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A short self-contained proof of the Commutation Theorem

by

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Abstract. Let G be a locally compact group. We give a self-contained elementary proof of the relation $\lambda(G)'' = \rho(G)'$ between the left and right regular representations of G.

Let G be a locally compact group. Denote by $\lambda : G \to \mathcal{U}(L^2(G))$ the left regular representation of G, and by $\rho : G \to \mathcal{U}(L^2(G))$ the right regular representation of G. The Commutation Theorem states:

THEOREM 1. $\lambda(G)'' = \rho(G)'$.

By the von Neumann double commutant theorem, $\lambda(G)''$ and $\rho(G)''$ are the weak*-closed subalgebras of $\mathcal{B}(L^2(G))$ generated by $\lambda(G)$ and $\rho(G)$, respectively; recall that the former is the group von Neumann algebra $VN(G)$ of G. Thus the Commutation Theorem says that these two von Neumann algebras are commutants of each other.

This theorem was first proved by Murray and von Neumann [5, Lemma 5.3.3] for discrete groups, by Segal [6] for unimodular groups, and finally by Dixmier [2] for general locally compact groups using the theory of quasi-Hilbert algebras. The aim of this short note is to give a self-contained elementary proof of Theorem 1. (For such a proof in the case of unimodular groups, see the recent book [1].)

As the aim is to give an elementary self-contained proof of Theorem 1, I would like to stress that all we need (besides some basic concepts from functional analysis, and the von Neumann double commutant theorem, which is only used in discussion of the immediate consequence of Theorem 1 above) is basic facts about integration theory on locally compact groups as can be found in any textbook on abstract harmonic analysis (such as [4, Chapter 2]).

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We shall even take care of distinguishing between a function and its equivalence class, and use convolution product only in the situation where the resulting product is an everywhere defined continuous function (see Lemma 2 below).

Let us now fix some more notation. A (left) Haar measure on G is fixed; the integral of a function f with respect to the Haar measure is written as $\int f(x)\,dx = \int f$. Let us denote by Δ the modular function of the group G. For any complex function f on G, we write $\widetilde{f}(x) := \overline{f(x^{-1})}$.

Denote by $\mathcal{C}(G)$ the algebra of all the complex-valued continuous functions defined on G, by $\mathcal{C}^b(G)$ the subalgebra consisting of those that are bounded, and by $\mathcal{C}_c(G)$ those with compact support. Denote by $\mathscr{B}(G)$ the collection of all complex-valued Borel functions on G. Given any $f \in \mathscr{B}(G)$, its equivalence class is

$$[f] := \{g \in \mathscr{B}(G) \colon g = f \text{ a.e. on } G\},$$

where a.e. means a.e. with respect to the Haar measure on G. We shall always distinguish between a function f and its equivalence class $[f]$.

For $1 \le p < \infty$, define as usual the spaces $\mathscr{L}^p(G)$ and $L^p(G)$ relative to the Haar measure; the former is a seminormed space of functions and the latter is a normed space (in fact, a Banach space) of equivalence classes of functions.

As we differentiate between functions and their equivalence classes, we have to differentiate between convolution products of functions and of equivalence classes of functions. But in fact, we shall only use the convolution product in one of the following situations.

LEMMA 2.

(i) *If $f, g \in \mathscr{L}^2(G)$, then $f * \widetilde{g}$ is an everywhere defined bounded continuous function on G with its uniform norm satisfying*

$$|f * \widetilde{g}|_G \le \|f\|_2 \|g\|_2.$$

(ii) *If $f \in \mathcal{C}_c(G)$ and $g \in \mathscr{L}^2(G)$, then $f * g$ is an everywhere defined (possibly unbounded) continuous function in $\mathscr{L}^2(G)$ such that*

$$\|f * g\|_2 \le \|f\|_1 \|g\|_2.$$

(iii) *If $f \in \mathscr{L}^2(G)$ and $g \in \mathcal{C}_c(G)$, then $f * g$ is an everywhere defined bounded continuous function in $\mathscr{L}^2(G)$ such that*

$$\|f * g\|_2 \le C_g \|g\|_1 \|f\|_2,$$

where $C_g := \sup\{\Delta(y)^{-1/2} \colon y \in \operatorname{supp} g\}$.

In particular, there will be no problems of measurability or of functions only defined a.e. in the discussion below.

Proof of Lemma 2. Most of the conclusions are standard. Statement (i) is a consequence of Cauchy–Schwarz–Hölder's inequality.

The inequality in (ii) can be found in, say, [4, Proposition 2.39]. For the continuity, notice that for $f \in \mathcal{C}_c(G)$ and $g \in \mathscr{L}^2(G)$,

$$\begin{aligned}\widetilde{(f*g)}(x) &= \overline{\int f(y)g(y^{-1}x^{-1})\,dy} = \int \widetilde{g}(xy)\widetilde{f}(y^{-1})\,dy \\ &= \Delta(x)^{1/2}\int \widetilde{g}(xy)\Delta(xy)^{-1/2} \cdot \widetilde{f}(y^{-1})\Delta(y^{-1})^{-1/2}\,dy \\ &= \Delta(x)^{1/2}\big((\widetilde{g}\Delta^{-1/2}) * (\widetilde{f}\Delta^{-1/2})\big)(x).\end{aligned}$$

Since $\widetilde{g}\Delta^{-1/2} \in \mathscr{L}^2(G)$ while $\widetilde{f}\Delta^{-1/2} \in \mathcal{C}_c(G) \subseteq \widetilde{\mathscr{L}^2(G)}$, the continuity of $f * g$ in (ii) then follows from that of (i).

Most of (iii) could also be found in [4, Proposition 2.39], but it could also be deduced from (i) and (ii) as follows. Suppose that $f \in \mathscr{L}^2(G)$ and $g \in \mathcal{C}_c(G)$. Then, first of all, $g = \widetilde{(\widetilde{g})}$ and $\widetilde{g} \in \mathcal{C}_c(G) \subseteq \mathscr{L}^2(G)$, and so $f * g = f * \widetilde{(\widetilde{g})}$ is an everywhere defined bounded continuous function on G by (i). Next, using the previous calculation,

$$\begin{aligned}\|f*g\|_2^2 &= \int |(f*g)(x)|^2\,dx = \int |\widetilde{(f*g)}(x)|^2\Delta(x)^{-1}\,dx \\ &= \int \big|\big((\widetilde{g}\Delta^{-1/2}) * (\widetilde{f}\Delta^{-1/2})\big)(x)\big|^2\,dx \\ &= \|(\widetilde{g}\Delta^{-1/2}) * (\widetilde{f}\Delta^{-1/2})\|_2^2 \le \|\widetilde{g}\Delta^{-1/2}\|_1^2\|\widetilde{f}\Delta^{-1/2}\|_2^2.\end{aligned}$$

This gives the inequality in (iii) because

$$\begin{aligned}\|\widetilde{g}\Delta^{-1/2}\|_1 &= \int |\widetilde{g}(x)\Delta(x)^{-1/2}|\,dx = \int |g(x^{-1})|\Delta(x^{-1})^{1/2}\,dx \\ &= \int |g(x)|\Delta(x)^{1/2}\Delta(x)^{-1}\,dx \le C_g\|g\|_1,\end{aligned}$$

while $\|\widetilde{f}\Delta^{-1/2}\|_2 = \|f\|_2$. ∎

LEMMA 3. *Let $f, g \in \mathscr{L}^2(G)$, and let $h \in \mathcal{C}_c(G)$. Then*

$$|(f*h)*\widetilde{g}|_G \le C_h\|h\|_1\|f\|_2\|g\|_2, \qquad |f*(h*\widetilde{g})|_G \le C_{\widetilde{h}}\|\widetilde{h}\|_1\|f\|_2\|g\|_2,$$

where C_h and $C_{\widetilde{h}}$ are as defined above.

Proof. For the first inequality,

$$|(f*h)*\widetilde{g}|_G \le \|f*h\|_2\|g\|_2 \le C_h\|f\|_2\|h\|_1\|g\|_2.$$

The second one is proved similarly, using in addition the following:

$$\widetilde{(h*\widetilde{g})}(x) = \overline{\int h(y)\widetilde{g}(y^{-1}x^{-1})\,dy} = \int g(xy)\widetilde{h}(y^{-1})\,dy = (g*\widetilde{h})(x).$$

Note that $g * \widetilde{h}$ is an everywhere defined (bounded continuous) function in $\mathscr{L}^2(G)$, and so

$$f * (h * \widetilde{g}) = f * \widetilde{(g * \widetilde{h})}$$

is an everywhere defined bounded continuous function on G. ∎

COROLLARY 4. *Let $f, g \in \mathscr{L}^2(G)$, and let $h \in \mathcal{C}_c(G)$. Then*

$$(f * h) * \widetilde{g} = f * (h * \widetilde{g}) \quad \textit{everywhere on } G.$$

Proof. Note from the preceding discussion that we are comparing two everywhere defined continuous functions. The statement follows from the special case where in addition $f, g \in \mathcal{C}_c(G)$ and the usual approximation using the previous lemma. ∎

We can now give our proof of the Commutation Theorem:

Proof of Theorem 1. It is easy to see that $\rho(G) \subseteq \lambda(G)'$, and so $\lambda(G)'' \subseteq \rho(G)'$. It remains to prove the reverse inclusion. A simple argument (e.g. approximating integrals by finite sums) shows that ([1])

$$\rho(G)' \subseteq \{S \in \mathcal{B}(L^2(G)) : S([f * g]) = (S[f]) * [g] \text{ for all } f, g \in \mathcal{C}_c(G)\},$$
$$\lambda(G)' \subseteq \{T \in \mathcal{B}(L^2(G)) : T([f * g]) = [f] * (T[g]) \text{ for all } f, g \in \mathcal{C}_c(G)\}.$$

We need to show that $ST = TS$ whenever $T \in \lambda(G)'$ and $S \in \rho(G)'$.

Indeed, for such T and S, we have

$$\|(S[f]) * [h]\|_2 \le \|S\|\, \|[f * h]\|_2 \le \|S\|\, \|f\|_1 \|h\|_2 \quad (f, h \in \mathcal{C}_c(G)).$$

For each neighbourhood V of $\mathbf{e}$, let $k_V \in \mathcal{C}_c(G)^+$ be such that $k_V = \widetilde{k_V}$ is supported in V and $\int k_V = 1$. Then we see that

$$\|(S[k_V]) * [h]\|_2 \le \|S\|\, \|[h]\|_2 \quad (h \in \mathcal{C}_c(G)).$$

It follows that there exists $A_V \in \mathcal{B}(L^2(G))$ of norm $\le \|S\|$ such that

$$A_V[h] = (S[k_V]) * [h] = S([k_V * h]) \quad (h \in \mathcal{C}_c(G));$$

consequently, A_V converges strongly to S as $V \to \{\mathbf{e}\}$.

Similarly, since $T^* \in \lambda(G)'$, we have

$$\|[h] * (T^*[g])\|_2 \le \|T\|\, \|[h * g]\|_2 \le C_g \|g\|_1 \|T\|\, \|h\|_2 \quad (g, h \in \mathcal{C}_c(G)),$$

where C_g is the constant depending on g defined earlier. So there exists $B_V \in \mathcal{B}(L^2(G))$ such that

$$B_V[h] = [h] * (T^*[k_V]) = T^*([h * k_V]) \quad (h \in \mathcal{C}_c(G)),$$

and

$$\limsup_{V \to \{\mathbf{e}\}} \|B_V\| \le \|T\|,$$

([1]) An argument using the standard approximate identity of $L^1(G)$ will show the reverse inclusion of the two relations below; however, we shall not need that.

and so B_V converges strongly to T^* as $V \to \{\mathbf{e}\}$. We deduce that $C_V := B_V^*$ converges (operator-)weakly to T as $V \to \{\mathbf{e}\}$.

Fix $f_V, g_V \in \mathscr{L}^2(G)$ such that

$$[f_V] = S[k_V] \quad \text{and} \quad [g_V] = T^*[k_V].$$

Then

$$A_V[h] = [f_V * h] \quad \text{and} \quad B_V[h] = [h * g_V] \qquad (h \in \mathcal{C}_c(G)). \tag{1}$$

For $h \in \mathscr{L}^2(G)$ and $h_1 \in \mathcal{C}_c(G)$, we see that

$$\begin{aligned}\langle C_V[h] \mid [h_1]\rangle &= \langle [h] \mid B_V[h_1]\rangle = \int h(x)\overline{(h_1 * g_V)(x)}\,dx \\ &= \iint h(x)\overline{h_1(y)g_V(y^{-1}x)}\,dy\,dx = \iint h(x)\overline{h_1(y)g_V(y^{-1}x)}\,dx\,dy \\ &= \iint h(x)\widetilde{g_V}(x^{-1}y)\overline{h_1(y)}\,dx\,dy = \int (h * \widetilde{g_V})(y)\overline{h_1(y)}\,dy,\end{aligned}$$

where the fourth equality follows from the Fubini theorem, whose hypothesis is easily checked to be satisfied. The last equality then shows that the everywhere defined (bounded) continuous function $h * \widetilde{g_V}$ must belong to $\mathscr{L}^2(G)$. Hence,

$$\langle C_V[h] \mid [h_1]\rangle = \langle [h * \widetilde{g_V}] \mid [h_1]\rangle,$$

and so

$$C_V[h] = [h * \widetilde{g_V}] \qquad (h \in \mathscr{L}^2(G)). \tag{2}$$

We remark that it is crucial for the argument below that in constructing g_V, we start with T^* instead of working directly with T.

Let U, V be neighbourhoods of $\mathbf{e}$, and let $h \in \mathcal{C}_c(G)$. By (1) and (2), we see that

$$C_U A_V[h] = C_U[f_V * h] = [(f_V * h) * \widetilde{g_U}]. \tag{3}$$

On the other hand, we have

$$C_U[h] = [h * \widetilde{g_U}];$$

but $g := h * \widetilde{g_U}$ belongs to $\mathscr{L}^2(G)$ but not to $\mathcal{C}_c(G)$, and so we cannot apply (1) yet. However, both g and $\widetilde{g}$ are still in $\mathscr{L}^2(G)$: the former represents $C_U[h]$, while for the latter, we see that $\widetilde{g} = g_U * \widetilde{h}$, and $g_U \in \mathscr{L}^2(G)$ while $\widetilde{h} \in \mathcal{C}_c(G)$, and so $g_U * \widetilde{h} \in \mathscr{L}^2(G)$ by Lemma 2(iii). Thus, there exist $g_n \in \mathcal{C}_c(G)$ such that $g_n \to g$ and $\widetilde{g}_n \to \widetilde{g}$, both in $\mathscr{L}^2(G)$ [say, by noting that $g \in \mathscr{L}^2(G, \mu)$ where $d\mu(x) = (1 + \Delta(x)^{-1})\,dx$, and then choosing $g_n \in \mathcal{C}_c(G)$ such that $g_n \to g$ in $\mathscr{L}^2(G, \mu)$]. Thus, for $\phi \in \mathcal{C}_c(G)$, we see that

$$\begin{aligned}\langle A_V[g] \mid [\phi]\rangle &= \lim_n \langle A_V[g_n] \mid [\phi]\rangle = \lim_n \langle [f_V * g_n] \mid [\phi]\rangle \\ &= \lim_n \int (f_V * g_n)\overline{\phi} = \int (f_V * g)\overline{\phi};\end{aligned}$$

the last equality is because $\widetilde{g}_n \to \widetilde{g}$ in $\mathscr{L}^2(G)$, and so

$$f_V * g_n = f_V * \widetilde{\widetilde{g}}_n \to f_V * \widetilde{\widetilde{g}} = f_V * g \quad \text{in } \mathcal{C}^b(G)$$

with respect to the uniform norm (by Lemma 2(i)). Since this is true for every $\phi \in \mathcal{C}_c(G)$, we see first that $f_V * g \in \mathscr{L}^2(G)$ and second that

$$A_V[g] = [f_V * g].$$

But this gives

$$A_V(C_U[h]) = A_V[g] = [f_V * g] = [f_V * (h * \widetilde{g_U})].$$

This, (3), and Corollary 4 then give $C_U A_V = A_V C_U$. Passing to the limit as $V \to \{\mathbf{e}\}$ and then as $U \to \{\mathbf{e}\}$, we conclude that $TS = ST$. ∎

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