COLLOQUIUM MATHEMATICUM

VOL. 147 2017 NO. 1

ON LIPSCHITZ $\tau(p)$-SUMMING OPERATORS

BY

LAHCÈNE MEZRAG and ABDELHAMID TALLAB (M'sila)

Abstract. We introduce and study a new concept of summability in the category of Lipschitz operators, which we call Lipschitz $\tau(p)$-summability. We give some characterizations in terms of a domination theorem and some properties of this concept. Also, connections with other summability notions are presented.

0. Introduction. The notion of p-summing linear operators goes back to Grothendieck in the 1950s, but only in 1967 and 1968, did the classical works of Pietsch [Pie67] and Lindenstrauss–Pełczyński [LP68] clarify Grothendieck's precious ideas and contributed to the vigorous development of this notion. Recently, Lipschitz versions of different types of linear operators were investigated in several papers, including [Cha11], [CZ11], [CZ12], [FJ09], [Saa15] and [YAR]. Farmer and Johnson [FJ09] introduced the notion of Lipschitz p-summing operators and showed that it is a good generalization of the concept of linear p-summing operators [FJ09, Theorem 2]. This notion marked the beginning of the theory of nonlinear summability. Motivated by the importance of this theory, several authors then developed and studied various related concepts. Chen and Zheng [CZ12] introduced (strongly) Lipschitz p-integral and p-nuclear operators. Chávez-Domínguez introduced the notion of Lipschitz (r,p,q)-summing operators in [Cha11] and Lipschitz (q,p)-mixing in [Cha12]. The latest papers in this domain are due to Yahi, Achour and Rueda [YAR] and Saadi [Saa15]. Independently, they introduced and studied the class of Lipschitz strongly p-summing operators. The former authors also introduced summing Lipschitz conjugates and (p,σ)-summability with an appropriate factorization. They also characterized those Lipschitz operators whose Lipschitz conjugates are absolutely p-summing.

In his book [Pie80], Pietsch introduced the notion of τ-summing linear operators. Mujica [Muj08] extended this notion to multilinear operators. In

2010 *Mathematics Subject Classification*: Primary 46B28, 46T99; Secondary 47H99, 47L20.
Key words and phrases: Cohen Lipschitz p-nuclear operators, Lipschitz p-summing operators, Lipschitz $\tau(p)$-summing operators, Lipschitz strongly p-summing operators.
Received 14 September 2015; revised 3 March 2016.
Published online 9 December 2016.

DOI: 10.4064/cm6763-3-2016

the present work, we generalize this class to Lipschitz operators. We give some characterizations and properties. We end the paper by studying some connections with other summability notions.

The paper is organized as follows.

In Section 1, we recall some basic definitions and properties concerning the Lipschitz dual and many concepts relating to summability.

In Section 2, we adapt the class of τ-summing linear operators introduced by Pietsch [Pie80] to the Lipschitz operator case, obtaining what we call Lipschitz $\tau(p,q)$-summing operators. We characterize this type of operators by proving a domination theorem, using the unified Pietsch domination theorem [BPR10] and Ky Fan's lemma. Also, we give some properties of this class.

In Section 3, we generalize to Lipschitz operators the class of Cohen p-nuclear operators introduced in [Coh73]. This class was extended by [AMS09] to sublinear operators and by [AA10] to multilinear operators. Some properties related to the class of strongly Lipschitz p-summing operators are also given.

Finally, in Section 4 we consider relations between various classes of Lipschitz summability.

1. Definitions and notations. Unless otherwise stated, X, Y, Z will always denote pointed metric spaces, i.e., with a distinguished element always denoted by 0, whereas E, F, G will denote real Banach spaces. As is customary, B_E denotes the closed unit ball of E and E^* its topological linear dual, and $\mathcal{L}(E,F)$ is the linear space of bounded linear maps from E to F.

$\mathrm{Lip}_0(X,E)$ is the Banach space of Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm given by

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)}.$$

We use the shorthand $X^{\#} := \mathrm{Lip}_0(X) := \mathrm{Lip}_0(X,\mathbb{R})$. The closed unit ball $B_{X^{\#}}$ is a compact Hausdorff space for the topology of pointwise convergence on X.

Let $(X, 0, d)$ be a pointed metric space. A *molecule* on X is a real valued function m on X with finite support and $\sum_{x \in \mathrm{supp}(m)} m(x) = 0$. Denote by $M(X)$ the real linear space of molecules on X. The condition $\sum_{j=1}^{l} m(x_j) = 0$ ensures that m can be represented by $m = \sum_{j=1}^{l} \lambda_j m_{x_j x_j'}$, where $m_{x_j x_j'} = \mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x_j'\}}$. Set

$$\|m\|_{M(X)} = \inf \Big\{ \sum_{j=1}^{l} |\lambda_j| d_X(x_j, x_j') \Big\},$$

where the infimum is taken over all representations $m = \sum_{j=1}^{l} \lambda_j(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}})$. It follows that $\|\cdot\|_{M(X)}$ is a norm on $M(X)$. Denote by Æ(X, d_X) the completion of $(M(X), \|\cdot\|_{M(X)})$. This space was first introduced by Arens and Eells [AE56] in 1956. The idea goes back to Kantorovich [Kan42]. The term "Arens–Eells space" and the notation Æ(X, d_X) are due to Weaver [Wea99]. The map $i_X : X \to$ Æ(X, d_X) defined by $i_X(x) = m_{x0} = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{0\}}$ is an isometric embedding of X into Æ(X, d_X). For more details on the basic theory of the spaces of Lipschitz operators and their preduals, one can consult for example the books of Benyamini–Lindenstrauss [BL00] or Weaver [Wea99].

The letters p, q, r, s will designate elements of $[1, \infty]$, and p^* denotes the exponent conjugate to p (i.e., $1/p + 1/p^* = 1$).

Inspired by the useful concept of absolutely summing operators, J. Farmer and W.-B. Johnson [FJ09] introduced the following definition: a Lipschitz map $T : X \to Y$ is called *Lipschitz p-summing* ($1 \le p < \infty$) if there exists a positive constant C such that for all $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X and all non-negative reals $\lambda_1, \dots, \lambda_n$ we have

$$\Big(\sum_{i=1}^{n} \lambda_i d_Y(T(x_i), T(x'_i))^p\Big)^{1/p} \le C \sup_{f \in B_{X^\#}} \Big(\sum_{i=1}^{n} \lambda_i |(f(x_i) - f(x'_i))|^p\Big)^{1/p}.$$

The infimum of such C is denoted by $\pi_p^L(T)$. This is indeed a generalization of the concept of linear p-summing operators, since it is shown in [FJ09] that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm. It will be useful to note that in the above definition we can restrict to $\lambda_i = 1$ (see [FJ09] for an implicit proof).

We now give some standard notation. We denote by $\|\cdot\|_p$ the l_p-norm of a sequence of real numbers. For a sequence $(x_i)_i$ of vectors in a Banach space E, its *strong p-norm* is the l_p-norm of the sequence $(\|x_i\|)_i$, and we denote its *weak p-norm* (cf. [DF93]) by

$$\omega_p((x_i)_i) = \sup_{x^* \in B_{E^*}} \|(x^*(x_i))_i\|_p.$$

We denote the relevant spaces respectively by $l_p(E)$ and $(l_p)^\omega(E)$ ($l_p^n(E)$ and $(l_p^n)^\omega(E)$ if we take finite sequences $(x_i)_{1 \le i \le n} \subset E$).

Analogously, for sequences $(\lambda_i)_i$ of real numbers and $(x_i)_i$, $(x'_i)_i$ of points in a metric space X, we denote the *weak Lipschitz p-norm* by

$$\omega_p^L((\lambda_i), (x_i)_i, (x'_i)_i) = \sup_{f \in B_{X^\#}} \|(\lambda_i |f(x_i) - f(x'_i)|)_i\|_p.$$

J. A. Chávez-Domínguez [Cha11] defined an operator $T : X \to E$ to be *Lipschitz (r, p, q)-summing* ($1 \le p, r < \infty$, $1 \le q \le \infty$ and $1/r \le 1/p + 1/q$) if there is a positive constant C such that for all $n \in \mathbb{N}$, $(x_i)_{1 \le i \le n}, (x'_i)_{1 \le i \le n} \subset X$,

$(a_i^*)_{1\le i\le n} \subset E^*$ and $\lambda_i \ge 0$, $k_i > 0$, we have

$$\|(\lambda_i \langle T(x_i) - T(x_i'), a_i^*\rangle)_i\|_r \le C\omega_p^L((\lambda_i k_i^{-1})_i, (x_i)_i, (x_i')_i)\omega_q((k_i a_i^*)_i). \tag{1.1}$$

The smallest constant C such that (1.1) holds is denoted by $\pi^L_{r,p,q}(T)$, and $\Pi^L_{r,p,q}(X,E)$ is the set of all such operators; it is a Banach space when equipped with the norm $\pi^L_{r,p,q}(\cdot)$. When E is a dual, we can consider the a_i^* in the predual. Also, the case (r,r,∞) corresponds to Lipschitz r-summing operators from X into E as in [FJ09], whereas (r,p,∞) corresponds to the Lipschitz (r,p)-summing operators from X into E as in [JS09]. Moreover, by the same argument as in [FJ09], in (1.1) we may take $\lambda_i = 1$ for all i.

2. Lipschitz $\tau(p)$-summing operators. The following definition was given by X. Mujica [Muj08] for multilinear operators; it generalizes absolutely τ-summing linear operators introduced by A. Pietsch [Pie80].

Definition 2.1. Let $T \in \mathcal{L}(E,F)$ and $1 \le q \le p < \infty$. We say that T is *absolutely $\tau(p,q)$-summing* if there is a positive constant C such that, for all $n \in \mathbb{N}$, $(a_i)_{1\le i\le n} \subset E$ and $(b_i^*)_{1\le i\le n} \subset F^*$, we have

$$\Big(\sum_{i=1}^n |\langle T(a_i), b_i^*\rangle|^p\Big)^{1/p} \le C \sup_{\substack{\|b\|_F\le 1\\ \|a^*\|_{E^*}\le 1}} \Big(\sum_{i=1}^n |\langle a_i, a^*\rangle\langle b_i^*, b\rangle|^q\Big)^{1/q}. \tag{2.1}$$

We denote by $\Pi_{\tau(p,q)}(E,F)$ the vector space of all $\tau(p,q)$-summing linear operators T from E into F; it is a Banach space with the norm $\pi_{\tau(p,q)}(T)$, the infimum of all C satisfying the above inequality. When $p=q$, we write $\Pi_{\tau(p)}$ and $\pi_{\tau(p)}(T)$ instead of $\Pi_{\tau(p,p)}$ and $\pi_{\tau(p,p)}(T)$ respectively. In this case, we say that T is *$\tau(p)$-summing.* If $p=q=1$, we simply write Π_τ and π_τ and we say that T is *τ-summing.*

The following theorem is due to Pietsch for $p=1$ and to Mujica for multilinear operators when $1 \le p < \infty$.

Theorem 2.2. *Let $1 \le p < \infty$. A mapping $T \in \mathcal{L}(E,F)$ is $\tau(p)$-summing if and only if there is a positive constant C and a Radon–Borel probability measure μ on $B_{E^*}\times B_{F^{**}}$ such that*

$$|\langle T(a), b^*\rangle| \le C\Big(\int_{B_{E^*}\times B_{F^{**}}} |\langle a, a^*\rangle\langle b^*, b^{**}\rangle|^p d\mu(a^*, b^{**})\Big)^{1/p} \tag{2.2}$$

for all $a \in E$ and $b^* \in F^*$. In this case, $\pi_{\tau(p)}(T) = \inf C$.

Straightforward calculations show that if T is $\tau(p)$-summing, then T and T^* are p-summing.

Now, we extend the preceding definition to Lipschitz operators.

Definition 2.3. Let T be in $\mathrm{Lip}_0(X,E)$ and $1 \le q \le p < \infty$. We say that T is *Lipschitz $\tau(p,q)$-summing* if there is a positive constant C such that, for all $n \in \mathbb{N}$, $(x_i), (x_i') \subset X$, $(a_i^*) \subset E^*$ and $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$, we have

$$\Big(\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x_i'), a_i^*\rangle|^p\Big)^{1/p} \le C \sup_{\substack{\|a\|\le 1\\ \|f\|\le 1}} \Big(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x_i'))\langle a_i^*, a\rangle|^q\Big)^{1/q} \tag{2.3}$$

where $f \in X^\#$ and $a \in E$. We will denote this class of mappings by $\Pi^L_{\tau(p,q)}(X,E)$. Equipped with the norm $\pi^L_{\tau(p,q)}(T) = \inf C$, it becomes a Banach space. When $p = q$, we write $\Pi^L_{\tau(p)}$ and $\pi^L_{\tau(p)}$ instead of $\Pi^L_{\tau(p,p)}$ and $\pi^L_{\tau(p,p)}$ respectively and we say that T is *Lipschitz $\tau(p)$-summing.* If $p = q = 1$, we simply write Π^L_τ and π^L_τ and we say that T is *Lipschitz τ-summing.* As in the linear case, if $1 \le s \le r \le q \le p$, then $\Pi^L_{\tau(q,r)} \subset \Pi^L_{\tau(p,s)}$ and $\pi^L_{\tau(p,s)}(T) \le \pi^L_{\tau(q,r)}(T)$ for all T in $\Pi^L_{\tau(q,r)}$. Moreover,

$$\Pi^L_{\tau(q,r)} \subset \Pi^L_{\tau(p,r)} \quad \text{and} \quad \pi^L_{\tau(p,r)}(T) \le \pi^L_{\tau(q,r)}(T) \quad \text{for all } T \text{ in } \Pi^L_{\tau(q,r)},$$
$$\Pi^L_{\tau(q,r)} \subset \Pi^L_{\tau(q,s)} \quad \text{and} \quad \pi^L_{\tau(q,s)}(T) \le \pi^L_{\tau(q,r)}(T) \quad \text{for all } T \text{ in } \Pi^L_{\tau(q,r)}.$$

REMARK 2.4. 1. In the definition we can restrict to $\lambda_i = 1$ (by the same argument cited implicitly in [FJ09]).

2. By Goldstine's theorem, we can replace a by $a^{**} \in E^{**}$ in (2.3).

3. If T is linear then "T is $\tau(p,q)$-summing" implies "T is Lipschitz $\tau(p,q)$-summing" and $\pi^L_{\tau(p,q)}(T) \le \pi_{\tau(p,q)}(T)$. We do not know if the converse is true, because we now do not have a factorization theorem and $B_{X^\#}$ is also difficult to handle. Is it a good generalization?

LEMMA 2.5. *Let $1 \le p < \infty$. For $n \in \mathbb{N}$, $(x_i)_{1\le i\le n}, (x_i')_{1\le i\le n} \subset X$, $(a_i^*)_{1\le i\le n} \subset E^*$ and $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$, let $v : l^n_{p^*} \to X \boxtimes_\varepsilon E^*$ be the linear operator such that $v(e_i) = \delta_{(x_i,x_i')} \boxtimes \lambda_i^{1/p} a_i^*$, where (e_i) denotes the unit vector basis of $l^n_{p^*}$ and $\boxtimes$ denotes the Lipschitz tensor product as introduced in* [CCJV]. *Then*

$$\|v\| = \sup_{\substack{\|a\|_E = 1\\ \|f\|_{X^\#} = 1}} \Big(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x_i'))\langle a_i^*, a\rangle|^p\Big)^{1/p}.$$

Proof. We have

$$\|v\| = \sup_{\|\alpha\|_{l^n_{p^*}} = 1} \|v(\alpha)\|_{X\boxtimes_\varepsilon E^*} = \sup_{\|\alpha\|_{l^n_{p^*}} = 1} \Big\|\sum_{i=1}^n \alpha_i v(e_i)\Big\|_{X\boxtimes_\varepsilon E^*} \qquad \Big(\alpha = \sum_{i=1}^n \alpha_i e_i\Big)$$
$$= \sup_{\|\alpha\|_{l^n_{p^*}} = 1} \Big\|\sum_{i=1}^n \alpha_i \delta_{(x_i,x_i')} \boxtimes \lambda_i^{1/p} a_i^*\Big\|_{X\boxtimes_\varepsilon E^*}$$

$$= \sup_{\|\alpha\|_{l_p^n}=1} \sup_{\substack{\|a\|_E=1 \\ \|f\|_{X^\#}=1}} \Big(\sum_{i=1}^n \alpha_i \lambda_i^{1/p} |(f(x_i) - f(x_i'))\langle a_i^*, a\rangle|\Big)$$

$$= \sup_{\substack{\|a\|_E=1 \\ \|f\|_{X^\#}=1}} \Big(\sum_{i=1}^n \lambda_i |(f(x_i) - f(x_i'))\langle a_i^*, a\rangle|^p\Big)^{1/p}. \ \blacksquare$$

PROPOSITION 2.6. *Let* $T \in \mathrm{Lip}_0(X,E)$*. Then* T *is Lipschitz* $\tau(p)$*-summing if and only if for all* $n \in \mathbb{N}$, $(x_i)_{1\le i\le n}, (x_i')_{1\le i\le n} \subset X$, $(a_i^*)_{1\le i\le n} \subset E^*$, $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$ *and the linear operator* $v : l_{p^*}^n \to X \boxtimes_\varepsilon E^*$ *such that* $v(e_i) = \delta_{(x_i,x_i')} \boxtimes \lambda_i^{1/p} a_i^*$*, we have*

$$\Big(\sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x_i'), a_i^*\rangle|^p\Big)^{1/p} \le C\|v\|. \tag{2.4}$$

We now prove the left ideal property in "Pietsch's sense".

PROPOSITION 2.7. *Let* $T \in \mathrm{Lip}_0(Y,E)$ *and* $R \in \mathrm{Lip}_0(X,Y)$*. If* T *is Lipschitz* $\tau(p)$*-summing, then so is* $T\circ R$*, and* $\pi_{\tau(p)}^L(T\circ R) \le \pi_{\tau(p)}^L(T)\,\mathrm{Lip}(R)$.

Proof. Let $n \in \mathbb{N}$, $(x_i)_{1\le i\le n}, (x_i')_{1\le i\le n} \subset X$, $(a_i^*)_{1\le i\le n} \subset E^*$ and $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$. It suffices by (2.4) to show that

$$\Big(\sum_{i=1}^n \lambda_i |\langle T\circ R(x_i) - T\circ R(x_i'), a_i^*\rangle|^p\Big)^{1/p} \le \pi_{\tau(p)}^L(T)\,\mathrm{Lip}(R)\|w\|$$

where $w : l_{p^*}^n \to X \boxtimes_\varepsilon E^*$ is such that $w(e_i) = \delta_{(x_i,x_i')} \boxtimes \lambda_i^{1/p} a_i^*$.

Consider the commutative diagram

$$\begin{array}{ccc} l_{p^*}^n & \xrightarrow{\ v\ } & Y \boxtimes_\varepsilon E^* \\ {\scriptstyle w}\downarrow & \nearrow{\scriptstyle R\boxtimes \mathrm{id}_{E^*}} & \\ X \boxtimes_\varepsilon E^* & & \end{array}$$

where

$$v(e_i) = \delta_{(R(x_i),R(x_i'))} \boxtimes \lambda_i^{1/p} a_i^*$$

and

$$R\boxtimes \mathrm{id}_{E^*}(\delta_{(x_i,x_i')} \boxtimes \lambda_i^{1/p} a_i^*) = \delta_{(R(x_i),R(x_i'))} \boxtimes \lambda_i^{1/p} a_i^*.$$

The Lipschitz injective norm ε is uniform by [CCJV, Theorem 7.1], and by [CCJV, Proposition 4.2] we have

$$\Big(\sum_{i=1}^n \lambda_i |\langle T\circ R(x_i) - T\circ R(x_i'), a_i^*\rangle|^p\Big)^{1/p}$$
$$\le \pi_{\tau(p)}^L(T)\|v\| \le \pi_{\tau(p)}^L(T)\|w\|\,\|R\boxtimes \mathrm{id}_{E^*}\| \le \pi_{\tau(p)}^L(T)\,\mathrm{Lip}(R)\|w\|.$$

This implies by (2.4) that $T \circ R$ is Lipschitz $\tau(p)$-summing and $\pi^L_{\tau(p)}(T \circ R) \leq \pi^L_{\tau(p)}(T)\,\mathrm{Lip}(R)$. ■

For the right ideal property, let us start with the following theorem.

THEOREM 2.8 ([Wea99]). *Let $T : X \to E$ be a Lipschitz map which preserves the base point (i.e., $T(0) = 0$). Then there is a unique bounded linear operator $T_L : \text{Æ}(X, d_X) \to E$ such that $T = T_L \circ i_X$, and $\|T_L\| =$* $\mathrm{Lip}(T)$ *(here $i_X : X \to \text{Æ}(X, d_X)$ is the natural inclusion).*

The linear operator T_L is called the *linearization* of T. We know that every molecule m is uniquely expressible in the form

$$m = \sum_{j=1}^{l} \lambda_j m_{x_j 0} \tag{2.5}$$

where the points x_j are all distinct and not 0. Then T_L is defined by

$$T_L(m) = \sum_{j=1}^{l} \lambda_j T(x_j). \tag{2.6}$$

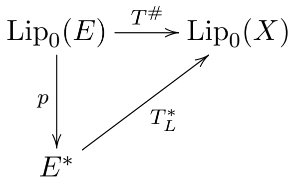
The dual E^* of E is naturally isometric to a subspace of $\mathrm{Lip}_0(E)$. By [BL00, p. 37], there is a norm one projection $p : \mathrm{Lip}_0(E) \to E^*$. Sawashima [Saw75] defined the *Lipschitz adjoint* (or *dual*) $T^\# : \mathrm{Lip}_0(E) \to \mathrm{Lip}_0(X)$ of $T \in \mathrm{Lip}_0(X, E)$ by

$$T^\#(f) = f \circ T, \quad f \in \mathrm{Lip}_0(E) \quad (T^\#(f)(x) = f(T(x))).$$

He showed that $T^\#$ is a continuous linear operator and

$$\|T^\#\| = \mathrm{Lip}(T) = \|T^\#|_{E^*}\|.$$

We notice that $T^\#|_{E^*}$ corresponds in a canonical way to the usual adjoint of the linear operator attached to T by Theorem 2.8, i.e., $T^\#|_{E^*} = T_L^*$:



Now, we can show the right ideal property for Banach spaces.

PROPOSITION 2.9. *Let $T \in \mathrm{Lip}_0(Y, E)$ and $S \in \mathrm{Lip}_0(E, F)$. If T is Lipschitz $\tau(p)$-summing, then so is $S \circ T$, and $\pi^L_{\tau(p)}(S \circ T) \leq \mathrm{Lip}(S)\pi^L_{\tau(p)}(T)$.*

Proof. Let $(y_i)_{1\le i\le n}, (y_i')_{1\le i\le n} \subset Y$, $(b_i^*)_{1\le i\le n} \subset F^*$ and $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$. We have

$$\Big(\sum_{i=1}^{n}\lambda_i|\langle S\circ T(y_i)-S\circ T(y_i'),b_i^*\rangle|^p\Big)^{1/p}$$
$$=\Big(\sum_{i=1}^{n}\lambda_i|\langle T(y_i),S^{\#}(b_i^*)\rangle-\langle T(y_i'),S^{\#}(b_i^*)\rangle|^p\Big)^{1/p}$$
$$=\Big(\sum_{i=1}^{n}\lambda_i|\langle T(y_i),S_L^*(b_i^*)\rangle-\langle T(y_i'),S_L^*(b_i^*)\rangle|^p\Big)^{1/p}$$

(S_L^* is the transpose of the linear operator attached to S)

$$=\Big(\sum_{i=1}^{n}\lambda_i|\langle T(y_i)-T(y_i'),S_L^*(b_i^*)\rangle|^p\Big)^{1/p}$$
$$\le\pi_{\tau(p)}^L(T)\sup_{\substack{\|a\|_E=1\\ \|f\|_{Y^\#}\le1}}\Big(\sum_{i=1}^{n}\lambda_i|(f(y_i)-f(y_i'))\langle S_L^*(b_i^*),a\rangle|^p\Big)^{1/p}$$
$$\le\pi_{\tau(p)}^L(T)\|S_L\|\sup_{\substack{\|a\|_E=1\\ \|f\|_{Y^\#}\le1}}\Big(\sum_{i=1}^{n}\lambda_i\Big|(f(y_i)-f(y_i'))\Big\langle\frac{S_L^*(b_i^*)}{\|S_L^*\|},a\Big\rangle\Big|^p\Big)^{1/p}$$
$$\le\pi_{\tau(p)}^L(T)\|S_L\|\sup_{\substack{\|a\|_E=1\\ \|f\|_{Y^\#}\le1}}\Big(\sum_{i=1}^{n}\lambda_i\Big|(f(y_i)-f(y_i'))\Big\langle b_i^*,\frac{S_L(a)}{\|S_L\|}\Big\rangle\Big|^p\Big)^{1/p}$$
$$\le\mathrm{Lip}(S)\pi_{\tau(p)}^L(T)\sup_{\substack{\|b\|_F=1\\ \|f\|_{B_{Y^\#}}\le1}}\Big(\sum_{i=1}^{n}\lambda_i|(f(y_i)-f(y_i'))\langle b_i^*,b\rangle|^p\Big)^{1/p}.$$

Thus, $S\circ T$ is Lipschitz $\tau(p)$-summing and $\pi_{\tau(p)}^L(S\circ T)\le\pi_{\tau(p)}^L(T)\,\mathrm{Lip}(S)$. ■

By the left ideal property, we have the following proposition. For the converse, see [Saa15, Remark 3.3].

PROPOSITION 2.10. *Let $1\le p<\infty$. Let $T:X\to E$ be a Lipschitz map and T_L its linearization. If T_L is $\tau(p)$-summing, then T is Lipschitz $\tau(p)$-summing. The converse is false.*

We will give an "alternative reciprocal" after the domination theorem.

Now, we characterize this type of operators by giving the Pietsch domination theorem. Recently, a general version of the Pietsch domination theorem was proved in [PSS12b], which is an improved version of a similar result in [BPR10] (see also [PSS12a]). Let X, Y and V be (arbitrary) non-void sets, $\mathcal{H}(X;Y)$ a non-void family of mappings from X to Y, G a Banach space, and K a compact Hausdorff topological space. Let

$$R\colon K\times V\times G\to[0,\infty)\quad\text{and}\quad S\colon\mathcal{H}(X;Y)\times V\times G\to[0,\infty)$$

be arbitrary mappings and $1 \leq q < \infty$. According to [PSS12b] and [BPR10], a mapping $f \in \mathcal{H}(X;Y)$ is *RS-abstract q-summing* if there is a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n} S(f, x_i, b_i)^q\right)^{1/q} \leq C \sup_{\varphi \in K} \left(\sum_{i=1}^{n} R(\varphi, x_i, b_i)^q\right)^{1/q} \tag{2.7}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in V$ and $b_1, \ldots, b_n \in G$. We define

$$\mathcal{H}_{RS,q}(X;Y) = \{f \in \mathcal{H}(X;Y) : f \text{ is } RS\text{-abstract } q\text{-summing}\}.$$

Suppose that R is such that the mapping

$$R_{x,b} \colon K \to [0,\infty) \quad \text{defined by} \quad R_{x,b}(\varphi) = R(\varphi, x, b) \tag{2.8}$$

is continuous for all $x \in V$ and $b \in G$. The Pietsch domination theorem from [PSS12b] reads as follows.

THEOREM 2.11. *Suppose that S is arbitrary, R satisfies* (2.8) *and let $1 \leq q < \infty$. A map $f \in \mathcal{H}(X;Y)$ is RS-abstract q-summing if and only if there is a constant $C \geq 0$ and a Borel probability measure μ on K such that*

$$S(f, x, b) \leq C\left(\int_K R(\varphi, x, b)^q \, d\mu(\varphi)\right)^{1/q} \tag{2.9}$$

for all $x \in V$ and $b \in G$.

We give a domination theorem for Lipschitz $\tau(p)$-summing operators by using the technique above.

THEOREM 2.12. *An operator $T \in \mathrm{Lip}_0(X,E)$ is Lipschitz $\tau(p)$-summing if and only if there exists a positive constant C and a probability μ on $K = B_{X^\#} \times B_{E^{**}}$ such that*

$$|\langle T(x) - T(x'), a^* \rangle| \leq C\left(\int_K |(f(x) - f(x'))\langle a^{**}, a^* \rangle|^p \, d\mu(f, a^{**})\right)^{1/p} \tag{2.10}$$

for all $x, x' \in X$ and $a^* \in E^*$. In this case, $\pi^L_{\tau(p)}(T) = \inf C$.

Proof. Note that T is Lipschitz $\tau(p)$-summing if and only if it is RS-abstract p-summing with

$$V = X \times X, \qquad G = E^*,$$

and $K = B_{X^\#} \times B_{E^{**}}$, which is a compact Hausdorff space in the topology of pointwise convergence on $X \times E^*$, $\mathcal{H} = \mathrm{Lip}_0(X,E)$ and R, S are defined as follows:

$$R : (B_{X^\#} \times B_{E^{**}}) \times (X \times X) \times E^* \to [0,\infty),$$
$$R((f, a^{**}), (x, x'), a^*) = \lambda^{1/p}|(f(x) - f(x'))\langle a^{**}, a^* \rangle|,$$

and

$$S : \mathcal{H} \times (X \times X) \times E^* \to [0, \infty),$$
$$S(T, (x, x', a^*)) = \lambda^{1/p} |\langle T(x) - T(x'), a^* \rangle|.$$

Now, as a consequence of Theorem 2.11,

$$|\langle T(x) - T(x'), a^* \rangle| \le C \Big(\int_{B_{X^\#} \times B_{E^{**}}} |(f(x) - f(x')) \langle a^{**}, a^* \rangle|^p \, d\mu(f, a^{**}) \Big)^{1/p}$$

and the proof is complete. ∎

As an immediate corollary of our theorem we obtain

COROLLARY 2.13. *$\Pi^L_{\tau(p)} \subseteq \Pi^L_{\tau(q)}$ when $1 \le p \le q < \infty$, and $\Pi^L_{\tau(p)} \subseteq \Pi^L_p$ for all $1 \le p < \infty$.*

We will present another characterization (Pietsch's domination theorem) for this class of Lipschitz operators. For the proof, we use the same idea as applied for example in [AMS09] and [Muj08, Theorem 3.6]. First, we recall Ky Fan's lemma. The proof can be found in [DJT95, p. 190].

LEMMA 2.14. *Let K be a Hausdorff topological vector space and $\mathfrak{C}$ a compact convex subset of K. Let $\mathcal{M}$ be a set of functions on $\mathfrak{C}$ with values in $(-\infty, \infty]$ having the following properties:*

(a) *each $f \in \mathcal{M}$ is convex and lower semicontinuous;*
(b) *if $g \in \operatorname{conv}(\mathcal{M})$, then there is an $f \in \mathcal{M}$ with $g(x) \le f(x)$ for all $x \in \mathfrak{C}$;*
(c) *there is an $r \in \mathbb{R}$ such that each $f \in \mathcal{M}$ has a value $\le r$.*

Then there is an $x_0 \in \mathfrak{C}$ such that $f(x_0) \le r$ for all $f \in \mathcal{M}$.

THEOREM 2.15. *Let $T \in \operatorname{Lip}_0(X, E)$ and C a positive constant. Then the following assertions are equivalent:*

(1) *The operator T is Lipschitz $\tau(p)$-summing and $\pi^L_{\tau(p)}(T) \le C$.*
(2) *There exist Radon probability measures μ_1 on $B_{X^\#}$ and μ_2 on $B_{E^{**}}$, such that for all x, x' in X and a^* in E^*, we have*

$$(2.11) \quad |\langle T(x) - T(x'), a^* \rangle| \le C \Big(\int_{B_{X^\#}} \int_{B_{E^{**}}} |(f(x) - f(x')) \langle a^*, a^{**} \rangle|^p \, d\mu_1(f) \, d\mu_2(a^{**}) \Big)^{1/p}.$$

Moreover, in this case $\pi^L_{\tau(p)}(T) = \inf C$.

Proof. We only need to prove (1)⇒(2), because (2.11) easily implies that T is Lipschitz $\tau(p)$-summing and $\pi^L_{\tau(p)}(T) \le C$.

Consider the sets $\mathcal{P}(B_{X^\#})$ and $\mathcal{P}(B_{E^{**}})$ of probability measures in $\mathcal{C}(B_{X^\#})^*$ and $\mathcal{C}(B_{E^{**}})^*$ respectively endowed with their weak* topologies.

These sets are compact and convex. We are going to apply Ky Fan's lemma with $K = \mathcal{C}(B_{X^\#})^* \times \mathcal{C}(B_{E^{**}})^*$ and $\mathfrak{C} = \mathcal{P}(B_{X^\#}) \times \mathcal{P}(B_{E^{**}})$, which is convex and compact.

Let $\mathcal{M}$ be the set of all functions $\varphi : \mathfrak{C} \to \mathbb{R}$ of the form

$$\begin{aligned}\varphi(\mu_1,\mu_2) &= \varphi_{((x_i),(x_i'),(a_i^*),(\lambda_i))}(\mu_1,\mu_2)\\ &= \sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x_i'), a_i^*\rangle|^p\\ &\quad - C \int\limits_{B_{X^\#}} \int\limits_{B_{E^{**}}} \lambda_i |(f(x_i) - f(x_i'))\langle a_i^*, a^{**}\rangle|^p \, d\mu_1(f)\, d\mu_2(a^{**})\end{aligned}$$

where $(x_i)_{1\le i\le n}, (x_i')_{1\le i\le n} \subset X$, $(a_i^*)_{1\le i\le n} \subset E^*$ and $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$. These functions are continuous and convex. The set $\mathcal{M}$ is a convex cone. We now apply Ky Fan's lemma (conditions (a) and (b) are satisfied). For (c), since $B_{X^\#}$ is a compact Hausdorff space in the topology of pointwise convergence on X and $B_{E^{**}}$ are weak* compact and norming sets, using the fact that X is isometrically embedded into $B_{X^\#}$ and by the classical Goldstine theorem, for $\varphi \in \mathcal{M}$ as above there exist f_0 in $B_{X^\#}$ and a_0^{**} in $B_{E^{**}}$ such that

$$\sup_{\substack{\|f\|_{X^\#}=1\\ \|a^{**}\|_{E^{**}}=1}} \big\|\big(\lambda_i^{1/p}(f(x_i)-f(x_i'))\langle a_i^*, a^{**}\rangle\big)\big\|_{l_p^n}^p = \sum_{i=1}^n \lambda_i |f_0(x_i)-f_0(x_i')\langle a_i^*, a_0^{**}\rangle|^p.$$

If δ_{f_0} and $\delta_{a_0^{**}}$ denote the Dirac measures supported by f_0 and a_0^{**} respectively, we have

$$\begin{aligned}&\varphi_{((x_i),(x_i'),(a_i^*),(\lambda_i))}(\delta_{f_0}, \delta_{a_0^{**}})\\ &\quad = \sum_{i=1}^n \lambda_i |\langle T(x_i) - T(x_i'), a_i^*\rangle|^p - C^p \sum_{i=1}^n \lambda_i |f_0(x_i) - f_0(x_i')\langle a_i^*, a_0^{**}\rangle|^p \le 0.\end{aligned}$$

Hypothesis (1) yields

$$\sup\{\varphi_{((x_i),(x_i'),(a_i^*),(\lambda_i))}(\mu_1,\mu_2) : (\mu_1,\mu_2) \in K\} \le 0.$$

By Ky Fan's lemma, there is $\mu = (\mu_1,\mu_2) \in \mathfrak{C}$ with $\mu(\varphi) \le 0$ for all φ in $\mathcal{M}$. If φ is generated by simple elements $x, x' \in X$, $a^* \in E^*$ and $\lambda = 1$, we find

$$\begin{aligned}\varphi_{(x,x',a^*,1)}(\mu_1,\mu_2) &= |\langle T(x) - T(x'), a^*\rangle|^p\\ &\quad - C^p \int\limits_{B_{X^\#}} \int\limits_{B_{E^{**}}} |(f(x) - f(x'))\langle a^*, a^{**}\rangle|^p \, d\mu_1(f)\, d\mu_2(a^{**}) \le 0.\end{aligned}$$

It follows that

$$\begin{aligned}&|\langle T(x) - T(x'), a^*\rangle|\\ &\quad \le C\Big(\int\limits_{B_{X^\#}} \int\limits_{B_{E^{**}}} |(f(x) - f(x'))\langle a^*, a^{**}\rangle|^p \, d\mu_1(f)\, d\mu_2(a^{**})\Big)^{1/p},\end{aligned}$$

and this completes the proof. ■

COROLLARY 2.16. *Let $T \in \mathrm{Lip}_0(X,E)$. If T is Lipschitz $\tau(p)$-summing then its linearization T_L is in $\mathcal{D}_{p^*}(\text{Æ}(X),E)$ in the sense of* [Coh73], *and consequently T_L^* is p-summing.*

Proof. Suppose that $T \in \Pi^L_{\tau(p)}(X,E)$. Let $m = \sum_{j=1}^l \lambda_j m_{x_j x_j'}$ be in $M(X)$ and $a^* \in E^*$. By Theorem 2.15, we have

$$\begin{aligned}
&|\langle T_L(m), a^*\rangle| \le \sum_{j=1}^{l} |\lambda_j|\,|\langle T(x_j) - T(x_j'), a^*\rangle| \\
&\le \pi^L_{\tau(p)}(T) \sum_{j=1}^{l} |\lambda_j| \Big(\int\limits_{B_{X^\#}} \int\limits_{B_{E^{**}}} |(f(x_j) - f(x_j'))\langle a^*, a^{**}\rangle|^p \, d\mu_1(f)\, d\mu_2(a^{**})\Big)^{1/p} \\
&\le \pi^L_{\tau(p)}(T) \sum_{j=1}^{l} |\lambda_j| d(x_j, x_j') \Big(\int\limits_{B_{E^{**}}} |\langle a^*, a^{**}\rangle|^p \, d\mu_2(a^{**})\Big)^{1/p} \\
&\le \pi^L_{\tau(p)}(T) \|m\|_{\text{Æ}(X)} \Big(\int\limits_{B_{E^{**}}} |\langle a^*, a^{**}\rangle|^p \, d\mu_2(a^{**})\Big)^{1/p}.
\end{aligned}$$

By density, this implies that $T_L \in \mathcal{D}_{p^*}(\text{Æ}(X),E)$ and $d_{p^*}(T_L) \le \pi^L_{\tau(p)}(T)$, hence $\pi_p(T_L^*) \le \pi^L_{\tau(p)}(T)$ by [Coh73, Theorem 2.2.2]. ∎

3. Cohen Lipschitz p-nuclear operators. We introduce the following generalization to Lipschitz operators of the class of Cohen p-nuclear operators studied in [Coh73]. It is a particular case of the class defined by J. A. Chávez-Domínguez [Cha11], which he called the Lipschitz (r,p,q)-summing operators, if we take $(r,p,q) = (1,p,p^*)$ and $k_i = 1$ for all i. The notion of p-nuclear operators was introduced by A. Persson and A. Pietsch [PP69]. Initially the definition of nuclear operators for Banach spaces was given by Grothendieck [Gro55]. J. S. Cohen [Coh73] introduced another concept of p-nuclear operators, which was generalized to (p,q)-nuclear operators ($1 \le q \le \infty$) by H. Apiola [Api76]. D. Chen and B. Zheng [CZ12] generalized this notion to Lipschitz operators. To distinguish these two notions, we will talk about *Cohen p-nuclear operators* and we generalize this notion to Lipschitz operators.

DEFINITION 3.1. A Lipschitz operator $T : X \to E$ is *Cohen Lipschitz p-nuclear* ($1 < p < \infty$) if there is a positive constant C such that for any $n \in \mathbb{N}$, $(x_i)_{1\le i\le n}, (x_i')_{1\le i\le n} \subset X$, $(a_i^*)_{1\le i\le n} \subset E^*$ and $(\lambda_i)_{1\le i\le n} \subset \mathbb{R}_+$, we have

$$\begin{aligned}
(3.1)\qquad &\Big|\sum_{i=1}^{n} \lambda_i \langle T(x_i) - T(x_i'), a_i^*\rangle\Big| \\
&\le C \sup_{f \in B_{X^\#}} \Big(\sum_{i=1}^{n} \lambda_i |f(x_i) - f(x_i')|^p\Big)^{1/p} \sup_{\|a\|_E \le 1} \Big(\sum_{i=1}^{n} |\langle a, a_i^*\rangle|^{p^*}\Big)^{1/p^*}.
\end{aligned}$$

The smallest constant C above, denoted by $\eta_p^L(T)$, is called the *Cohen Lipschitz p-nuclear norm*; it makes the space $\mathcal{N}_p^L(X,E)$ of all Cohen Lipschitz p-nuclear operators from X into E a Banach space. For $p=1$ and for $p=\infty$ we have, as in the linear case, $\mathcal{N}_1^L(X,E)=\Pi_1^L(X,E)$ and $\mathcal{N}_\infty^L(X,E)=\mathcal{D}_{\mathrm{st},\infty}^L(X,E)$ (see below). In the definition we can restrict to $\lambda_i=1$, as in [FJ09]. We use this definition with the λ_i only in the proof of Pietsch's domination theorem.

We know (see [DJT95]) that $l_p(E)\equiv l_p^\omega(E)$ ($\equiv$ indicates isometric isomorphism) for some $1\le p<\infty$ if and only if $\dim(E)$ is finite. If $p=\infty$, we have $l_\infty(E)\equiv l_\infty^\omega(E)$. Moreover, if $1<p\le\infty$, then $l_p^\omega(E)\equiv\mathcal{L}(l_{p^*},E)$. In other words, if we let $v:l_{p^*}\to E$ be a linear operator such that $v(e_i)=a_i$ (that is, $v=\sum_{i=1}^\infty e_i\otimes a_i$, where e_i denotes the unit vector basis of l_p), then

$$\|v\|=\|(x_i)\|_{l_p^\omega(E)}. \tag{3.2}$$

Let $T:X\to E$ be a Lipschitz operator and $v:l_{p^*}^n\to E^*$ a bounded linear operator. By (3.2), T is Cohen Lipschitz p-nuclear if and only if

$$\Big|\sum_{i=1}^n\lambda_i\langle T(x_i)-T(x_i'),v(e_i)\rangle\Big| \le C\sup_{f\in B_{X^\#}}\Big(\sum_{i=1}^n\lambda_i|f(x_i)-f(x_i')|^p\Big)^{1/p}\|v\|. \tag{3.3}$$

PROPOSITION 3.2. *Let $T\in\mathrm{Lip}_0(X,E)$, $R\in\mathrm{Lip}_0(E,F)$ and S in $\mathrm{Lip}_0(Y,X)$. If T is Cohen Lipschitz p-nuclear, then so is $R\circ T\circ S$, and $\eta_p^L(R\circ T\circ S)\le\mathrm{Lip}(R)\eta_p^L(T)\,\mathrm{Lip}(S)$.*

Proof. Let $n\in\mathbb{N}$, $(y_i)_{1\le i\le n},(y_i')_{1\le i\le n}\subset Y$ and $(a_i^*)_{1\le i\le n}\subset E^*$. By using (3.3), we first prove that

$$\Big|\sum_{i=1}^n\langle TS(y_i)-TS(y_i'),a_i^*\rangle\Big|\le\eta_p^L(T)\sup_{f\in B_{Y^\#}}\mathrm{Lip}(S)\Big(\sum_{i=1}^n|f(y_i)-f(y_i')|^p\Big)^{1/p}\|v\|$$

where $v:E\to l_{p^*}^n$ is defined by $v(a)=\sum_{i=1}^n a_i^*(a)e_i$. Indeed,

$$\begin{aligned}\Big|\sum_{i=1}^n\langle TS(y_i)-TS(y_i'),a_i^*\rangle\Big|&\\ \le\eta_p^L(T)&\sup_{f\in B_{X^\#}}\Big(\sum_{i=1}^n|f(S(y_i))-f(S(y_i'))|^p\Big)^{1/p}\|v\|,\\ \le\eta_p^L(T)&\,\mathrm{Lip}(S)\sup_{f\in B_{X^\#}}\bigg(\sum_{i=1}^n\bigg|\frac{f(S(y_i))}{\mathrm{Lip}(S)}-\frac{f(S(y_i'))}{\mathrm{Lip}(S)}\bigg|^p\bigg)^{1/p}\|v\|\\ \le\eta_p^L(T)&\,\mathrm{Lip}(S)\sup_{f\in B_{Y^\#}}\Big(\sum_{i=1}^n|f(y_i)-f(y_i')|^p\Big)^{1/p}\|v\|.\end{aligned}$$

This implies that
$$\eta_p^L(T\circ S)\leq \eta_p^L(T)\operatorname{Lip}(S).$$

Let now $n\in\mathbb{N}$, $(x_i)_{1\leq i\leq n},(x_i')_{1\leq i\leq n}\subset X$ and $(b_i^*)_{1\leq i\leq n}\subset F^*$. Secondly, using (3.3) we prove that

$$\Big|\sum_{i=1}^{n}\langle RT(x_i)-RT(x_i'),b_i^*\rangle\Big| \leq \eta_p^L(T)\operatorname{Lip}(R)\sup_{f\in B_{X^\#}}\Big(\sum_{i=1}^{n}|f(x_i)-f(x_i')|^p\Big)^{1/p}\|w\|,$$

where $w:F\to l_{p^*}^n$ is defined by $w(b)=\sum_{i=1}^{n}b_i^*(b)e_i$. Indeed,

$$\begin{aligned}\Big|\sum_{i=1}^{n}\langle RT(x_i)-RT(x_i'),b_i^*\rangle\Big| &= \Big|\sum_{i=1}^{n}\langle T(x_i)-T(x_i'),R^\#(b_i^*)\rangle\Big|\\ &\leq \eta_p^L(T)\sup_{f\in B_{E^\#}}\Big(\sum_{i=1}^{n}|f(x_i)-f(x_i')|^p\Big)^{1/p}\|u\|,\\ &\leq \eta_p^L(T)\operatorname{Lip}(R)\sup_{f\in B_{E^\#}}\Big(\sum_{i=1}^{n}|f(e_i)-f(e_i')|^p\Big)^{1/p}\|w\|,\end{aligned}$$

where $u(y)=\sum_{i=1}^{n}\langle R^\#(b_i^*),y\rangle e_i=\sum_{i=1}^{n}\langle b_i^*,R(y)\rangle e_i$. This implies that $R\circ T$ is Cohen Lipschitz p-nuclear and $\eta_p^L(R\circ T)\leq\|R\|\eta_p^L(T)$. ∎

Let us prove the Pietsch domination theorem for this class of Lipschitz operators. The proof is like that in [AMS09]. In [Cha11], J. A. Chávez-Domínguez gives a domination theorem for Lipschitz (r,p,q)-summing operators from X into E^* such that $1/r+1/p+1/q=1$.

Theorem 3.3. *Let $T\in\operatorname{Lip}_0(X,E)$ and C a positive constant. Then the following assertions are equivalent:*

(1) *T is Cohen Lipschitz p-nuclear and $\eta_p^L(T)\leq C$.*

(2) *For any $n\in\mathbb{N}$, $(x_i)_{1\leq i\leq n},(x_i')_{1\leq i\leq n}\subset X$, $(a_i^*)_{1\leq i\leq n}\subset E^*$ and $(\lambda_i)_{1\leq i\leq n}\subset\mathbb{R}_+$, we have*

$$\sum_{i=1}^{n}\lambda_i|\langle T(x_i)-T(x_i'),a_i^*\rangle| \leq C\sup_{f\in B_{X^\#}}\Big(\sum_{i=1}^{n}\lambda_i|f(x_i)-f(x_i')|^p\Big)^{1/p}\sup_{\|a\|_E\leq1}\Big(\sum_{i=1}^{n}|\langle a,a_i^*\rangle|^{p^*}\Big)^{1/p^*}. \tag{3.4}$$

(3) *There exist Radon probability measures μ_1 on $B_{X^\#}$ and μ_2 on $B_{E^{**}}$ such that for all $x,x'\in X$ and $a^*\in E^*$, we have*

$$|\langle T(x)-T(x'),a^*\rangle| \leq C\Big(\int_{B_{X^\#}}|f(x)-f(x')|^p\,d\mu_1(f)\Big)^{1/p}\Big(\int_{B_{E^{**}}}|a^*(a^{**})|^{p^*}\,d\mu_2(a^{**})\Big)^{1/p^*}. \tag{3.5}$$

Moreover, in this case $\eta_p^L(T)=\inf C$.

The following definition was introduced independently by [Saa15] and [YAR]. For our convenience, we adopt the notation of [YAR].

Definition 3.4. A Lipschitz map $T : X \to E$ is *Lipschitz strongly p-summing* $(1 < p \leq \infty)$ if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n}, (x'_i)_{1\leq i\leq n} \subset X$, $(a^*_i)_{1\leq i\leq n} \subset E^*$ and $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}_+$, we have

$$\sum_{i=1}^{n} \lambda_i |\langle T(x_i) - T(x'_i), a^*_i\rangle| \leq C\Big(\sum_{i=1}^{n} \lambda_i d_X(x_i, x'_i)^p\Big)^{1/p} \omega_{p^*}((a^*_i)_i). \tag{3.6}$$

We denote by $\mathcal{D}^L_{\mathrm{st},p}(X, E)$ the class of all Lipschitz strongly p-summing operators from X into E, and by $d^L_{\mathrm{st},p}(T)$ the smallest C such that (3.6) holds. This generalizes the definition introduced in [Coh73] in the linear case. If T is linear, then $\mathcal{D}^L_{st,p}(X, E) = \mathcal{D}_p(X, E)$ because $B_{X^\#}$ is not involved in the definition.

Let $T \in \mathrm{Lip}_0(X; E)$ and $v : l^n_p \to E^*$ be a bounded linear operator. Then T is strongly Lipschitz p-summing if and only if

$$\sum_{i=1}^{n} \lambda_i |\langle T(x_i) - T(x'_i), v(e_i)\rangle| \leq C\Big(\sum_{i=1}^{n} \lambda_i d_X(x_i, x'_i)^p\Big)^{1/p} \|v\|. \tag{3.7}$$

Now, we give the domination theorem for Lipschitz strongly p-summing operators (see [Saa15] and [YAR]).

Theorem 3.5. *A Lipschitz operator $T : X \to E$ is Lipschitz strongly p-summing $(1 < p < \infty)$ if and only if there exist a positive constant C and a Radon probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, we have*

$$|\langle T(x) - T(x'), a^*\rangle| \leq C d_X(x, x')\Big(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu(a^{**})\Big)^{1/p^*}. \tag{3.8}$$

Moreover, in this case $d^L_{\mathrm{st},p}(T) = \inf C$.

From the above result, we deduce (see also [Saa15, Theorem 3.8] and [Cha11]).

Corollary 3.6. *Let C be a positive constant. A Lipschitz map $T : X \to E$ is Cohen Lipschitz p-nuclear and $\eta^L_p(T) \leq C$ if and only if there exist a Banach space Z, a Lipschitz p-summing operator $R : X \to Z^*$ and a linear p^*-summing operator $S : E^* \to Z$ (i.e., S^* is strongly p-summing) such that $T = S^* \circ R$ and $\pi^L_p(R)\pi_{p^*}(S) \leq C$.*

4. Relationships between $\Pi^L_p(X, E), \mathcal{D}^L_{\mathrm{st},p}(X, E), \Pi^L_{\tau(p)}(X, E)$ and $\mathcal{N}^L_p(X, E)$. In this section we investigate, as in the linear case, the relationships between various classes of Lipschitz operators.

THEOREM 4.1.

(1) $\mathcal{N}_p^L(X,E) \subseteq \mathcal{D}_{\mathrm{st},p}^L(X,E)$ *and* $d_{\mathrm{st},p}^L(\cdot) \le \eta_p^L(\cdot)$ for $1 < p \le \infty$.

(2) $\mathcal{N}_p^L(X,E) \subseteq \Pi_p^L(X,E)$ *and* $\pi_p^L(\cdot) \le \eta_p^L(\cdot)$ *for* $1 \le p < \infty$.

(3) $\Pi_{\tau(p)}^L(X,E) \subseteq \mathcal{D}_{\mathrm{st},p^*}^L(X,E)$ *and* $d_{\mathrm{st},p^*}^L(\cdot) \le \pi_{\tau(p)}^L(\cdot)$ *for* $1 \le p < \infty$.

(4) $\Pi_\tau^L(X,E) \subset \mathcal{N}_p^L(X,E)$ *and* $\eta_p^L(\cdot) \le \pi_\tau^L(\cdot)$ *for* $1 \le p \le \infty$.

Proof. Let $T \in \mathcal{N}_p^L(X,E)$, $x, x' \in X$ and $a^* \in E^*$. By (3.5),

$$\begin{aligned}
&|\langle T(x) - T(x'), a^*\rangle| \\
&\quad \le \eta_p^L(T)\Big(\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\mu_1(f)\Big)^{1/p} \Big(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu_2(a^{**})\Big)^{1/p^*} \\
&\quad \le \eta_p^L(T)\Big(\int_{B_{X^\#}} d(x,x')^p \, d\mu_1(f)\Big)^{1/p} \Big(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu_2(a^{**})\Big)^{1/p^*} \\
&\quad \le \eta_p^L(T) d(x,x') \Big(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu_2(a^{**})\Big)^{1/p^*}.
\end{aligned}$$

Hence by (3.8), T is Lipschitz strongly p-summing and $d_{\mathrm{st},p}^L(T) \le \eta_p^L(T)$.

(2) Let $T \in \mathcal{N}_p^L(X,E)$. By (3.5),

$$\begin{aligned}
\|T(x) - T(x')\| &= \sup_{a^* \in B_{E^*}} |\langle T(x) - T(x'), a^*\rangle| \\
&\le \sup_{a^* \in B_{E^*}} \eta_p^L(T)\Big(\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\mu_1(f)\Big)^{1/p} \\
&\qquad \times \Big(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu_2(a^{**})\Big)^{1/p^*} \\
&\le \eta_p^L(T)\Big(\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\mu_1(f)\Big)^{1/p}.
\end{aligned}$$

By the Pietsch domination theorem [FJ09], T is Lipschitz p-summing and $\pi_p^L(T) \le \eta_p^L(T)$.

(3) Let $T \in \Pi_{\tau(p)}^L(X,E)$, $x, x' \in E$ and $a^* \in E^*$. By (2.11),

$$\begin{aligned}
&|\langle T(x) - T(x'), a^*\rangle| \\
&\quad \le \pi_{\tau(p)}^L(T)\Big(\int_{B_{X^\#}} \int_{B_{E^{**}}} |(f(x) - f(x'))\langle a^*, a^{**}\rangle|^p \, d\mu_1(f) \, d\mu_2(a^{**})\Big)^{1/p} \\
&\quad \le \pi_{\tau(p)}^L(T) d(x,x') \Big(\int_{B_{X^\#}} \int_{B_{E^{**}}} \Big|\frac{f(x) - f(x')}{d(x,x')}\langle a^*, a^{**}\rangle\Big|^p \, d\mu_1(f) \, d\mu_2(a^{**})\Big)^{1/p}
\end{aligned}$$

$$\leq \pi^L_{\tau(p)}(T)d(x,x')\Big(\int\limits_{B_{X^\#}}\int\limits_{B_{E^{**}}}\sup_{x\neq x'}\Big|\frac{f(x)-f(x')}{d(x,x')}\langle a^*,a^{**}\rangle\Big|^p d\mu_1(f)\,d\mu_2(a^{**})\Big)^{1/p}$$

$$\leq \pi^L_{\tau(p)}(T)d(x,x')\Big(\int\limits_{B_{E^{**}}}|\langle a^*,a^{**}\rangle|^p\,d\mu_2(a^{**})\Big)^{1/p}.$$

This implies by (3.8) that T is Lipschitz strongly p^*-summing and $d^L_{\mathrm{st},p^*}(T)\leq \pi^L_{\tau(p)}(T)$.

(4) Let $T\in \Pi^L_\tau(X,E)$. For $n\in\mathbb{N}$, $(x_i)_{1\leq i\leq n},(x'_i)_{1\leq i\leq n}\subset X$ and $(a^*_i)_{1\leq i\leq n}\subset E^*$, we have

$$\sum_{i=1}^n|\langle T(x_i)-T(x'_i),a^*_i\rangle|\leq \pi^L_\tau(T)\sup_{\substack{\|f\|\leq1\\ \|a^{**}\|\leq1}}\Big(\sum_{i=1}^n|(f(x_i)-f(x'_i))\langle a^{**},a^*_i\rangle|\Big)$$
$$\leq \pi^L_\tau(T)\omega^L_p(1,(x_i),(x'_i))\omega_{p^*}((a^*_i)_i)$$

by Hölder's inequality. This proves $T\in\mathcal{N}^L_p(X,E)$ and $\eta^L_p(T)\leq\pi^L_\tau(T)$. ∎

THEOREM 4.2. *Let $1\leq p\leq\infty$, $T\in \mathrm{Lip}_0(X,E)$ and $L\in\mathrm{Lip}_0(E,F)$. If L is Lipschitz strongly p-summing and T is Lipschitz p-summing, then $L\circ T$ is Cohen Lipschitz p-nuclear and $\eta^L_p(L\circ T)\leq d^L_{\mathrm{st},p}(L)\pi^L_p(T)$.*

Proof. Let $x,x'\in E$ and $b^*\in F^*$. By (3.8), we have

$$|\langle L\circ T(x)-L\circ T(x),b^*\rangle| = |\langle L(T(x))-L(T(x')),b^*\rangle|$$
$$\leq d^L_{\mathrm{st},p}(L)\|T(x)-T(x')\|\Big(\int\limits_{B_{F^{**}}}|b^*(b^{**})|^{p^*}\,d\mu_2(b^{**})\Big)^{1/p^*},$$

and by the Pietsch domination theorem [FJ09],

$$|\langle L\circ T(x)-L\circ T(x),b^*\rangle|$$
$$\leq d^L_{\mathrm{st},p}(L)\pi^L_p(T)\Big(\int\limits_{B_{X^\#}}|f(x)-f(x')|^p\,d\mu_1(f)\Big)^{1/p}\Big(\int\limits_{B_{F^{**}}}|b^*(b^{**})|^{p^*}\,d\mu_2(b^{**})\Big)^{1/p^*}.$$

This shows that $L\circ T\in\mathcal{N}^L_p(X,F)$ and $\eta^L_p(L\circ T)\leq d^L_{\mathrm{st},p}(L)\pi^L_p(T)$. ∎

COROLLARY 4.3. *If $p\geq2$, then $\pi^L_{\tau(p)}(L\circ T)\leq d^L_{\mathrm{st},p}(L)\pi^L_p(T)$.*

THEOREM 4.4. *Let $1\leq r,p,q<\infty$ with $1/r=1/p+1/q$ and let $T\in\mathrm{Lip}_0(X,E)$ and $L\in\mathrm{Lip}_0(E,F)$. If L is Lipschitz $\tau(r)$-summing and T is Lipschitz p-summing, then $L\circ T$ is Lipschitz (r,p,q)-summing and $\pi^L_{(r,p,q)}(L\circ T)\leq\pi^L_{\tau(r)}(L)\pi^L_p(T)$.*

Proof. Let $x, x' \in X$ and $b^* \in F^*$. By (2.11),

$$|\langle L \circ T(x) - L \circ T(x'), b^* \rangle| \leq \pi^L_{\tau(r)}(L) \Big(\int_{B_{E^\#}} \int_{B_{F^{**}}} |f(T(x)) - f(T(x')) \langle b^*, b^{**} \rangle|^r \, d\mu_1(f) \, d\mu_2(b^{**}) \Big)^{1/r}.$$

Using the general Hölder inequality and the fact that T is Lipschitz p-summing, we get

$$\begin{aligned} &|\langle L \circ T(x) - L \circ T(x'), b^* \rangle| \\ &\leq \pi^L_{\tau(r)}(L) \Big(\int_{B_{E^\#}} |f(T(x)) - f(T(x'))|^p \, d\mu_1(f) \Big)^{1/p} \Big(\int_{B_{F^{**}}} |\langle b^*, b^{**} \rangle|^q \, d\mu_2(b^{**}) \Big)^{1/q} \\ &\leq \pi^L_{\tau(r)}(L) \|T(x) - T(x')\| \Big(\int_{B_{F^{**}}} |\langle b^*, b^{**} \rangle|^q \, d\mu_2(b^{**}) \Big)^{1/q} \\ &\leq \pi^L_{\tau(r)}(L) \pi^L_p(T) \Big(\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\mu(f) \Big)^{1/p} \Big(\int_{B_{F^{**}}} |\langle b^*, b^{**} \rangle|^q \, d\mu_2(b^{**}) \Big)^{1/q}. \end{aligned}$$

This implies $L \circ T \in \Pi^L_{(r,p,q)}(X, F)$ and $\pi^L_{(r,p,q)}(L \circ T) \leq \pi^L_{\tau(r)}(L) \pi^L_p(T)$. ■

COROLLARY 4.5. *Let $1 < p < \infty$, $T \in \mathrm{Lip}_0(X, E)$ and $L \in \mathrm{Lip}_0(E, F)$. If L is Lipschitz τ-summing and T is Lipschitz p-summing, then $L \circ T$ is Cohen Lipschitz p-nuclear and $\eta^L_p(L \circ T) \leq \pi^L_\tau(L) \pi^L_p(T)$.*

Acknowledgments. The authors are grateful to an anonymous referee for pointing out some mistakes in the previous version of this paper, and for a careful reading that led to many improvements.

REFERENCES

[AA10] A. Achour and A. Alouani, *On multilinear generalizations of the concept of nuclear operators*, Colloq. Math. 120 (2010), 85–102.

[AM07] D. Achour and L. Mezrag, *On the Cohen strongly p-summing multilinear operators*, J. Math. Anal. Appl. 327 (2007), 55–63.

[AMS09] D. Achour, L. Mezrag and K. Saadi, *On the Cohen p-nuclear sublinear operators*, J. Inequal. Pure Appl. Math. 10 (2009), 1–13.

[Api76] H. Apiola, *Duality between spaces of p-summable sequences, (p, q)-summing operators and characterizations of nuclearity*, Math. Ann. 219 (1976), 53–64.

[AE56] R. F. Arens and J. Eels, *On embedding uniform and topological spaces*, Pacific J. Math 6 (1956), 397–403.

[BL00] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Amer. Math. Soc., Providence, 2000.

[BPR10] G. Botelho, D. Pellegrino and P. Rueda, *A unified Pietsch Domination Theorem*, J. Math. Anal. Appl. 365 (2010), 269–276.

[CCJV] M. G. Cabrera-Padilla, J. A. Chávez-Domínguez, A. Jiménez-Vargas and M. Villegas-Vallecillos, *Lipschitz tensor product*, Khayyam J. Math. 1 (2015), 185–218.

[Cha11] J. A. Chávez-Domínguez, *Duality for Lipschitz p-summing operators*, J. Funct. Anal. 261 (2011), 387–407.

[Cha12] J. A. Chávez-Domínguez, *Lipschitz (q,p)-mixing operators*, Proc. Amer. Math. Soc. 140 (2012), 3101–3115.

[CZ11] D. Chen and B. Zheng, *Remarks on Lipschitz p-summing operators*, Proc. Amer. Math. Soc. 139 (2011), 2891–2898.

[CZ12] D. Chen and B. Zheng, *Lipschitz p-integral operators and Lipschitz p-nuclear operators*, Nonlinear Anal. 75 (2012), 5270–5282.

[Coh73] J. S. Cohen, *Absolutely p-summing, p-nuclear operators and their conjugates*, Math. Ann. 201 (1973), 177–200.

[DF93] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Stud. 176, North-Holland, Amsterdam, 1993.

[DJT95] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge Univ. Press, Cambridge, 1995.

[FJ09] J.-D. Farmer and W.-B. Johnson, *Lipschitz p-summing operators*, Proc. Amer. Math. Soc. 137 (2009), 2989–2995.

[Gro55] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[JS09] W. B. Johnson and G. Schechtman, *Diamond graphs and super-reflexivity*, J. Topol. Anal. 1 (2009), 177–189.

[Kan42] L. V. Kantorovich, *On the translocation of masses*, Dokl. Akad. Nauk SSSR 37 (1942), 227–229.

[LP68] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. 29 (1968), 275–326.

[Muj08] X. Mujica, *$\tau(p;q)$-summing mappings and the domination theorem*, Portugal. Math. (N.S.) 65 (2008), 211–226.

[PSS12a] D. Pellegrino, J. Santos and J.-B. Seoane-Sepúlveda, *Some techniques on nonlinear analysis and applications*, Adv. Math. 229 (2012), 1235–1265.

[PSS12b] D. Pellegrino, J. Santos and J.-B. Seoane-Sepúlveda, *A general extrapolation theorem for absolutely summing operators*, Bull. London Math. Soc. 44 (2012), 1292–1302.

[PP69] A. Persson and A. Pietsch, *p-nukleare and p-integrale Abbildungen in Banachräumen*, Studia Math. 33 (1969), 213–222.

[Pie67] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. 28 (1967), 333–353.

[Pie80] A. Pietsch, *Operator Ideals*, North-Holland Math. Library 20, North Holland, Amsterdam, 1980.

[PSS12a] D. Pellegrino, J. Santos and J.-B. Seoane-Sepúlveda, *Some techniques on nonlinear analysis and applications*, Adv. Math. 229 (2012), 1235–1265.

[PSS12b] D. Pellegrino, J. Santos and J.-B. Seoane-Sepúlveda, *A general extrapolation theorem for absolutely summing operators*, Bull. London Math. Soc. 44 (2012), 1292–1302.

[Saa15] K. Saadi, *Some properties of Lipschitz strongly p-summing operators*, J. Math. Anal. Appl. 423 (2015), 1410–1426.

[Saw75] I. Sawashima, *Methods of duals in nonlinear analysis—Lipschits duals of Banach spaces and some applications*, in: Nonlinear and Convex Analysis in Economic Theory (Tokyo, 1993) Lecture Notes in Econom. Math. Systems 419, Springer, 1975, 247–259.

[Wea99] N. Weaver, *Lipschitz Algebras*, World Sci., Singapore, 1999.
[YAR] R. Yahi, D. Achour and P. Rueda, *Absolutely summing Lipschitz conjugates*, submitted.

Lahcène Mezrag, Abdelhamid Tallab
Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces
University of M'sila
Box 166
Ichbilia, M'sila, 28000, Algeria
E-mail: lamezrag@yahoo.fr
hamidtallab@yahoo.fr