes by the method described in Section 5. Suppose that, contrary to the statement, it is impossible to pack $Q_1, Q_2, \ldots$ into J_n by this method. Let Q_z be the cube which stops the packing process and let

$$\zeta = \sum_{i=1}^{z}|Q_i|.$$

We show that this leads to the false inequality

$$\zeta > (n+1)\cdot 2^{-d}.$$

Denote by n_s, n_l the number of s- and l-cubes, respectively. Obviously we have $n_s + n_l = n$. It is easy to verify that

$$\left(1 - \frac{1}{4}\sqrt[d]{2}\right)^d > 2\left(\frac{1}{2}\right)^d$$

for $d \geq 5$. This implies that the total volume of the cubes packed into l-cubes is greater than $n_l \cdot 2(1/2)^d$. We consider two cases.

CASE 1: *Q_z is f-small.* By Lemma 6.2 we get

$$\zeta > (n_s+1)\left(\frac{1}{2}\right)^d + n_l\cdot 2\left(\frac{1}{2}\right)^d \geq (n+1)\left(\frac{1}{2}\right)^d.$$

CASE 2: *Q_z is f-big.* Obviously $q_z^d > 2(1/2)^d$. It is possible that one of the s-cubes is almost empty.

SUBCASE 2a: $n = 1$. We get

$$\zeta > q_z^d > 2\left(\frac{1}{2}\right)^d = (n+1)\left(\frac{1}{2}\right)^d.$$

SUBCASE 2b: $n \geq 2$. By Lemma 6.2 we get

$$\zeta > (n_s-1)\cdot\left(\frac{1}{2}\right)^d + n_l\cdot 2\left(\frac{1}{2}\right)^d + q_z^d > (n-1)\left(\frac{1}{2}\right)^d + 2\left(\frac{1}{2}\right)^d$$
$$= (n+1)\left(\frac{1}{2}\right)^d. \ \blacksquare$$

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