

Sigma-continuity with closed witnesses

by

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Abstract. We use variants of the \mathbb{G}_0 dichotomy to establish a refinement of Solecki's basis theorem for the family of Baire-class one functions which are not σ -continuous with closed witnesses.

Introduction. A subset of a topological space is F_σ if it is a union of countably-many closed sets, *Borel* if it is in the σ -algebra generated by the closed sets, and *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

A function between topological spaces is *σ -continuous with closed witnesses* if its domain is a union of countably-many closed sets on which it is continuous, *Baire class one* if preimages of open sets are F_σ , *strongly σ -closed-to-one* if its domain is a union of countably-many analytic sets intersecting the preimage of each singleton in a closed set, *F_σ -to-one* if the preimage of each singleton is F_σ , and *Borel* if preimages of open sets are Borel.

A *topological embedding* of a topological space X into a topological space Y is a function $\pi: X \rightarrow Y$ which is a homeomorphism onto its image, where the latter is endowed with the subspace topology. A *topological embedding* of a set $A \subseteq X$ into a set $B \subseteq Y$ is a topological embedding π of X into Y such that $A = \pi^{-1}(B)$. A *topological embedding* of a function $f: X \rightarrow Y$ into a function $f': X' \rightarrow Y'$ is a pair (π_X, π_Y) consisting of topological embeddings π_X of X into X' and π_Y of $f(X)$ into $f'(X')$ with $f' \circ \pi_X = \pi_Y \circ f$.

Some time ago, Jayne–Rogers showed that a function between Polish spaces is σ -continuous with closed witnesses if and only if preimages of closed sets are F_σ (see [JR82, Theorem 1]). Solecki later refined this result by providing a two-element basis, under topological embeddability, for the family of Baire-class one functions which do not have this property (see [Sol98,

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Theorem 3.1]). Here we use variants of the \mathbb{G}_0 dichotomy (see [KST99]) to establish a pair of dichotomies which together refine Solecki's theorem.

In §1, we provide a simple characterization of Baire-class one functions that is used throughout the remainder of the paper. As a first application, we use the Lecomte–Zelený Δ_2^0 -measurable analog of the \mathbb{G}_0 dichotomy theorem (see [LZ14, Corollary 4.5]) to establish that the property of being Baire class one is determined by behaviour on countable sets.

In §2, we use the Hurewicz dichotomy theorem to provide a one-element basis, under topological embeddability, for the family of Baire-class one functions which are not F_σ -to-one. To be precise, let $\mathbb{N}_*^{\leq \mathbb{N}}$ denote the set $\mathbb{N}^{\leq \mathbb{N}}$ equipped with the smallest topology making the sets $\mathcal{N}_s^* = \{t \in \mathbb{N}^{\leq \mathbb{N}} \mid s \sqsubseteq t\}$ clopen for all $s \in \mathbb{N}^{< \mathbb{N}}$, and fix a function $f_0: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow \mathbb{R}$ such that $f_0 \upharpoonright \mathbb{N}^{\mathbb{N}}$ has constant value zero and $f_0 \upharpoonright \mathbb{N}^{< \mathbb{N}}$ is an injection into $\{1/n \mid n \in \mathbb{N}\}$.

THEOREM 1. *Suppose that X and Y are Polish spaces and $f: X \rightarrow Y$ is a Baire-class one function. Then exactly one of the following holds:*

- (1) *The function f is F_σ -to-one.*
- (2) *There is a topological embedding of f_0 into f .*

In §3, we use the sequential \aleph_0 -dimensional analog of the \mathbb{G}_0 dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) to provide a one-element basis, under topological embeddability, for the family of F_σ -to-one Baire-class one functions which are not σ -continuous with closed witnesses. To be precise, let $\mathbb{N}_{**}^{\leq \mathbb{N}}$ denote the set $\mathbb{N}^{\leq \mathbb{N}}$ equipped with the smallest topology making the sets \mathcal{N}_s^* and $\{s\}$ clopen for all $s \in \mathbb{N}^{< \mathbb{N}}$, and define $f_1: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow \mathbb{N}_{**}^{\leq \mathbb{N}}$ by $f_1(s) = s$.

THEOREM 2. *Suppose that X and Y are Polish spaces and $f: X \rightarrow Y$ is an F_σ -to-one Baire-class one function. Then exactly one of the following holds:*

- (1) *The function f is σ -continuous with closed witnesses.*
- (2) *There is a topological embedding of f_1 into f .*

Theorem 2 trivially yields the following.

THEOREM 3 (Jayne–Rogers). *Suppose that X and Y are Polish spaces, and $f: X \rightarrow Y$ is a function with the property that $f^{-1}(C)$ is F_σ for all closed subsets C of Y . Then f is σ -continuous with closed witnesses.*

Theorems 1 and 2 trivially yield the following.

THEOREM 4 (Solecki). *Suppose that X and Y are Polish spaces and f is a Baire-class one function. Then exactly one of the following holds:*

- (1) *The function f is σ -continuous with closed witnesses.*
- (2) *There is a topological embedding of f_0 or f_1 into f .*

We close the paper with an appendix in which we prove several auxiliary facts needed for the proof of the main results above. Along the way, we use Lecomte's \aleph_0 -dimensional analog of the \mathbb{G}_0 dichotomy theorem (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) to give a new proof of a special case of Hurewicz's dichotomy theorem (see, for example, [Kec95, Theorem 21.18]), yielding the existence of a one-element basis, under topological embeddability, for the family of Borel sets which are not F_σ . To be precise, we show that if X is a Polish space and $B \subseteq X$ is Borel, then either B is F_σ , or there is a topological embedding $\pi: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ of $\mathbb{N}^{\mathbb{N}}$ into B . We note that the same argument, using the parametrized \aleph_0 -dimensional analog of the \mathbb{G}_0 dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 31]) in lieu of its non-parametrized counterpart, yields a slight weakening of Saint Raymond's parametrized analog of Hurewicz's result (see, for example, [Kec95, Theorem 35.45]). As a corollary, we show that F_σ -to-one Borel functions between Polish spaces are strongly σ -closed-to-one.

1. Baire-class one functions. Throughout the rest of the paper, we will rely on the following characterization of Baire-class one functions. Given a metric space X , a point $x \in X$, and a real number $\epsilon > 0$, set $\mathcal{B}_{d_X}(x, \epsilon) = \{y \in X \mid d_X(x, y) < \epsilon\}$.

PROPOSITION 1.1. *Suppose that X is a topological space, Y is a second countable metric space, and $f: X \rightarrow Y$ is a function. Then the following are equivalent:*

- (1) *The function f is Baire class one.*
- (2) *For all $\epsilon > 0$, there is a cover of X by countably-many closed subsets whose f -images have d_Y -diameter strictly less than ϵ .*

Proof. To see (1) \Rightarrow (2), it is sufficient to show that for all real numbers $\epsilon > 0$ and open sets $V \subseteq Y$ of d_Y -diameter strictly less than ϵ , the set $f^{-1}(V)$ is a union of countably-many closed subsets of X . But this follows from the fact that $f^{-1}(V)$ is F_σ .

To see (2) \Rightarrow (1), it is sufficient to show that for all real numbers $\epsilon > 0$ and open sets $V \subseteq Y$, there is an F_σ set $F \subseteq X$ such that $f^{-1}(V_\epsilon) \subseteq F \subseteq f^{-1}(V)$, where $V_\epsilon = \{y \in Y \mid \mathcal{B}_{d_Y}(y, \epsilon) \subseteq V\}$. Towards this end, fix a cover $(C_n)_{n \in \mathbb{N}}$ of X by closed sets whose f -images have d_Y -diameter strictly less than ϵ , define $N = \{n \in \mathbb{N} \mid f(C_n) \cap V_\epsilon \neq \emptyset\}$, and observe that the set $F = \bigcup_{n \in N} C_n$ is as desired. ■

As a corollary, we obtain the following.

THEOREM 1.2. *Suppose that X and Y are Polish spaces, d_Y is a compatible metric on Y , and $f: X \rightarrow Y$ is Borel. Suppose further that for all count-*

able sets $C \subseteq X$ and real numbers $\epsilon > 0$, there is a Baire-class one function $g: X \rightarrow Y$ such that $\sup_{x \in C} d_Y(f(x), g(x)) \leq \epsilon$. Then f is Baire class one.

Proof. Suppose, towards a contradiction, that f is not Baire class one, fix a compatible metric d_Y on Y , and appeal to Proposition 1.1 to find $\delta > 0$ for which there is no cover of X by countably-many closed subsets whose f -images have d_Y -diameter at most δ .

A *digraph* on a set X is an irreflexive subset of $X \times X$. A *homomorphism* from a digraph G on X to a digraph H on Y is a function $\phi: X \rightarrow Y$ sending G -related points to H -related points.

Let $G_{\delta, f}$ denote the digraph on X consisting of all $(w, x) \in X \times X$ for which $d_Y(f(w), f(x)) > \delta$. We say that a set $W \subseteq X$ is $G_{\delta, f}$ -*independent* if $G_{\delta, f}|_W = \emptyset$. Our choice of δ ensures that X is not the union of countably-many closed $G_{\delta, f}$ -independent sets.

Fix $s_n^{\Delta_2^0} \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N} s \sqsubseteq s_n^{\Delta_2^0}$, as well as $z_n \in 2^{\mathbb{N}}$ for all $n \in \mathbb{N}$. Now define a digraph on $2^{\mathbb{N}}$ by setting

$$\mathbb{G}_0^{\Delta_2^0} = \{(s_n^{\Delta_2^0} \frown (0) \frown z_n, s_n^{\Delta_2^0} \frown (1) \frown z_n) \mid n \in \mathbb{N}\}.$$

The Lecomte–Zelený dichotomy theorem characterizing analytic graphs of uncountable Δ_2^0 -measurable chromatic number (see [LZ14, Corollary 4.5]) yields a continuous homomorphism $\phi: 2^{\mathbb{N}} \rightarrow X$ from this digraph to $G_{\delta, f}$. Set

$$C = \phi(\{s_n^{\Delta_2^0} \frown (i) \frown z_n \mid i < 2 \text{ and } n \in \mathbb{N}\})$$

and $\epsilon = \delta/3$.

It only remains to check that no function $g: X \rightarrow Y$ with the property that $\sup_{x \in C} d_Y(f(x), g(x)) \leq \epsilon$ is Baire class one. As ϕ is necessarily a homomorphism from the above digraph to the digraph $G_{\epsilon, g}$ associated with such a function, there can be no cover of X by countably-many closed subsets whose g -images have d_Y -diameter at most ϵ , so one more appeal to Proposition 1.1 ensures that g is not Baire class one. ■

2. F_σ -to-one functions. The proof of Theorem 1 is based on a technical but useful sufficient condition for the topological embeddability of f_0 .

PROPOSITION 2.1. *Suppose that Y is a Polish space and $f: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow Y$ is a Baire-class one function for which there exists $y \in Y$ such that $\mathbb{N}^{\mathbb{N}} = f^{-1}(y)$. Then there is a topological embedding of f_0 into f .*

Proof. Fix a compatible metric d_Y on Y .

LEMMA 2.2. *Suppose that $\epsilon > 0$. Then there is a dense open subset U of $\mathbb{N}_*^{\leq \mathbb{N}}$ such that $f(U) \subseteq \mathcal{B}_{d_Y}(y, \epsilon)$.*

Proof. By Proposition 1.1, there is a partition $(C_n)_{n \in \mathbb{N}}$ of $\mathbb{N}_*^{\leq \mathbb{N}}$ into closed sets whose f -images have d_Y -diameter strictly less than ϵ . Then for each non-

empty open set $V \subseteq \mathbb{N}_*^{\leq \mathbb{N}}$, there exists $n \in \mathbb{N}$ for which C_n is non-meager in V , so there is a non-empty open set $W \subseteq V$ such that C_n is comeager in W . As C_n is closed, it follows that $W \subseteq C_n$, thus the diameter of $f(W)$ is strictly less than ϵ . As W necessarily contains a point of $\mathbb{N}^{\mathbb{N}}$, it follows that $f(W) \subseteq \mathcal{B}_{d_Y}(y, \epsilon)$. The union of the non-empty open sets $W \subseteq \mathbb{N}_*^{\leq \mathbb{N}}$ obtained in this way from non-empty open sets $V \subseteq \mathbb{N}_*^{\leq \mathbb{N}}$ is therefore as desired. ■

Fix an injective enumeration $(s_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{< \mathbb{N}}$ with the property that $s_m \sqsubseteq s_n \Rightarrow m \leq n$ for all $m, n \in \mathbb{N}$, fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of strictly positive real numbers such that $0 = \lim_{n \rightarrow \infty} \epsilon_n$, and for each $n \in \mathbb{N}$, set $m(n+1) = \max\{m \leq n \mid s_m \sqsubseteq s_{n+1}\}$. Define $u_\emptyset = \emptyset$, and recursively appeal to Lemma 2.2 to obtain sequences $u_{s_{n+1}} \in \mathbb{N}^{< \mathbb{N}}$, for all $n \in \mathbb{N}$, with the property that $u_{s_{m(n+1)}} \cap s_{n+1}(|s_{m(n+1)}|) \sqsubseteq u_{s_{n+1}}$ and $f(\mathcal{N}_{u_{s_{n+1}}}^*) \subseteq \mathcal{B}_{d_Y}(y, \min\{\epsilon_n, d_Y(y, f(u_{s_n}))\})$.

Define $\pi_X: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow \mathbb{N}_*^{\leq \mathbb{N}}$ by

$$\pi_X(s) = \begin{cases} u_s & \text{if } s \in \mathbb{N}^{< \mathbb{N}}, \\ \bigcup_{n \in \mathbb{N}} u_{s \upharpoonright n} & \text{otherwise.} \end{cases}$$

Define $\pi_Y: f_0(\mathbb{N}_*^{\leq \mathbb{N}}) \rightarrow f(X)$ by $\pi_Y(0) = y$ and $\pi_Y(f_0(s)) = f(u_s)$ for all $s \in \mathbb{N}^{< \mathbb{N}}$. As both of these functions are continuous injections with compact domains, they are necessarily topological embeddings, thus (π_X, π_Y) is a topological embedding of f_0 into f . ■

Proof of Theorem 1. Proposition A.1 ensures that conditions (1) and (2) are mutually exclusive. To see $\neg(1) \Rightarrow (2)$, suppose that there exists $y \in Y$ such that $f^{-1}(y)$ is not F_σ , and appeal to Theorem A.4 to obtain a topological embedding $\pi: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ of $\mathbb{N}^{\mathbb{N}}$ into $f^{-1}(y)$. Proposition 2.1 then yields a topological embedding (π_X, π_Y) of f_0 into $f \circ \pi$, and it follows that $(\pi \circ \pi_X, \pi_Y)$ is a topological embedding of f_0 into f . ■

3. Sigma-continuous functions. We begin with a technical but useful sufficient condition for the topological embeddability of f_1 .

PROPOSITION 3.1. *Suppose that X and Y are metric spaces, $f: X \rightarrow Y$, and there are a dense G_δ set $C \subseteq \mathbb{N}^{\mathbb{N}}$, a set $W \subseteq X$ intersecting the f -preimage of every singleton in a closed set, and a function $\phi: C \rightarrow W$, such that both ϕ and $f \circ \phi$ are continuous, which is a homomorphism from $\mathbb{G}_{0,m}^{\mathbb{N}} \upharpoonright C$ to the \aleph_0 -dimensional dihypergraph G_m consisting of all convergent sequences $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ with $f(\lim_{n \rightarrow \infty} x_n) \neq \lim_{n \rightarrow \infty} f(x_n)$ but $\{f(x_n) \mid n \in \mathbb{N}\} \subseteq \mathcal{B}_{d_Y}(f(\lim_{n \rightarrow \infty} x_n), 1/m)$ for all $m \in \mathbb{N}$. Then there is a topological embedding of f_1 into f .*

Proof. Fix dense open sets $U_n \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. We will recursively construct sequences $(u_s)_{s \in \mathbb{N}^n}$ of elements of $\mathbb{N}^{< \mathbb{N}}$, sequences $(V_s)_{s \in \mathbb{N}^n}$ of open subsets of Y , and sequences $(x_s)_{s \in \mathbb{N}^n}$ of elements of X , for all $n \in \mathbb{N}$, such that:

- (1) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} u_s \sqsubset u_{s \frown (i)}$.
- (2) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \mathcal{N}_{u_{s \frown (i)}} \subseteq U_{|s|}$.
- (3) $\forall s \in \mathbb{N}^{< \mathbb{N}} (f \circ \phi)(\mathcal{N}_{u_s}) \cup \{f(x_s)\} \subseteq V_s$.
- (4) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} V_{s \frown (i)} \subseteq V_s$.
- (5) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \text{diam}_{d_X}(\phi(\mathcal{N}_{u_{s \frown (i)}})) < 1/|s|$.
- (6) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \text{diam}_{d_Y}(V_{s \frown (i)}) < 1/|s|$.
- (7) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})} \subseteq \mathcal{B}_{d_X}(x_s, 1/i)$.
- (8) $\forall s \in \mathbb{N}^{< \mathbb{N}} f(x_s) \notin \overline{\bigcup_{i \in \mathbb{N}} V_{s \frown (i)}}$.
- (9) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} V_{s \frown (i)} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} V_{s \frown (j)}} = \emptyset$.

We begin by setting $u_\emptyset = \emptyset$ and $V_\emptyset = Y$. Suppose now that $n \in \mathbb{N}$ and we have already found $(u_s)_{s \in \mathbb{N}^{\leq n}}$, $(V_s)_{s \in \mathbb{N}^{\leq n}}$, and $(x_s)_{s \in \mathbb{N}^{< n}}$. For each $s \in \mathbb{N}^n$, fix $\delta_s > 0$ as well as $u'_s \in \mathbb{N}^{< \mathbb{N}}$ such that $u_s \sqsubset u'_s$, $\mathcal{N}_{u'_s} \subseteq U_n$, $\text{diam}_{d_X}(\phi(\mathcal{N}_{u'_s})) < 1/n$, $\text{diam}_{d_Y}((f \circ \phi)(\mathcal{N}_{u'_s})) < 3/(2n)$, and $\mathcal{B}_{d_Y}((f \circ \phi)(\mathcal{N}_{u'_s}), \delta_s) \subseteq V_s$. Fix a natural number $n_s \geq 1/\delta_s$ such that $u'_s \sqsubset s_{n_s}^{\mathbb{N}}$, appeal to the Baire category theorem to find $z_s \in \mathbb{N}^{\mathbb{N}}$ with the property that $s_{n_s}^{\mathbb{N}} \frown (i) \frown z_s \in C$ for all $i \in \mathbb{N}$, and define $x_{i,s} = \phi(s_{n_s}^{\mathbb{N}} \frown (i) \frown z_s)$ and $y_{i,s} = f(x_{i,s})$ for all $i \in \mathbb{N}$, as well as $x_s = \lim_{i \rightarrow \infty} x_{i,s}$. The fact that $f(x_s) \neq \lim_{i \rightarrow \infty} y_{i,s}$ ensures the existence of an infinite set $I_s \subseteq \mathbb{N}$ for which $f(x_s) \notin \overline{\{y_{i,s} \mid i \in I_s\}}$. Note that there can be no infinite set $J \subseteq I_s$ such that $(y_{j,s})_{j \in J}$ is constant, since otherwise the fact that $\phi(C) \subseteq W$ would imply that $f(x_s) = y_{j,s}$ for all $j \in J$. So by passing to an infinite subset of I_s , we can assume that $(y_{i,s})_{i \in I_s}$ is injective. By passing to a further infinite subset of I_s , we can ensure that $(y_{i,s})_{i \in I_s}$ has at most one limit point. By eliminating this limit point from the sequence if necessary, we can therefore ensure that $y_{i,s} \notin \overline{\{y_{j,s} \mid j \in I_s \setminus \{i\}\}}$ for all $i \in I_s$. Similarly, we can assume that $x_s \notin \overline{\{x_{i,s} \mid i \in I_s\}}$. By passing one last time to an infinite subset of I_s , we can assume that $d_X(x_s, x_{i_{k,s},s}) < 1/k$ for all $k \in \mathbb{N}$, where $(i_{k,s})_{k \in \mathbb{N}}$ is the strictly increasing enumeration of I_s . For each $k \in \mathbb{N}$, fix $\epsilon_{k,s}^X > 0$ strictly less than $1/k - d_X(x_s, x_{i_{k,s},s})$, and fix $\epsilon_{k,s}^Y > 0$ strictly less than $d_Y(f(x_s), y_{i_{k,s},s})/2$ and $d_Y(y_{i_{k,s},s}, y_{i_{k,s},s})/3$ for all $i \in I_s \setminus \{i_{k,s}\}$. Set $V_{s \frown (k)} = \mathcal{B}_{d_Y}(y_{i_{k,s},s}, \epsilon_{k,s}^Y) \cap V_s$, and fix an initial segment $u_{s \frown (k)}$ of $s_{n_s}^{\mathbb{N}} \frown (i_{k,s}) \frown z_s$ of length at least $n_s + 1$ with the property that $\phi(\mathcal{N}_{u_{s \frown (k)}}) \subseteq \mathcal{B}_{d_X}(x_{i_{k,s},s}, \epsilon_{k,s}^X)$ and $(f \circ \phi)(\mathcal{N}_{u_{s \frown (k)}}) \subseteq V_{s \frown (k)}$. Our choice of u'_s ensures that conditions (1), (2), and (5) hold, and along with the fact that ϕ is a homomorphism, that condition (3) holds as well. Condition (4) holds trivially, and the re-

maining conditions follow from our upper bounds on $\epsilon_{k,s}^X$ and $\epsilon_{k,s}^Y$. This completes the recursive construction.

By condition (1), we obtain a continuous function $\psi: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ by setting $\psi(s) = \bigcup_{n \in \mathbb{N}} u_{s \upharpoonright n}$. Condition (2) ensures that $\psi(\mathbb{N}^{\mathbb{N}}) \subseteq C$. Set $x_s = (\phi \circ \psi)(s)$ for $s \in \mathbb{N}^{\mathbb{N}}$, and define $\pi_X: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ and $\pi_Y: \mathbb{N}_{**}^{\leq \mathbb{N}} \rightarrow Y$ by $\pi_X(s) = x_s$ and $\pi_Y = f \circ \pi_X$. We will show that (π_X, π_Y) is a topological embedding of f_1 into f .

LEMMA 3.2. *Suppose that $s \in \mathbb{N}^{< \mathbb{N}}$. Then $\pi_X(\mathcal{N}_s^*) \subseteq \overline{\phi(\mathcal{N}_{u_s})}$.*

Proof. Simply observe that

$$\begin{aligned} \pi_X(\mathcal{N}_s^*) &= (\phi \circ \psi)(\mathcal{N}_s) \cup \{x_t \mid t \in \mathcal{N}_s^* \setminus \mathcal{N}_s\} \\ &\subseteq \phi(\mathcal{N}_{u_s}) \cup \bigcup_{t \in \mathcal{N}_s^* \setminus \mathcal{N}_s} \overline{\phi(\mathcal{N}_{u_t})} \subseteq \overline{\phi(\mathcal{N}_{u_s})}, \end{aligned}$$

by conditions (1) and (7). ■

LEMMA 3.3. *Suppose that $s \in \mathbb{N}^{< \mathbb{N}}$. Then $\pi_Y(\mathcal{N}_s^*) \subseteq V_s$.*

Proof. Simply observe that

$$\begin{aligned} \pi_Y(\mathcal{N}_s^*) &= (f \circ \phi \circ \psi)(\mathcal{N}_s) \cup \{f(x_t) \mid t \in \mathcal{N}_s^* \setminus \mathcal{N}_s\} \\ &\subseteq (f \circ \phi)(\mathcal{N}_{u_s}) \cup \{f(x_t) \mid t \in \mathcal{N}_s^* \setminus \mathcal{N}_s\} \\ &\subseteq V_s \cup \bigcup_{t \in \mathcal{N}_s^* \setminus \mathcal{N}_s} V_t \subseteq V_s, \end{aligned}$$

by conditions (3) and (4). ■

To see that π_X and π_Y are injective, it is enough to check that the latter is injective. Towards this end, suppose that $s, t \in \mathbb{N}^{\leq \mathbb{N}}$ are distinct. If there is a least $n \leq \min\{|s|, |t|\}$ with $s \upharpoonright n \neq t \upharpoonright n$, then condition (9) ensures that $V_{s \upharpoonright n}$ and $V_{t \upharpoonright n}$ are disjoint, and since Lemma 3.3 implies that $\pi_Y(s)$ is in the former and $\pi_Y(t)$ is in the latter, it follows that they are distinct. Otherwise, after reversing the roles of s and t if necessary, we can assume that there exists $n < |t|$ for which $s = t \upharpoonright n$. But then condition (8) ensures that $\pi_Y(s) \notin V_{t \upharpoonright (n+1)}$, while Lemma 3.3 implies that $\pi_Y(t) \in V_{t \upharpoonright (n+1)}$, thus $\pi_Y(s) \neq \pi_Y(t)$.

To see that π_X is a topological embedding, it only remains to show that it is continuous (since $\mathbb{N}_*^{\leq \mathbb{N}}$ is compact). And for this, it is enough to check that for all $n \in \mathbb{N}$ and $s \in \mathbb{N}_*^{\leq \mathbb{N}}$, there is an open neighborhood of s whose image under π_X is a subset of $\mathcal{B}_{d_X}(\pi_X(s), 1/n)$. Towards this end, observe that if $s \in \mathbb{N}^{\mathbb{N}}$, then Lemma 3.2 ensures that $\pi_X(\mathcal{N}_{s \upharpoonright (n+1)}^*) \subseteq \overline{\phi(\mathcal{N}_{u_{s \upharpoonright (n+1)}})}$, so condition (5) implies that $\mathcal{N}_{s \upharpoonright (n+1)}^*$ is an open neighborhood of s whose image under π_X is a subset of $\mathcal{B}_{d_X}(\pi_X(s), 1/n)$. And if $s \in \mathbb{N}^{< \mathbb{N}}$, then Lemma 3.2

ensures that

$$\begin{aligned} \pi_X \left(\mathcal{N}_s^* \setminus \bigcup_{i < n} \mathcal{N}_{s \frown (i)}^* \right) &= \pi_X \left(\{s\} \cup \bigcup_{i \geq n} \mathcal{N}_{s \frown (i)}^* \right) \\ &\subseteq \{ \pi_X(s) \} \cup \bigcup_{i \geq n} \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})} \end{aligned}$$

so condition (7) implies that $\mathcal{N}_s^* \setminus \bigcup_{i < n} \mathcal{N}_{s \frown (i)}^*$ is an open neighborhood of s whose image under π_X is a subset of $\mathcal{B}_{d_X}(\pi_X(s), 1/n)$.

To see that π_Y is continuous, it is sufficient to check that for all $n \in \mathbb{N}$ and $s \in \mathbb{N}_{**}^{\leq \mathbb{N}}$, there is an open neighborhood of s whose image under π_Y is contained in $\mathcal{B}_{d_Y}(\pi_Y(s), 1/n)$. Towards this end, observe that if $s \in \mathbb{N}^{\mathbb{N}}$, then Lemma 3.3 ensures that $\pi_Y(\mathcal{N}_{s \upharpoonright (n+1)}^*) \subseteq V_{s \upharpoonright (n+1)}$, so condition (6) implies that $\mathcal{N}_{s \upharpoonright (n+1)}^*$ is an open neighborhood of s whose image under π_Y is contained in $\mathcal{B}_{d_Y}(\pi_Y(s), 1/n)$. And if $s \in \mathbb{N}^{< \mathbb{N}}$, then $\{s\}$ is an open neighborhood of s whose image under π_Y is a subset of $\mathcal{B}_{d_Y}(\pi_Y(s), 1/n)$.

Before showing that π_Y is a topological embedding, we first establish several lemmas.

LEMMA 3.4. *Suppose that $s \in \mathbb{N}^{< \mathbb{N}}$. Then $\pi_Y(\mathcal{N}_s^*) = \overline{V_s} \cap \pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}})$.*

Proof. Lemma 3.3 ensures that $\pi_Y(\mathcal{N}_s^*) \subseteq \overline{V_s} \cap \pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}})$, so it is enough to show that $\pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}} \setminus \mathcal{N}_s^*) \cap \overline{V_s} = \emptyset$. Towards this end, note that if $t \in \mathbb{N}_{**}^{\leq \mathbb{N}} \setminus \mathcal{N}_s^*$, then either there exists a least $n \leq \min\{|s|, |t|\}$ for which $s \upharpoonright n$ and $t \upharpoonright n$ are incompatible, or $t \sqsubset s$. In the former case, condition (9) implies that $\overline{V_{s \upharpoonright n}}$ and $V_{t \upharpoonright n}$ are disjoint, and since Lemma 3.3 implies that $\pi_Y(t)$ is in the latter, it is not in the former. But then it is not in $\overline{V_s}$ either, by condition (4). In the latter case, set $n = |t|$, and appeal to condition (8) to see that $\pi_Y(t)$ is not in $\overline{V_{s \upharpoonright (n+1)}}$. But then it is not in $\overline{V_s}$ either, by condition (4). ■

LEMMA 3.5. *Suppose that $s \in \mathbb{N}^{< \mathbb{N}}$. Then*

$$\pi_Y(\mathcal{N}_s^*) = \pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}}) \setminus \left(\bigcup_{t \perp s} \overline{V_t} \cup \{ \pi_Y(t) \mid t \sqsubset s \} \right).$$

Proof. To see that $\pi_Y(\mathcal{N}_s^*) \cap \left(\bigcup_{t \perp s} \overline{V_t} \cup \{ \pi_Y(t) \mid t \sqsubset s \} \right) = \emptyset$, note that if $t \perp s$, then there is a maximal $n < \min\{|s|, |t|\}$ with the property that $s \upharpoonright n = t \upharpoonright n$, in which case t is an extension of $(s \upharpoonright n) \frown (j)$ for some $j \in \mathbb{N} \setminus \{s(n)\}$. Condition (4) therefore ensures that

$$\bigcup_{t \perp s} \overline{V_t} = \bigcup_{n < |s|} \bigcup_{j \in \mathbb{N} \setminus \{s(n)\}} \overline{V_{(s \upharpoonright n) \frown (j)}}.$$

As Lemma 3.3 implies that $\pi_Y(\mathcal{N}_s^*) \subseteq V_s$, and condition (4) ensures that $V_s \subseteq V_{s \upharpoonright (n+1)}$ for all $n < |s|$, it follows from condition (9) that $\pi_Y(\mathcal{N}_s^*) \cap \bigcup_{t \perp s} \overline{V_t} = \emptyset$.

To see that $\pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}} \setminus \mathcal{N}_s^*) \subseteq \overline{\bigcup_{t \perp s} V_t} \cup \{\pi_Y(t) \mid t \sqsubset s\}$, note that if $t \in \mathbb{N}_{**}^{\leq \mathbb{N}} \setminus \mathcal{N}_s^*$ and $t \not\sqsubset s$, then there exists $n \leq \min\{|s|, |t|\}$ such that $s \upharpoonright n$ and $t \upharpoonright n$ are incompatible, so Lemma 3.3 ensures that $\pi_Y(t) \in \overline{\bigcup_{t \perp s} V_t}$. ■

LEMMA 3.6. *Suppose that $s \in \mathbb{N}^{< \mathbb{N}}$. Then $\pi_Y(s)$ is the unique element of $\pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}}) \setminus (\overline{\bigcup_{t \perp s} V_t} \cup \{\pi_Y(t) \mid t \sqsubset s\})$.*

Proof. As $\overline{\bigcup_{t \perp s} V_t} = \overline{\bigcup_{t \perp s} V_t} \cup \overline{\bigcup_{i \in \mathbb{N}} V_{s \frown (i)}}$ by condition (4), Lemma 3.5 ensures that we need only show that $\pi_Y(s)$ is the unique element of $\pi_Y(\mathcal{N}_s^*) \setminus \overline{\bigcup_{i \in \mathbb{N}} V_{s \frown (i)}}$. Condition (8) ensures that $\pi_Y(s)$ is in this set, while Lemma 3.3 implies that the other points of $\pi_Y(\mathcal{N}_s^*)$ are not. ■

It remains to show that $\pi_Y(\mathcal{N}_s^*)$ and $\{\pi_Y(s)\}$ are clopen in $\pi_Y(\mathbb{N}_{**}^{\leq \mathbb{N}})$ for all $s \in \mathbb{N}^{\mathbb{N}}$. The former is a consequence of Lemmas 3.4 and 3.5, while the latter follows from Lemma 3.6. ■

PROPOSITION 3.7. *Suppose that X and Y are Polish spaces, $f: X \rightarrow Y$, G is the \aleph_0 -dimensional dihypergraph on X consisting of all convergent sequences $(x_n)_{n \in \mathbb{N}}$ such that $f(\lim_{n \rightarrow \infty} x_n) \neq \lim_{n \rightarrow \infty} f(x_n)$, and $W \subseteq X$ is G -independent. Then \overline{W} is G -independent.*

Proof. Fix compatible metrics d_X and d_Y on X and Y , respectively. We must show that if $x = \lim_{n \rightarrow \infty} \overline{w}_n$ and each \overline{w}_n is in \overline{W} , then $f(x) = \lim_{n \rightarrow \infty} f(\overline{w}_n)$. For each $n \in \mathbb{N}$, write $\overline{w}_n = \lim_{m \rightarrow \infty} w_{m,n}$, where each $w_{m,n}$ is in W . The fact that W is G -independent then ensures that $f(\overline{w}_n) = \lim_{m \rightarrow \infty} f(w_{m,n})$. Fix $m_n \in \mathbb{N}$ with the property that both $d_X(w_{m_n,n}, \overline{w}_n)$ and $d_Y(f(w_{m_n,n}), f(\overline{w}_n))$ are at most $1/n$. It follows that $x = \lim_{n \rightarrow \infty} w_{m_n,n}$, so one more appeal to the fact that W is G -independent yields the equality $f(x) = \lim_{n \rightarrow \infty} f(w_{m_n,n}) = \lim_{n \rightarrow \infty} f(\overline{w}_n)$. ■

Proof of Theorem 2. Clearly, if (1) holds then $f^{-1}(C)$ is F_σ for every closed set $C \subseteq Y$. Hence conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, let G denote the \aleph_0 -dimensional dihypergraph on X consisting of all convergent sequences $(x_n)_{n \in \mathbb{N}}$ such that $f(\lim_{n \rightarrow \infty} x_n) \neq \lim_{n \rightarrow \infty} f(x_n)$.

As f is continuous on a closed set if and only if the set in question is G -independent, Proposition 3.7 ensures that if X is a union of countably many G -independent sets, then f is σ -continuous with closed witnesses. We can therefore focus on the case that X is not a union of countably many G -independent sets. While it is not difficult to see that condition (1) fails in this case, simply applying the dichotomy for \aleph_0 -dimensional analytic dihypergraphs of uncountable chromatic number (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) will not yield the sort of homomorphism we require. So instead, we will use our further assumptions to obtain a homomorphism with stronger properties.

Fix a compatible metric d_Y on Y , and for each $\epsilon > 0$, let G_ϵ denote the \aleph_0 -dimensional dihypergraph on X consisting of all sequences $(x_n)_{n \in \mathbb{N}} \in G$ with $\{f(x_n) \mid n \in \mathbb{N}\} \subseteq \mathcal{B}_{d_Y}(f(\lim_{n \rightarrow \infty} x_n), \epsilon)$. Note that if $B \subseteq X$ is a G_ϵ -independent set and $C \subseteq X$ is a closed set whose f -image has d_Y -diameter strictly less than ϵ , then $B \cap C$ is G -independent. As Proposition 1.1 ensures that X is a union of countably-many closed sets whose f -images have d_Y -diameter strictly less than ϵ , it follows that every G_ϵ -independent set is a union of countably-many G -independent sets.

We say that a set $W \subseteq X$ is *eventually $(G_\epsilon)_{\epsilon > 0}$ -independent* if there exists $\epsilon > 0$ for which it is G_ϵ -independent. As X is not a union of countably-many G -independent sets, it follows that it is not a union of countably-many eventually $(G_\epsilon)_{\epsilon > 0}$ -independent sets. Again, however, simply applying the sequential \aleph_0 -dimensional analog of the \mathbb{G}_0 dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) will not yield the sort of homomorphism we require, and we must once more appeal to our further assumptions.

Theorem A.6 ensures that X is a union of countably-many analytic sets whose intersection with the f -preimage of each singleton is closed. As X is not a union of countably-many eventually $(G_\epsilon)_{\epsilon > 0}$ -independent sets, it follows that there is an analytic set $A \subseteq X$, whose intersection with the f -preimage of each singleton is closed, that is not a union of countably-many eventually $(G_\epsilon)_{\epsilon > 0}$ -independent sets.

At long last, we now appeal to the sequential \aleph_0 -dimensional analog of the \mathbb{G}_0 dichotomy theorem to obtain a dense G_δ set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous function $\phi: C \rightarrow A$ which is a homomorphism from $\mathbb{G}_{0,n}^{\mathbb{N}} \upharpoonright C$ to $G_{1/n}$ for all $n \in \mathbb{N}$. In fact, by first replacing the given topology of X with a finer Polish topology consisting only of Borel sets but with respect to which f is continuous (see, for example, [Kec95, Theorem 13.11]), we can ensure that $f \circ \phi$ is continuous as well. An application of Proposition 3.1 therefore yields the desired topological embedding of f_1 into f . ■

Appendix: F_σ sets. We begin the appendix with a straightforward observation.

PROPOSITION A.1.

- (a) *The set $\mathbb{N}^{\mathbb{N}}$ is not an F_σ subspace of $\mathbb{N}_*^{\leq \mathbb{N}}$.*
- (b) *The set $\mathbb{N}^{\mathbb{N}}$ is a closed subspace of $\mathbb{N}_{**}^{\leq \mathbb{N}}$.*

Proof. To see (a), note that a subset of a topological space is G_δ if it is an intersection of countably-many open sets. As $\mathbb{N}^{< \mathbb{N}}$ is countable and $\mathbb{N}^{\mathbb{N}}$ is dense in $\mathbb{N}_*^{\leq \mathbb{N}}$, it follows that $\mathbb{N}^{\mathbb{N}}$ is a dense G_δ subspace of $\mathbb{N}_*^{\leq \mathbb{N}}$. As $\mathbb{N}^{< \mathbb{N}}$ is also dense in $\mathbb{N}_*^{\leq \mathbb{N}}$, the Baire category theorem (see, for example, [Kec95,

Theorem 8.4]) ensures that it is not a G_δ subspace of $\mathbb{N}_*^{\leq \mathbb{N}}$, thus $\mathbb{N}^{\mathbb{N}}$ is not an F_σ subspace of $\mathbb{N}_*^{\leq \mathbb{N}}$.

To see (b), note that $\{s\}$ is clopen in $\mathbb{N}_{**}^{\leq \mathbb{N}}$ for all $s \in \mathbb{N}^{< \mathbb{N}}$, so $\mathbb{N}^{< \mathbb{N}}$ is open in $\mathbb{N}_{**}^{\leq \mathbb{N}}$, thus $\mathbb{N}^{\mathbb{N}}$ is closed in $\mathbb{N}_{**}^{\leq \mathbb{N}}$. ■

An \aleph_0 -dimensional dihypergraph on a set X is a set of non-constant elements of $X^{\mathbb{N}}$. A homomorphism from an \aleph_0 -dimensional dihypergraph G on X to an \aleph_0 -dimensional dihypergraph H on Y is a function $\phi: X \rightarrow Y$ sending elements of G to elements of H .

Fix sequences $s_n^{\mathbb{N}} \in \mathbb{N}^n$ such that $\forall s \in \mathbb{N}^{< \mathbb{N}} \exists n \in \mathbb{N} s \sqsubseteq s_n^{\mathbb{N}}$, and define \aleph_0 -dimensional dihypergraphs on $\mathbb{N}^{\mathbb{N}}$ by setting

$$\mathbb{G}_{0,n}^{\mathbb{N}} = \{(s_n^{\mathbb{N}} \frown (i) \frown z)_{i \in \mathbb{N}} \mid z \in \mathbb{N}^{\mathbb{N}}\}$$

for all $n \in \mathbb{N}$, and $\mathbb{G}_0^{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} \mathbb{G}_{0,n}^{\mathbb{N}}$.

We now establish a technical but useful sufficient condition for the topological embeddability of $\mathbb{N}^{\mathbb{N}}$.

PROPOSITION A.2. *Suppose that X is a metric space, $Y \subseteq X$ is a set, and there are a dense G_δ set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous homomorphism $\phi: C \rightarrow Y$ from $\mathbb{G}_0^{\mathbb{N}} \upharpoonright C$ to the \aleph_0 -dimensional dihypergraph*

$$G = \left\{ (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}} \mid \exists x \in X \setminus Y x = \lim_{n \rightarrow \infty} y_n \right\}.$$

Then there is a topological embedding $\pi: \mathbb{N}_^{\leq \mathbb{N}} \rightarrow X$ of $\mathbb{N}^{\mathbb{N}}$ into Y .*

Proof. Fix dense open sets $U_n \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. We will recursively construct sequences $(u_s)_{s \in \mathbb{N}^n}$ of elements of $\mathbb{N}^{< \mathbb{N}}$ and sequences $(x_s)_{s \in \mathbb{N}^n}$ of elements of X , for all $n \in \mathbb{N}$, such that:

- (1) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} u_s \sqsubset u_{s \frown (i)}$.
- (2) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \mathcal{N}_{u_{s \frown (i)}} \subseteq U_{|s|}$.
- (3) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \text{diam}_{d_X}(\phi(\mathcal{N}_{u_{s \frown (i)}})) < 1/|s|$.
- (4) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})} \subseteq \mathcal{B}_{d_X}(x_s, 1/i)$.
- (5) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} x_s \notin \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})}$.
- (6) $\forall i, j \in \mathbb{N} \forall s \in \mathbb{N}^{< \mathbb{N}} (i \neq j \Rightarrow \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})} \cap \overline{\phi(\mathcal{N}_{u_{s \frown (j)}})} = \emptyset)$.

We begin by setting $u_\emptyset = \emptyset$. Suppose now that $n \in \mathbb{N}$ and we have already found $(u_s)_{s \in \mathbb{N}^{\leq n}}$ and $(x_s)_{s \in \mathbb{N}^{< n}}$. For each $s \in \mathbb{N}^n$, fix $u'_s \in \mathbb{N}^{< \mathbb{N}}$ such that $u_s \sqsubseteq u'_s$, $\mathcal{N}_{u'_s} \subseteq U_n$, and $\text{diam}_{d_X}(\phi(\mathcal{N}_{u'_s})) < 1/n$, fix $n_s \in \mathbb{N}$ for which $u'_s \sqsubseteq s_{n_s}^{\mathbb{N}}$, and appeal to the Baire category theorem to find $z_s \in \mathbb{N}^{\mathbb{N}}$ with the property that $s_{n_s}^{\mathbb{N}} \frown (i) \frown z_s \in C$ for all $i \in \mathbb{N}$. Set $y_{i,s} = \phi(s_{n_s}^{\mathbb{N}} \frown (i) \frown z_s)$ for all $i \in \mathbb{N}$, as well as $x_s = \lim_{n \rightarrow \infty} y_{i,s}$. As $x_s \notin \{y_{i,s} \mid i \in \mathbb{N}\}$, there is an infinite set $I_s \subseteq \mathbb{N}$ for which $(y_{i,s})_{i \in I_s}$ is injective. By passing to an infinite subset of I_s , we can assume that $d_X(x_s, y_{i_{k,s},s}) < 1/k$ for all $k \in \mathbb{N}$, where $(i_{k,s})_{k \in \mathbb{N}}$ is the strictly increasing enumeration of I_s . For each

$k \in \mathbb{N}$, fix $\epsilon_{k,s} > 0$ strictly less than $1/k - d_X(x_s, y_{i_{k,s},s})$, $d_X(x_s, y_{i_{k,s},s})$, and $d_X(y_{i_{k,s},s}, y_{i_{k,s},s})/2$ for all $i \in I_s \setminus \{i_{k,s}\}$, and fix an initial segment $u_{s \frown (k)}$ of $s_{n_s}^{\mathbb{N}} \frown (i_{k,s}) \frown z_s$ of length at least $n_s + 1$ with the property that $\phi(\mathcal{N}_{u_{s \frown (k)}}) \subseteq \mathcal{B}_{d_X}(y_{i_{k,s},s}, \epsilon_{k,s})$. Our choice of u'_s ensures that conditions (1)–(3) hold, and our strict upper bounds on $\epsilon_{k,s}$ yield the remaining conditions. This completes the recursive construction.

Condition (1) ensures that we obtain a function $\psi: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ by setting $\psi(s) = \bigcup_{n \in \mathbb{N}} u_{s \upharpoonright n}$, and condition (2) implies that $\psi(\mathbb{N}^{\mathbb{N}}) \subseteq C$. Set $x_s = (\phi \circ \psi)(s)$ for $s \in \mathbb{N}^{\mathbb{N}}$, and define $\pi: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ by $\pi(s) = x_s$. We will show that π is a topological embedding of $\mathbb{N}^{\mathbb{N}}$ into Y .

LEMMA A.3. *Suppose that $s \in \mathbb{N}^{< \mathbb{N}}$. Then $\pi(\mathcal{N}_s^*) \subseteq \overline{\phi(\mathcal{N}_{u_s})}$.*

Proof. Simply observe that

$$\begin{aligned} \pi(\mathcal{N}_s^*) &= (\phi \circ \psi)(\mathcal{N}_s) \cup \{x_t \mid t \in \mathcal{N}_s^* \setminus \mathcal{N}_s\} \\ &\subseteq \phi(\mathcal{N}_{u_s}) \cup \bigcup_{t \in \mathcal{N}_s^* \setminus \mathcal{N}_s} \overline{\phi(\mathcal{N}_{u_t})} \subseteq \overline{\phi(\mathcal{N}_{u_s})}, \end{aligned}$$

by conditions (1) and (4). ■

To see that π is injective, suppose that $s, t \in \mathbb{N}^{\leq \mathbb{N}}$ are distinct. If there is a least $n \leq \min\{|s|, |t|\}$ with $s \upharpoonright n \neq t \upharpoonright n$, then condition (6) ensures that $\overline{\phi(\mathcal{N}_{u_{s \upharpoonright n}})}$ and $\overline{\phi(\mathcal{N}_{u_{t \upharpoonright n}})}$ are disjoint, and since Lemma A.3 implies that $\pi(s)$ is in the former and $\pi(t)$ is in the latter, it follows that they are distinct. Otherwise, after reversing the roles of s and t if necessary, we can assume that there exists $n < |t|$ for which $s = t \upharpoonright n$. But then condition (5) ensures that $\pi(s) \notin \overline{\phi(\mathcal{N}_{u_{t \upharpoonright (n+1)}})}$, while Lemma A.3 implies that $\pi(t) \in \overline{\phi(\mathcal{N}_{u_{t \upharpoonright (n+1)}})}$, thus $\pi(s) \neq \pi(t)$.

As $\mathbb{N}_*^{\leq \mathbb{N}}$ is compact, it only remains to check that π is continuous. And for this, it is enough to check that for all $n \in \mathbb{N}$ and $s \in \mathbb{N}_*^{\leq \mathbb{N}}$, there is an open neighborhood of s whose image under π is a subset of $\mathcal{B}_{d_X}(\pi(s), 1/n)$. Towards this end, note first that if $s \in \mathbb{N}^{\mathbb{N}}$, then Lemma A.3 ensures that $\pi(\mathcal{N}_{s \upharpoonright (n+1)}^*) \subseteq \overline{\phi(\mathcal{N}_{u_{s \upharpoonright (n+1)}})}$, so condition (3) implies that $\mathcal{N}_{s \upharpoonright (n+1)}^*$ is an open neighborhood of s whose image under π is a subset of $\mathcal{B}_{d_X}(\pi(s), 1/n)$. On the other hand, if $s \in \mathbb{N}^{< \mathbb{N}}$, then Lemma A.3 ensures that

$$\pi\left(\mathcal{N}_s^* \setminus \bigcup_{i < n} \mathcal{N}_{s \frown (i)}^*\right) = \pi\left(\{s\} \cup \bigcup_{i \geq n} \mathcal{N}_{s \frown (i)}^*\right) \subseteq \{\pi(s)\} \cup \bigcup_{i \geq n} \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})},$$

so condition (4) implies that $\mathcal{N}_s^* \setminus \bigcup_{i < n} \mathcal{N}_{s \frown (i)}^*$ is an open neighborhood of s whose image under π is a subset of $\mathcal{B}_{d_X}(\pi(s), 1/n)$. ■

As a corollary, we obtain the following dichotomy theorem characterizing the family of Borel sets which are F_σ .

THEOREM A.4 (Hurewicz). *Suppose that X is a Polish space and $B \subseteq X$ is Borel. Then exactly one of the following holds:*

- (1) *The set B is F_σ .*
- (2) *There is a topological embedding $\pi: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ of $\mathbb{N}^{\mathbb{N}}$ into B .*

Proof. Proposition A.1 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, let G denote the \aleph_0 -dimensional dihypergraph consisting of all sequences $(y_n)_{n \in \mathbb{N}}$ of points of B converging to a point of $X \setminus B$. We say that a set $W \subseteq X$ is G -independent if $G \upharpoonright W = \emptyset$. Note that the closure of every such subset of B is contained in B . In particular, it follows that if B is a union of countably-many G -independent sets, then it is F_σ . Otherwise, Lecomte's dichotomy theorem for \aleph_0 -dimensional dihypergraphs of uncountable chromatic number (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) yields a dense G_δ set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous homomorphism $\phi: C \rightarrow B$ from $\mathbb{G}_0^{\mathbb{N}} \upharpoonright C$ to G , in which case Proposition A.2 yields a topological embedding $\pi: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ of $\mathbb{N}^{\mathbb{N}}$ into B . ■

There is also a parametrized form of this theorem.

THEOREM A.5 (Saint Raymond). *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set with F_σ horizontal sections. Then R is a union of countably-many analytic subsets with closed horizontal sections.*

Proof. The parametrized form of our earlier dihypergraph is given by

$$G = \left\{ ((x_n)_{n \in \mathbb{N}}, y) \in (R^y)^{\mathbb{N}} \times Y \mid \exists x \in X \setminus R^y \ x = \lim_{n \rightarrow \infty} x_n \right\}.$$

We say that a set $S \subseteq X \times Y$ is G -independent if S^y is G^y -independent for all $y \in Y$. Note that the closure of every horizontal section of every such subset of R is contained in the corresponding horizontal section of R . Moreover, if $S \subseteq R$ is analytic, then so too is the set $\{(x, y) \in X \times Y \mid x \in \overline{S^y}\}$. In particular, it follows that if R is a union of countably-many G -independent analytic subsets, then it is a union of countably-many analytic subsets with closed horizontal sections. Otherwise, the parametrized form of the dichotomy theorem for \aleph_0 -dimensional dihypergraphs of uncountable Borel chromatic number (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 31]) yields a dense G_δ set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and $y \in Y$ for which there is a continuous homomorphism $\phi: C \rightarrow R^y$ from $\mathbb{G}_0^{\mathbb{N}} \upharpoonright C$ to G^y , in which case Proposition A.2 yields a continuous embedding $\pi: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow X$ of $\mathbb{N}^{\mathbb{N}}$ into R^y . As the latter is F_σ , this contradicts Proposition A.1. ■

Our use of this result is via the following corollary.

THEOREM A.6. *Suppose that X and Y are Polish spaces and $f: X \rightarrow Y$ is an F_σ -to-one Borel function. Then f is strongly σ -closed-to-one.*

Proof. As the set $R = \text{graph}(f)$ is Borel (see, for example, [Kec95, Proposition 12.4]) and has F_σ horizontal sections, an application of Theorem A.5 ensures that it is a union of countably-many analytic sets with closed horizontal sections. As the projections of these sets onto X intersect the preimage of each singleton in a closed set, it follows that f is strongly σ -closed-to-one. ■

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References

- [JR82] J. E. Jayne and C. A. Rogers, *First level Borel functions and isomorphisms*, J. Math. Pures Appl. (9) 61 (1982), 177–205.
- [Kec95] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, New York, 1995.
- [KST99] A. S. Kechris, S. Solecki, and S. Todorcevic, *Borel chromatic numbers*, Adv. Math. 141 (1999), 1–44.
- [Lec09] D. Lecomte, *A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension*, Trans. Amer. Math. Soc. 361 (2009), 4181–4193.
- [LZ14] D. Lecomte and M. Zelený, *Baire-class ξ colorings: the first three levels*, Trans. Amer. Math. Soc. 366 (2014), 2345–2373.
- [Mil11] B. D. Miller, *Dichotomy theorems for countably infinite dimensional analytic hypergraphs*, Ann. Pure Appl. Logic 162 (2011), 561–565.
- [Mil12] B. D. Miller, *The graph-theoretic approach to descriptive set theory*, Bull. Symbolic Logic 18 (2012), 554–575.
- [Sol98] S. Solecki, *Decomposing Borel sets and functions and the structure of Baire class 1 functions*, J. Amer. Math. Soc. 11 (1998), 521–550.

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