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ERGODICITY CONDITIONS ON THE GROUP OF 3-ADIC INTEGERS

BY

NACIMA MEMIĆ (Sarajevo)

Abstract. We provide necessary and sufficient conditions for ergodicity on $\mathbb{Z}_3$ and deduce an alternative proof for the ergodicity of 3-adic affine dynamical systems. Moreover, we prove that the ergodicity characterization by means of the van der Put series known for the group of 2-adic integers has no equivalent on $\mathbb{Z}_3$.

1. Introduction. Let $\mathbb{Z}_p$ be the group of p-adic integers endowed with its ultrametric norm $|\cdot|$ and natural probability measure μ.

We recall that a bijective function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is *measure preserving* if $\mu(f^{-1}(S)) = \mu(S)$ for every measurable subset S of $\mathbb{Z}_p$. A description of 1-Lipschitz measure preserving functions on $\mathbb{Z}_p$ was given in [Y1] and [KY].

Also, a measure preserving function is said to be *ergodic* if it has no proper invariant subset.

A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be *bijective modulo* p^n where n is a positive integer if for each $x \in \mathbb{Z}_p$ the elements $x, f(x), \ldots, f^{p^n-1}(x)$ are representatives of distinct classes of $\mathbb{Z}_p/p^n\mathbb{Z}_p$.

A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be *transitive modulo* p^n if it is bijective modulo p^n and the set $x, f(x), \ldots, f^{p^n-1}(x)$ is composed of only one cycle. In other words, $f^{p^n}(x) = x \pmod{p^n}$, but $f^r(x) \neq x \pmod{p^n}$ for all $r < p^n$.

It can be proved (see [DS, Theorem 3.2]) that an isometric function is ergodic if and only if it is transitive modulo p^n for every positive integer n.

Some equivalent definitions of measure preserving and ergodic 1-Lipschitz functions are given in [A], [AKY2] and [DP].

We use the same notation as in [J]. Namely, if the p-adic expansion of the positive integer m is given by

$$m = \sum_{i=0}^{s} a_i p^i, \qquad 0 \leq a_i < p,\ a_s \neq 0,$$

then we define $q(m) = a_s p^s$.

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For every 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ we define the coefficients

$$B_m = \begin{cases} f(m), & m \in \{0, \ldots, p-1\}, \\ f(m) - f(m - q(m)), & m \geq p. \end{cases}$$

In this way the function f can be represented in the so called *van der Put basis* as follows:

$$f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x),$$

where if $m > 0$,

$$\chi(m, x) = \begin{cases} 1, & |x - m| \leq p^{-[\log_p m]-1}, \\ 0, & \text{otherwise.} \end{cases}$$

For $m = 0$ we have

$$\chi(0, x) = \begin{cases} 1, & |x| \leq p^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Ergodicity of 1-Lipschitz functions on 2-adic integers was characterized by means of the coefficients $(B_m)_m$:

THEOREM 1.1 (Anashin, Khrennikov and Yurova [Y2], [AKY1], [AKY2]). *A 1-Lipschitz function f is ergodic on $\mathbb{Z}_2$ if and only if:*

(1) $B_0 = 1 \pmod 2$,
(2) $B_0 + B_1 = 3 \pmod 4$,
(3) $(B_2 + B_3)/2 = 2 \pmod 4$,
(4) $|B_m| = 2^{-n}$ *for all* $m \in \{2^n, \ldots, 2^{n+1} - 1\}$,
(5) $2^{-n+1} \sum_{m=2^{n-1}}^{2^n-1} B_m = 0 \pmod 4$ *for all* $n \geq 3$.

On the other hand, S. Jeong proved some sufficient conditions for ergodicity of 1-Lipschitz measure preserving functions on general p-adic integers expressed by the coefficients $(B_m)_m$.

THEOREM 1.2 (S. Jeong [J, Theorem 3.8]). *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be 1-Lipschitz and measure preserving. If*

(1) $B_0 = s \pmod p$, $0 < s < p$,
(2) $\sum_{m=0}^{p-1} B_m = ps + \frac{1}{2}(p-1)p \pmod{p^2}$,
(3) $\sum_{m=p}^{p^2-1} B_m = \frac{1}{2}(p-1)p^3 \pmod{p^3}$,
(4) $B_m = q(m) \pmod{p^{[\log_p m]+1}}$,
(5) $B_m = B_0 + m \pmod p$,
(6) $\sum_{m=p^{n-1}}^{p^n-1} B_m = 0 \pmod{p^{n+1}}$, $n \geq 3$,

then f is ergodic.

In Section 2 we provide some necessary and sufficient conditions for ergodicity of 1-Lipschitz invertible functions on the set of 3-adic integers. In

Theorem 2.2 the conditions involve some specific powers of f. In Theorem 2.5 we prove sufficient and necessary ergodicity conditions by means of the coefficients $(B_m)_m$ for 1-Lipschitz invertible functions already satisfying condition (4) of Theorem 1.2.

In Section 3 we discuss this last condition. Namely, Example 3.1 shows that without this condition it is impossible to express necessary and sufficient conditions by means of sums $(\sum_{m=3^{n-1}}^{3^n-1} B_m)_n$ as in Theorems 1.2 and 1.1. On the other hand, Example 3.3 shows that condition (4) of Theorem 1.2 is not necessary for ergodicity of 1-Lipschitz invertible functions.

2. Necessary and sufficient conditions

LEMMA 2.1. *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a 1-Lipschitz and measure preserving function. Then f is isometric.*

Proof. It was proved in [DS, Theorem 3.1] that such a function is locally isometric and invertible. Suppose there exist $x, y \in \mathbb{Z}_p$ such that $|f(x) - f(y)| < |x - y|$. Let $|x - y| = p^{-l}$ where l is a nonnegative integer. Since f is 1-Lipschitz, for all $z \in x + p^{l+1}\mathbb{Z}_p$ we have $|f(x) - f(z)| \le p^{-l-1}$. Then

$$f^{-1}(f(x) + p^{l+1}\mathbb{Z}_p) \supseteq x + p^{l+1}\mathbb{Z}_p.$$

Similarly,

$$f^{-1}(f(y) + p^{l+1}\mathbb{Z}_p) \supseteq y + p^{l+1}\mathbb{Z}_p.$$

However, this contradicts the fact that f is measure preserving because

$$f^{-1}(f(x) + p^{l+1}\mathbb{Z}_p) = f^{-1}(f(y) + p^{l+1}\mathbb{Z}_p),$$

while

$$x + p^{l+1}\mathbb{Z}_p \cap y + p^{l+1}\mathbb{Z}_p = \emptyset. \ \blacksquare$$

THEOREM 2.2. *Let $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ be a 1-Lipschitz measure preserving function. Then f is ergodic if and only if*

$$f^{3^{n-1}}(x) \neq x \ (\mathrm{mod}\ 3^n) \quad \textit{for every } x \in \mathbb{Z}_3 \textit{ and } n \in \mathbb{N}. \tag{2.1}$$

Proof. If f is ergodic then (2.1) follows immediately from the definition of transitivity.

Assume that (2.1) holds. We know from Lemma 2.1 that f is isometric. By [DS, Theorem 3.2] it suffices to prove that f is transitive modulo 3^n for every positive integer n.

We first prove this for $n = 1$. Namely, we will show that

$$f(1) = 1 + f(0) \ (\mathrm{mod}\ 3), \qquad f(2) = 2 + f(0) \ (\mathrm{mod}\ 3).$$

Suppose that

$$f(1) \neq 1 + f(0) \ (\mathrm{mod}\ 3).$$

Since f is isometric and measure preserving, this implies that

$$f(1) = 2 + f(0) \pmod 3, \qquad f(2) = 1 + f(0) \pmod 3.$$

Meanwhile, in this case the following equivalences hold:

$$f(1) = 0 \pmod 3 \;\Leftrightarrow\; f(0) = 1 \pmod 3,$$
$$f(2) = 0 \pmod 3 \;\Leftrightarrow\; f(0) = 2 \pmod 3.$$

Moreover, since $f(0) \neq 0 \pmod 3$, we have either

$$f(1) = 0 \pmod 3 \quad \text{or} \quad f(2) = 0 \pmod 3.$$

In case $f(1) = 0 \pmod 3$, we have $f(0) = 1 \pmod 3$, and then $f(2) = 2 \pmod 3$, which contradicts the assumption. Similarly, $f(2) = 0 \pmod 3$ leads to a contradiction.

Suppose that f is transitive modulo 3^{n-1} for some $n \geq 2$. Then f is not transitive modulo 3^n only if $f^{2\cdot 3^{n-1}}(x) = x \pmod{3^n}$ for some x. Assume that there exists such an x. Define $z \in x + 3^{n-1}\mathbb{Z}_3$, so that

$$x + 3^{n-1}\mathbb{Z}_3 = (x + 3^n\mathbb{Z}_3) \uplus (f^{3^{n-1}}(x) + 3^n\mathbb{Z}_3) \uplus (z + 3^n\mathbb{Z}_3).$$

Since f is transitive modulo 3^{n-1}, we have $f^{3^{n-1}}(z) \in x + 3^{n-1}\mathbb{Z}_3$, which is only possible if $f^{3^{n-1}}(z) \in x + 3^n\mathbb{Z}_3$. Therefore,

$$z = f^{3^n}(z) \pmod{3^n} = f^{2\cdot 3^{n-1}}(x) \pmod{3^n} = x \pmod{3^n},$$

which is absurd. ■

As an application of this result we see that ergodicity of p-adic dynamical systems studied in [FLYZ] can be characterized by means of modularity methods. The following example is a special case of the results proved in [FLYZ]. Here we give a proof using Theorem 2.2.

EXAMPLE 2.3. *On $\mathbb{Z}_3$ consider the affine 3-adic dynamical system f defined by $x \mapsto \alpha x + \beta$, where α and β are 3-adic integers. Then the dynamical system f is ergodic if and only if $\alpha = 1 \pmod 3$ and $\beta \neq 0 \pmod 3$.*

Proof. We may assume that α is a unit, because otherwise f is not measure preserving. Moreover, f is by definition 1-Lipschitz, hence Theorem 2.2 can be applied.

Since

$$f^k(x) = \alpha^k x + (\alpha^{k-1} + \cdots + \alpha + 1)\beta, \qquad k \geq 1,$$

Theorem 2.2 shows that f is ergodic if and only if for every $n \geq 1$ and every $x \in \mathbb{Z}_3$ we have

$$\alpha^{3^{n-1}} x + (\alpha^{3^{n-1}-1} + \cdots + \alpha + 1)\beta \neq x \pmod{3^n}. \tag{2.2}$$

Suppose that f is ergodic. Then (2.2) for $n = 1$ immediately yields $\alpha = 1 \pmod 3$ and $\beta \neq 0 \pmod 3$.

On the other hand, if we assume that $\alpha = 1 \pmod 3$, $\beta \neq 0 \pmod 3$ and (2.2) is not valid for some $n \geq 1$ and $x \in \mathbb{Z}_3$, then since

$$\alpha^{3^{n-1}-1} + \cdots + \alpha + 1 \neq 0 \pmod{3^n}, \qquad \beta \neq 0 \pmod 3,$$

and since recursively on n we can see that $\alpha^{3^{n-1}} = 1 \pmod{3^n}$, it follows that the equality

$$(\alpha^{3^{n-1}} - 1)x + (\alpha^{3^{n-1}-1} + \cdots + \alpha + 1)\beta = 0 \pmod{3^n}$$

is impossible. Hence (2.2) is true. ■

Lemma 2.4. *Let $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ be a 1-Lipschitz measure preserving function satisfying $B_m = q(m) \pmod{3^{[\log_3 m]+1}}$ for every $m \geq 3$. Suppose that f is transitive modulo 3^n. Then the following assertions are equivalent:*

(1) $f^{3^n}(x) \neq x \pmod{3^{n+1}}$ *for all* $x \in \mathbb{Z}_3$,
(2) $f^{3^n}(x) \neq x \pmod{3^{n+1}}$ *for some* $x \in \mathbb{Z}_3$,
(3)

$$\sum_{m=3^{n-1}}^{3^n-1} B_m + 3\sum_{m=3^{n-2}}^{3^{n-1}-1} B_m + \cdots + 3^{n-1}\sum_{m=0}^{2} B_m \neq 3^n \pmod{3^{n+1}},\ n \geq 2.$$

If $n = 1$ assertion (3) should be replaced by $\sum_{m=0}^{2} B_m \neq 3 \pmod{3^2}$.

Proof. Applying induction on n gives

$$\sum_{m=3^{n-1}}^{3^n-1} B_m + 3\sum_{m=3^{n-2}}^{3^{n-1}-1} B_m + \cdots + 3^{n-1}\sum_{m=0}^{2} B_m = \sum_{m=0}^{3^n-1} f(m). \tag{2.3}$$

Fix $x \in \mathbb{Z}_3$. Define integers $y_0, \ldots, y_{3^n-1}$ as follows: $y_0 \in \{0, \ldots, 3^n - 1\}$ is such that $x \in y_0 + 3^n\mathbb{Z}_3$, and for every $i \in \{1, \ldots, 3^n - 1\}$,

$$y_i + 3^n\mathbb{Z}_3 = f(y_{i-1} + 3^n\mathbb{Z}_3).$$

Hence, clearly

$$y_0 + 3^n\mathbb{Z}_3 = f(y_{3^n-1} + 3^n\mathbb{Z}_3).$$

For every $i \in \{0, \ldots, 3^n - 1\}$, $k \in \{0, 1, 2\}$ and $z \in y_i + k \cdot 3^n + 3^{n+1}\mathbb{Z}_3$ we have

$$\begin{aligned} f(z) - f(y_i) &= f(y_i + k \cdot 3^n) - f(y_i) \pmod{3^{n+1}} \\ &= \begin{cases} B_{y_i + k\cdot 3^n} \pmod{3^{n+1}}), & k \neq 0, \\ 0 \pmod{3^{n+1}}, & k = 0. \end{cases} \end{aligned}$$

Therefore,

$$f(z) = f(y_i) + z - y_i \pmod{3^{n+1}} \quad \text{for every } z \in y_i + 3^n\mathbb{Z}_3.$$

We deduce that

$$f^{i+1}(x) = f(f^i(x)) = f(y_i) + f^i(x) - y_i \pmod{3^{n+1}},$$

hence

$$f^{3^n}(x) = \sum_{i=0}^{3^n-1}(f(y_i)-y_i)+x = \sum_{m=0}^{3^n-1} f(m) - \frac{3^n(3^n-1)}{2} + x.$$

The result follows immediately by applying (2.3) and

$$\frac{3^n(3^n-1)}{2} = 3^n \pmod{3^{n+1}}. \ \blacksquare$$

THEOREM 2.5. *Let $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ be a 1-Lipschitz invertible function satisfying $B_m = q(m) \pmod{3^{[\log_3 m]+1}}$ for every $m \geq 3$. Then f is ergodic if and only if:*

(1) $B_m \neq m \pmod 3$ *for* $m < 3$,
(2) $\sum_{m=0}^{2} B_m \neq 3 \pmod{3^2}$,
(3)

$$\sum_{m=3^{n-1}}^{3^n-1} B_m + 3\sum_{m=3^{n-2}}^{3^{n-1}-1} B_m + \cdots + 3^{n-1}\sum_{m=0}^{2} B_m \neq 3^n \pmod{3^{n+1}},\ n \geq 2.$$

Proof. It was proved in Theorem 2.2 that condition (1) of Theorem 2.5 implies that f is transitive modulo 3. Applying Lemma 2.4 we see that conditions (2) and (3) lead to transitivity of all orders. Conversely, transitivity modulo 3 immediately gives condition (1), and from Lemma 2.4 we obtain (2) and (3). $\blacksquare$

3. Discussion and examples. Although, as we will see in the following examples, the property $B_m = q(m) \pmod{3^{[\log_3 m]+1}}$ is not necessary for ergodicity, it can be seen in Example 3.1 that if this condition is not satisfied then ergodicity cannot be characterized by means of sums appearing in conditions (2) and (3) of Theorem 2.5 or conditions (ii)(2), (3) and (5) of [J, Theorem 3.8].

EXAMPLE 3.1. *Let $s \in \{0,1,2\}$ and let $(k_n)_{n\geq 3}$ be any sequence of integers satisfying $k_n = 0 \pmod{3^n}$. Suppose there exists an invertible isometric function $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ which is transitive modulo 3 and*

(1) $B_0 + B_1 + B_2 = 3(1+s) \pmod 9$,
(2) $\sum_{m=3^{n-1}}^{3^n-1} B_m = k_n \pmod{3^{n+1}}$ *for all* $n \geq 3$.

Then there exists an invertible, isometric function $\tilde{f} : \mathbb{Z}_3 \to \mathbb{Z}_3$ satisfying (1) and (2) but not transitive modulo 9.

Proof. We construct $\tilde{f}$ for each of the three possible values of s.

If

$$f(x) = \sum_{i=0}^{\infty} a_i 3^i,$$

then we write

$$F_n(x) = \sum_{i=n}^{\infty} a_i 3^i.$$

Let $s = 1$. Then we define

$$\begin{aligned}
&\tilde{f}(x) = 7 + F_2(x),\ \forall x \in B_{3^{-2}}(0); \quad &&\tilde{f}(x) = 2 + F_2(x),\ \forall x \in B_{3^{-2}}(1);\\
&\tilde{f}(x) = 6 + F_2(x),\ \forall x \in B_{3^{-2}}(2); \quad &&\tilde{f}(x) = 4 + F_2(x),\ \forall x \in B_{3^{-2}}(3);\\
&\tilde{f}(x) = 5 + F_2(x),\ \forall x \in B_{3^{-2}}(4); \quad &&\tilde{f}(x) = 3 + F_2(x),\ \forall x \in B_{3^{-2}}(5);\\
&\tilde{f}(x) = 1 + F_2(x),\ \forall x \in B_{3^{-2}}(6); \quad &&\tilde{f}(x) = 8 + F_2(x),\ \forall x \in B_{3^{-2}}(7);\\
&\tilde{f}(x) = F_2(x),\ \forall x \in B_{3^{-2}}(8).
\end{aligned}$$

It is easily verified that $\tilde{f}$ is invertible and isometric but not transitive modulo 9. Moreover, (1) is satisfied because

$$\tilde{f}(0) + \tilde{f}(1) + \tilde{f}(2) = 6 \pmod 9.$$

On the other hand, from the definition of the van der Put coefficients, if $(B_m)_m$ are the coefficients corresponding to f, and $(\tilde{B}_m)_m$ those for $\tilde{f}$, it can be seen that $\tilde{B}_m = B_m$ whenever $m \geq 9$. Indeed, since f is isometric, for every $m \geq 9$ we have either $F_2(m) \neq 0$ or $F_2(m - q(m)) \neq 0$, because otherwise either $f(m) - f(m - q(m)) = 0$ or $|f(m) - f(m - q(m))| > 3^{-2}$, which contradicts isometricity. Hence,

$$\tilde{B}_m = \tilde{f}(m) - \tilde{f}(m - q(m)) = F_2(m) - F_2(m - q(m)) = f(m) - f(m - q(m)),$$

and (2) follows immediately.

For $s = 2$ define

$$\begin{aligned}
&\tilde{f}(x) = 4 + F_2(x),\ \forall x \in B_{3^{-2}}(0); \quad &&\tilde{f}(x) = 2 + F_2(x),\ \forall x \in B_{3^{-2}}(1);\\
&\tilde{f}(x) = 3 + F_2(x),\ \forall x \in B_{3^{-2}}(2); \quad &&\tilde{f}(x) = 1 + F_2(x),\ \forall x \in B_{3^{-2}}(3);\\
&\tilde{f}(x) = 5 + F_2(x),\ \forall x \in B_{3^{-2}}(4); \quad &&\tilde{f}(x) = F_2(x),\ \forall x \in B_{3^{-2}}(5);\\
&\tilde{f}(x) = 7 + F_2(x),\ \forall x \in B_{3^{-2}}(6); \quad &&\tilde{f}(x) = 8 + F_2(x),\ \forall x \in B_{3^{-2}}(7);\\
&\tilde{f}(x) = 6 + F_2(x),\ \forall x \in B_{3^{-2}}(8).
\end{aligned}$$

Finally, if $s = 0$ define

$$\begin{aligned}
&\tilde{f}(x) = 1 + F_2(x),\ \forall x \in B_{3^{-2}}(0); \quad &&\tilde{f}(x) = 5 + F_2(x),\ \forall x \in B_{3^{-2}}(1);\\
&\tilde{f}(x) = 6 + F_2(x),\ \forall x \in B_{3^{-2}}(2); \quad &&\tilde{f}(x) = 4 + F_2(x),\ \forall x \in B_{3^{-2}}(3);\\
&\tilde{f}(x) = 8 + F_2(x),\ \forall x \in B_{3^{-2}}(4); \quad &&\tilde{f}(x) = F_2(x),\ \forall x \in B_{3^{-2}}(5);\\
&\tilde{f}(x) = 7 + F_2(x),\ \forall x \in B_{3^{-2}}(6); \quad &&\tilde{f}(x) = 2 + F_2(x),\ \forall x \in B_{3^{-2}}(7);\\
&\tilde{f}(x) = 3 + F_2(x),\ \forall x \in B_{3^{-2}}(8).
\end{aligned}$$

The same conclusions as above can also be made for these cases. ■

Remark 3.2. Example 3.1 shows that Theorem 1.1 has no equivalent on $\mathbb{Z}_3$ without considering the sum $\sum_{m=3}^{8} B_m \pmod{3^3}$. In fact, this sum does not need to be mentioned since for isometric functions it only depends on s and F_2, which are common for f and $\tilde{f}$. Indeed,

$$\begin{aligned}\sum_{m=3}^{8} B_m &= f(3)+\cdots+f(8)-2(f(0)+f(1)+f(2))\\ &= \sum_{m=0}^{8} f(m)-3(f(0)+f(1)+f(2))\\ &= 1+\ldots+8+\sum_{m=0}^{8} F_2(m)-9(1+s) \pmod{3^3}\\ &= \sum_{m=0}^{8} F_2(m)-9s \pmod{3^3}.\end{aligned}$$

As said above, the property $B_m = q(m) \pmod{3^{[\log_3 m]+1}}$ allows a direct characterization of ergodicity by means of sums in conditions (2) and (3) of Theorem 2.5. In the following example we can see that this condition is not necessary for ergodicity.

Example 3.3. *Let f be any function satisfying all conditions of Theorem 2.5, with the additional assumption $f(0) = 1 \pmod 3$. Let the function g be defined as follows:*

For every $x \in 3\mathbb{Z}_3$ set $g(x) = f(x)$. For $x \in 1+3\mathbb{Z}_3$ define

$$g(x) = \begin{cases} f(x+3), & x \in 4+9\mathbb{Z}_3,\\ f(x-3), & x \in 7+9\mathbb{Z}_3,\\ f(x), & x \in 1+9\mathbb{Z}_3.\end{cases}$$

For $x \in 2+3\mathbb{Z}_3$ define

$$g(x) = f^2(g^{-1}(x)).$$

Then g is ergodic and

$$B_m = -q(m) \pmod{3^{[\log_3 m]+1}}, \qquad m \in \{4,7\}. \tag{3.1}$$

Proof. The function g is obviously invertible and isometric. Property (3.1) is easily verified from the definition of g. We can prove ergodicity by using Theorem 2.2. Indeed, in order to see that g satisfies (2.1) notice that for every $x \in 3\mathbb{Z}_3$, since $g^2(x) \in 2+3\mathbb{Z}_3$ and $g(x) = f(x)$, we have

$$g^3(x) = g(g^2(x)) = f^2(g^{-1}(g^2(x))) = f^2(g(x)) = f^3(x).$$

Now let $x \in 1+3\mathbb{Z}_3$. Since $g(x) \in 2+3\mathbb{Z}_3$ we have

$$g^2(x) = g(g(x)) = f^2\left(g^{-1}(g(x))\right) = f^2(x) \in 3\mathbb{Z}_3.$$

This yields $g^3(x) = f^3(x)$, from which we deduce by induction that $g^{3^n} = f^{3^n}$ for every positive integer n and every $x \in 3\mathbb{Z}_3 \cup 1 + 3\mathbb{Z}_3$.

Now, let $x \in 2 + 3\mathbb{Z}_3$. We have $g(x) = f^2(g^{-1}(x)) \in 3\mathbb{Z}_3$, so $g^2(x) = f^3(g^{-1}(x))$, and therefore

$$g^3(x) = g\left(f^3(g^{-1}(x))\right).$$

Recursively on positive integers k we deduce that

$$g^{3k}(x) = g(f^{3k}(g^{-1}(x))).$$

Applying (2.1) on f and isometricity of g we find for every positive integer n that

$$g^{3^n}(x) = g\left(f^{3^n}(g^{-1}(x))\right) \neq g(g^{-1}(x)) \pmod{3^{n+1}} = x \pmod{3^{n+1}}. \blacksquare$$

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Nacima Memić
Department of Mathematics
Faculty of Natural Sciences and Mathematics
University of Sarajevo
Zmaja od Bosne 33-35
Sarajevo, Bosnia and Herzegovina
E-mail: nacima.o@gmail.com