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SUBMODULE-CLOSED SUBCATEGORIES OF FINITE TYPE

BY

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Abstract. Let R be a left pure semisimple ring, and $\mathcal{C}$ a full subcategory of finitely generated left R-modules such that $\mathcal{C}$ is closed under finite direct sums and submodules. It is shown that if $\mathcal{C}$ has an infinite number of non-isomorphic indecomposable modules, then $\mathcal{C}$ contains a submodule-closed subcategory of finite type $\mathcal{A}$ (i.e. $\mathcal{A}$ has only finitely many non-isomorphic indecomposable modules) which is maximal among all submodule-closed subcategories of finite type in $\mathcal{C}$, and moreover $\mathcal{A}$ contains an indecomposable module which is not the source of a left almost split morphism in R-mod. If R is an indecomposable hereditary left pure semisimple ring, a maximal submodule-closed subcategory of finite type of R-mod always contains the preprojective component of R-mod, and if such a ring R has only two simple modules, the unique maximal submodule-closed subcategory of finite type in R-mod can be described explicitly.

1. Introduction. Let R be a left artinian ring, and R-mod the category of all finitely generated left R-modules. We will always consider full additive subcategories $\mathcal{C}$ of R-mod, i.e. $\mathcal{C}$ is closed under finite direct sums, direct summands and isomorphic images. $\mathcal{C}$ is called *submodule-closed* if any submodule of a module in $\mathcal{C}$ also belongs to $\mathcal{C}$. We say that $\mathcal{C}$ is of *finite type* if $\mathcal{C}$ has only finitely many non-isomorphic indecomposable modules; otherwise $\mathcal{C}$ is of *infinite type*. In a recent work [38], Ringel proved the following rather surprising result: If R is an artin algebra, and $\mathcal{C}$ is a submodule-closed subcategory of infinite type in R-mod, then $\mathcal{C}$ contains a minimal submodule-closed subcategory $\mathcal{B}$ of infinite type (i.e. $\mathcal{B}$ does not contain any proper submodule-closed subcategory of infinite type). Ringel's proof was based on the concept of the Gabriel–Roiter measure, introduced and studied in [36, 37]. Krause and Prest [32] provided an alternative proof, using the compactness of the Ziegler spectrum over an artin algebra R.

In this paper, we consider submodule-closed subcategories over left pure semisimple rings, and obtain a (modified) version of Ringel's result in our

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context. A ring R is called *left pure semisimple* if every left R-module is a direct sum of finitely generated left R-modules, or equivalently, if R has left pure global dimension zero [23, 40, 41]. It is well known that a ring R is left and right pure semisimple if and only if R is of *finite representation type*, i.e. R is left (and right) artinian with only finitely many non-isomorphic finitely generated indecomposable left (and right) modules (see [7, 24, 39]). The problem whether left pure semisimple rings are always of finite representation type, known as the *pure semisimplicity conjecture*, still remains open. The reader is referred to [30, 35, 45] for the history and basic background, and [26, 27, 43, 44, 46] for potential counter-examples to the conjecture.

Submodule-closed subcategories occur naturally in the study of left pure semisimple rings (see, e.g., [18, 21]). As pointed out in [21, Remark 3.10(a)], if R is a left pure semisimple ring, each submodule-closed subcategory of infinite type in R-mod contains a proper submodule-closed subcategory of infinite type. Thus, in contrast to the artin algebra case, minimal submodule-closed subcategories of infinite type over a left pure semisimple ring do not exist. It is well known that if R is an artin algebra, then every indecomposable module in R-mod is the source of a left almost split morphism in R-mod (see, e.g., [9]). On the other hand, if R is a left pure semisimple ring, then every indecomposable module in R-mod is the source of a left almost split morphism in R-mod if and only if R is of finite representation type (see [29]). Indeed, if R is a left pure semisimple ring of infinite representation type, the lack of left almost split morphisms in R-mod is the crucial reason why the proofs of Ringel's result in [32, 38] do not apply in our context. It is therefore of interest to identify, inside submodule-closed subcategories of infinite type, those indecomposable modules that are not sources of left almost split morphisms in R-mod.

We will show that if R is any left pure semisimple ring, and $\mathcal{C}$ is a submodule-closed subcategory of infinite type in R-mod, then $\mathcal{C}$ contains a submodule-closed subcategory $\mathcal{A}$ of finite type which is maximal in $\mathcal{C}$ with these properties, i.e. there are no submodule-closed subcategories of finite type in $\mathcal{C}$ that properly contain $\mathcal{A}$. Moreover, the subcategory $\mathcal{A}$ contains an indecomposable module M which is not the source of a left almost split morphism in R-mod. We provide two different approaches for a construction of such a subcategory $\mathcal{A}$. The first approach uses results on the Gabriel–Roiter measure over a left pure semisimple ring, as developed in [22], and the second approach is based on the strong preinjective partition in R-mod (see [29]).

Our results show, in particular, that if R is a left pure semisimple ring, then there are maximal submodule-closed subcategories of finite type in R-mod, and each submodule-closed subcategory of finite type in R-mod is contained in a maximal one. When R is an indecomposable hereditary left pure semisimple ring of infinite representation type, we show that a maximal submodule-closed subcategory of finite type always contains the preprojective component of R-mod. If, in addition, such a ring R has only two simple modules (as in the case of Simson's potential counter-examples to the pure semisimplicity conjecture [43, 44, 46]; see also García [26, 27]), then there is a unique maximal submodule-closed subcategory of finite type in R-mod whose indecomposable modules consist of all preprojective modules and the unique simple injective module. We also provide some sufficient conditions for a submodule-closed subcategory over a general left artinian ring to be of finite type, implying in particular that if R is an artin algebra, a submodule-closed subcategory $\mathcal{C}$ of R-mod is of finite type if and only if the direct sum of all indecomposable modules in $\mathcal{C}$ is pure-injective.

We refer the reader to [1, 6, 9] for general properties of rings and modules, and for basic representation-theoretic notions and results used in the text.

2. Definitions and preliminaries. Throughout this paper, R is an associative ring with identity. We denote by R-mod the category of all finitely generated left R-modules, and by R-Mod the category of all left R-modules. The corresponding categories of right R-modules are denoted by mod-R and Mod-R.

Let $\mathcal{B}$ be a family of finitely generated left R-modules. By $\mathrm{Add}(\mathcal{B})$ (respectively, $\mathrm{add}(\mathcal{B})$) we denote the class consisting of all left R-modules that are isomorphic to direct summands of (respectively, finite) direct sums of modules in $\mathcal{B}$. Similarly, $\mathrm{ind}(\mathcal{B})$ will stand for the set of all non-isomorphic indecomposable summands of modules belonging to $\mathcal{B}$. In particular, we set $R\text{-ind} = \mathrm{ind}(R\text{-mod})$. Given families $\mathcal{A}$ and $\mathcal{B}$ of modules in R-mod, $\mathcal{A}$ is called a *cogenerating set* for $\mathcal{B}$ if every module in $\mathcal{B}$ can be embedded into a module in $\mathrm{add}(\mathcal{A})$. If $\mathcal{A}$ is a cogenerating set for $\mathcal{B}$, and no proper subfamily of $\mathcal{A}$ is a cogenerating set for $\mathcal{B}$, then we say $\mathcal{A}$ is a *minimal cogenerating set* for $\mathcal{B}$. Following [10], an indecomposable module M in $\mathcal{B}$ is said to be *splitting injective* in $\mathcal{B}$ if any monomorphism $f : M \to N$ with $N \in \mathrm{add}(\mathcal{B})$ splits.

Following [29] (see also [10]), if $\mathcal{C}$ is a family of finitely generated left R-modules, then $\mathcal{C}$ is said to have a *strong preinjective partition* if $\mathrm{ind}(\mathcal{C}) = \bigcup_{\alpha<\rho} \mathcal{I}_\alpha$, for an ordinal number ρ, such that all sets $\mathcal{I}_\alpha$ are non-empty finite and pairwise disjoint, and $\mathcal{I}_\gamma$ is a minimal cogenerating set for $\bigcup_{\beta\geq\gamma} \mathcal{I}_\beta$ for each $\gamma < \rho$.

Recall that if $\mathcal{C}$ is a class of left R-modules, a module X in $\mathcal{C}$ is said to be *Ext-injective* in $\mathcal{C}$ if $\mathrm{Ext}^1_R(C, X) = 0$ for all $C \in \mathcal{C}$. Following [20], we say that a family $\mathcal{C}$ of non-isomorphic indecomposable modules in R-mod has an *Ext-injective partition* if $\mathcal{C} = \bigcup_{\alpha<\rho} \mathcal{U}_\alpha$, for an ordinal number ρ, such that all sets $\mathcal{U}_\alpha$ are non-empty finite and pairwise disjoint, and for each ordinal number $\gamma < \rho$, $\mathcal{U}_\gamma$ is the set of all Ext-injective modules of $\bigcup_{\beta\geq\gamma} \mathcal{U}_\beta$.

We first observe the following result due to Huisgen-Zimmermann [29].

THEOREM 2.1. *Let R be a left pure semisimple ring. Then each subfamily of indecomposable modules in R-*mod *has a strong preinjective partition.*

We will frequently use the following fact which connects the splitting injectives with the minimal cogenerating set of some families of indecomposable modules.

LEMMA 2.2. *Let R be any left artinian ring and $\mathcal{C} = \{M_i \mid i \in I\}$ a family of finitely generated indecomposable left R-modules. If $\mathcal{C}$ contains a minimal cogenerating set $\mathcal{A}$, then $\mathcal{A}$ coincides with the set of all splitting injective modules in $\mathcal{C}$.*

Proof. See [15, Lemma 2.1] (cf. [10, Theorem 2.3], [29, Theorem 4(2)]). ∎

The next result was obtained by Prest [34, Theorem 3.8] and Huisgen-Zimmermann and Zimmermann [31, Corollary 10].

THEOREM 2.3. *If R is a left pure semisimple ring then, for every positive integer n, the number of isomorphism classes of the indecomposable left R-modules of length n is finite.*

Let $\mathcal{A}$ be a family of modules closed under finite direct sums and direct summands, and $\mathcal{B}$ a subfamily of $\mathcal{A}$. For a module $M \in \mathcal{A}$, a homomorphism $f : M \to N$ with $N \in \text{add}(\mathcal{B})$ is a *left $\mathcal{B}$-approximation* of M if each map in $\text{Hom}(M, X)$ with $X \in \text{add}(\mathcal{B})$ factors through f. If every module M in $\mathcal{A}$ has a left $\mathcal{B}$-approximation, then we say that $\mathcal{B}$ is *covariantly finite* in $\mathcal{A}$ (see [10]). A left $\mathcal{B}$-approximation $f : M \to N$ is said to be *minimal* if for any endomorphism $g : N \to N$, $g \circ f = f$ implies that g is an automorphism of N. *Minimal right $\mathcal{B}$-approximations* $f : N \to M$ are defined dually, and then f is called a *$\mathcal{B}$-cover* of M. For a ring R and a left R-module M, $\text{sub}(M)$ will denote the family of all submodules of finite direct sums of copies of M.

LEMMA 2.4. *If R is a left pure semisimple ring, and $\mathcal{C}$ a subcategory of R-*mod*, then $\mathcal{C}$ is submodule-closed if and only if $\mathcal{C} = \text{sub}(M)$ for some finitely generated left R-module M. In this case, $\mathcal{C}$ is covariantly finite in R-*mod.

Proof. If M is a finitely generated left R-module, it is clear that $\text{sub}(M)$ is a submodule-closed subcategory of R-mod. Conversely, suppose $\mathcal{C}$ is a submodule-closed subcategory of R-mod. By Theorem 2.1, $\mathcal{C}$ has a strong preinjective partition, in particular it has a finite cogenerating set of indecomposable modules $\{M_1, \ldots, M_n\}$, so each module in $\mathcal{C}$ is isomorphic to a submodule of a finite direct sum of modules in the family $\{M_1, \ldots, M_n\}$. It follows that $\mathcal{C} = \mathrm{sub}(M)$, where $M = \bigoplus_{i=1}^n M_i$ is a finitely generated module. Finally, the proof of [10, Proposition 4.8] shows that if $\mathcal{C}$ is a submodule-closed subcategory of R-mod, then $\mathcal{C}$ is covariantly finite in R-mod (this is true for any left artinian ring). ∎

Now we recall the definition of the Gabriel–Roiter measure, following Gabriel [25] and Ringel [36, 37]. We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of all positive integers, and by $\mathcal{P}(\mathbb{N})$ the set of all subsets of $\mathbb{N}$ equipped with the total order $<$ by setting $I < J$ if the smallest element of the symmetric difference $(I \setminus J) \cup (J \setminus I)$ belongs to J, for a pair I, J of different subsets of $\mathbb{N}$. For a module M of finite length, its length is denoted by $|M|$. The *Gabriel–Roiter measure* of an indecomposable module M of finite length, denoted by $\mu(M)$, is defined to be the maximum (in $\mathcal{P}(\mathbb{N})$) of the sets $\{|M_1|, \ldots, |M_n|\}$, where $M_1 \subset \cdots \subset M_n$ is a chain of indecomposable submodules of M. We say that q is a *Gabriel–Roiter measure* for a ring R if $q = \mu(M)$, where M is an indecomposable module of finite length in R-mod.

Following [36, 37], an inclusion $M \subset N$ of indecomposable R-modules of finite length is said to be a *Gabriel–Roiter inclusion* if $\mu(N') \leq \mu(M)$ for every proper indecomposable submodule N' of N. In this case, we say that M is a *Gabriel–Roiter submodule* of N. A sequence $M_1 \subset \cdots \subset M_n = M$ of indecomposable modules in R-mod is called a *Gabriel–Roiter filtration* of M if M_1 is simple and M_i is a Gabriel–Roiter submodule of M_{i+1} for $i = 1, \ldots, n-1$.

We now recall some basic properties of the Gabriel–Roiter measure that we need. The next result is the *main property* of Gabriel–Roiter measures, essentially due to Gabriel [25], and proved by Ringel [36, Main Property, p. 728] for arbitrary modules.

PROPOSITION 2.5. *Let R be a left artinian ring. Assume $M, N_1, \ldots, N_m$ are indecomposable modules in R-mod and there is a monomorphism $f : M \to N_1 \oplus \cdots \oplus N_m$. Then:*

(a) $\mu(M) \leq \max\{\mu(N_1), \ldots, \mu(N_m)\}$.
(b) *If $\mu(M) = \max\{\mu(N_1), \ldots, \mu(N_m)\}$, then the monomorphism f splits.* ∎

PROPOSITION 2.6. *Let R be a left artinian ring. Then:*

(a) *If M and N are indecomposable modules in R-mod and $\mu(M) = \mu(N)$, then M and N have the same length.*
(b) *Every non-simple indecomposable module N in R-mod has a Gabriel–Roiter submodule M. The module M might not be unique, but the value of $\mu(M)$ is uniquely determined by N.*

(c) *Every indecomposable module M in R-*mod *has a Gabriel–Roiter filtration $M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$, and the values $\mu(M_1), \ldots, \mu(M_n)$ are uniquely determined by M.*

(d) *Let $M \subset N$ be a Gabriel–Roiter inclusion of indecomposable R-modules of finite length. If L is a proper submodule of N containing M, then M is a direct summand of L.*

Proof. Apply [22, Lemma 2.7] (see also [36, 37] for the original proofs). ∎

For Gabriel–Roiter measures p, q we call q a *direct successor* of p if $p < q$ and there is no Gabriel–Roiter measure q' such that $p < q' < q$. A family $\mathcal{I}$ of Gabriel–Roiter measures of R-mod is said to satisfy the *noetherian condition* if there does not exist an infinite strictly ascending chain $q_1 < q_2 < \cdots$ of Gabriel–Roiter measures in $\mathcal{I}$, or equivalently, if every non-empty subset of Gabriel–Roiter measures in $\mathcal{I}$ has a maximal element.

We will need the following characterization of left pure semisimple rings in terms of Gabriel–Roiter measures, obtained in [22].

PROPOSITION 2.7. *Let R be a left artinian ring. Then R is left pure semisimple if and only if the family of all Gabriel–Roiter measures $\mu(M)$ with indecomposable modules $M \in R$-*mod *satisfies the noetherian condition.*

Proof. Apply [22, Theorem 3.2]. ∎

If $\mathcal{C}$ is a subfamily in R-mod, then for an indecomposable module M in $\mathcal{C}$ and N in add$(\mathcal{C})$, a homomorphism $f : M \to N$ is called a *left almost split morphism* in add$(\mathcal{C})$ provided f is not a split monomorphism, and for any module K in add$(\mathcal{C})$ and any homomorphism $g : M \to K$ which is not a split monomorphism, there is a homomorphism $h : N \to K$ such that $g = h \circ f$. If $f : M \to N$ is a left almost split morphism in add$(\mathcal{C})$, then we say that M is the *source* of a left almost split morphism in add$(\mathcal{C})$.

It is clear (by Theorem 2.3) that if M is an indecomposable left R-module, where R is left pure semisimple, then M contains only finitely many non-isomorphic Gabriel–Roiter submodules. It is natural to ask if an indecomposable left R-module M can admit a finite or an infinite number of different Gabriel–Roiter inclusions $f : M \to N$. The next result, inspired by ideas in [37, Section 3], will provide an answer to this question when M is the source of a left almost split morphism in R-mod.

LEMMA 2.8. *Let R be a left pure semisimple ring and M an indecomposable module which is the source of a left almost split morphism in R-*mod*. Then there is only a finite number of non-isomorphic indecomposable modules N in R-*mod *such that there is a Gabriel–Roiter inclusion $f : M \to N$.*

Proof. Let $g : M \to L$ be a left almost split morphism in R-mod. For any Gabriel–Roiter inclusion $f : M \to N$, since f is non-split, there is a homomorphism $h : L \to N$ such that $h \circ g = f$. Then $f(M) \subseteq h(L)$, and if $h(L) \neq N$, then by Proposition 2.6(d), $f(M)$ is a direct summand of $h(L)$. But then $g : M \to L$ would split, a contradiction. Hence $h(L) = N$, so the length of N does not exceed the length of L. By Theorem 2.3, there are only finitely many non-isomorphic indecomposable left R-modules of a given length, and the result follows. ■

It would be interesting to know whether (or when) the converse of Lemma 2.8 is true for an indecomposable module M over a left pure semisimple ring R.

3. Main results. In this section we study submodule-closed subcategories over a left pure semisimple ring R. Our first theorem may be regarded as a (modified) version of Ringel's result in [38] on submodule-closed subcategories of infinite type over artin algebras. We know, by [21, Remark 3.10(a)], that minimal submodule-closed subcategories of infinite type do not exist over a left pure semisimple ring, and maximal submodule-closed subcategories of finite type appear to be a relevant concept in our setting.

THEOREM 3.1. *Let R be a left pure semisimple ring and $\mathcal{C}$ a submodule-closed subcategory of infinite type in R-*mod*. Then $\mathcal{C}$ contains a submodule-closed subcategory of finite type $\mathcal{A}$, satisfying the following conditions:*

- (a) *$\mathcal{A}$ contains an indecomposable module M such that there is an infinite number of non-isomorphic indecomposable modules N with Gabriel–Roiter inclusions $f : M \to N$ (in particular, M is not the source of a left almost split morphism in R-*mod*).*
- (b) *$\mathcal{A}$ is maximal among all submodule-closed subcategories of finite type in $\mathcal{C}$, i.e. there are no submodule-closed subcategories of finite type in $\mathcal{C}$ that properly contain $\mathcal{A}$.*
- (c) *$\mathcal{A}$ coincides with the intersection of all submodule-closed subcategories of infinite type in R-*mod* that contain $\mathcal{A}$.*

Proof. Since $\mathcal{C}$ is submodule-closed and R is left artinian, $\mathcal{C}$ contains simple modules. By Proposition 2.7, the family of all Gabriel–Roiter measures of R-mod satisfies the noetherian condition. An easy induction implies that there is a finite ascending chain of Gabriel–Roiter measures $I_0 < I_1 < \cdots < I_n$ of modules in $\mathrm{ind}(\mathcal{C})$ such that $I_0 = \mu(X)$ for a simple module $X \in \mathcal{C}$, I_{k+1} is a direct successor of I_k for $k = 0, 1, \ldots, n-1$, and I_n has no direct successor in the set of Gabriel–Roiter measures of $\mathrm{ind}(\mathcal{C})$. Let $\mathcal{B}$ be the subcategory of R-mod consisting of all finite direct sums of modules M in $\mathrm{ind}(\mathcal{C})$ with $\mu(M) \in \{I_0, I_1, \ldots, I_n\}$. If $B \in \mathcal{B}$, and A is an indecomposable submodule of B, then $A \in \mathcal{C}$ because $\mathcal{C}$ is submodule-closed. By Proposition 2.5, $\mu(A)$ does not exceed the maximum of Gabriel–Roiter measures of indecomposable summands of B, thus $\mu(A) \le I_n$, implying that $A \in \mathcal{B}$. It follows that each submodule of B is a finite direct sum of modules in $\mathrm{ind}(\mathcal{B})$, hence $\mathcal{B}$ is a submodule-closed subcategory of R-mod. Since, by Proposition 2.6(a), indecomposable modules with the same Gabriel–Roiter measure have the same composition length, we deduce by Theorem 2.3 that $\mathcal{B}$ has only finitely many non-isomorphic indecomposable modules, i.e. $\mathcal{B}$ is of finite type.

Next, we show that $\mathcal{B}$ contains an indecomposable module M such that there is an infinite number of non-isomorphic indecomposable modules N with Gabriel–Roiter inclusions $f : M \to N$. We will use the idea of the proof of [37, Successor Lemma]. Assume, on the contrary, that for each indecomposable module M in $\mathcal{B}$ there is only a finite number of non-isomorphic indecomposable modules N with Gabriel–Roiter inclusions $f : M \to N$. Take any Gabriel–Roiter measure $I = \mu(K)$ with $K \in \mathrm{ind}(\mathcal{C})$ such that $I_n < I$. By Proposition 2.6(c), there are indecomposable submodules $Y \subset X \subseteq K$ in a Gabriel–Roiter filtration of K such that $\mu(Y) \le I_n < \mu(X)$ and $Y \subset X$ is a Gabriel–Roiter inclusion. Since $\mathcal{C}$ is submodule-closed, X and Y belong to $\mathcal{C}$, and clearly $Y \in \mathcal{B}$. Consider the (non-empty) finite set $\mathcal{F}$ of all non-isomorphic indecomposable modules $X \in \mathcal{C}$ admitting Gabriel–Roiter inclusions $f : Y \to X$ for some $Y \in \mathrm{ind}(\mathcal{B})$, such that $I_n < \mu(X)$. Choose $L \in \mathcal{F}$ with a smallest Gabriel–Roiter measure $\mu(L)$. If $\mu(L)$ is not a direct successor of I_n in the set of Gabriel–Roiter measures of $\mathrm{ind}(\mathcal{C})$, then there is a Gabriel–Roiter measure $\mu(Z)$ with $Z \in \mathrm{ind}(\mathcal{C})$ and $I_n < \mu(Z) < L$, and an application of the above argument will give an indecomposable module in $\mathcal{F}$ with Gabriel–Roiter measure less than $\mu(L)$. This would contradict the choice of $\mu(L)$, completing the proof of our assertion.

Now, it follows by Lemma 2.8 that if $M \in \mathcal{B}$ is an indecomposable module such that there is an infinite number of non-isomorphic indecomposable modules N with Gabriel–Roiter inclusions $f : M \to N$, then M is not the source of a left almost split morphism in R-mod.

Let $\mathcal{B}_0 = \mathcal{B}$ be the submodule-closed subcategory of finite type constructed above. If $\mathcal{B}_0$ is not maximal in $\mathcal{C}$, there is a proper inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$, where $\mathcal{B}_1$ is a submodule-closed subcategory of finite type in $\mathcal{C}$. Again, if $\mathcal{B}_1$ is not maximal in $\mathcal{C}$, there is a proper inclusion $\mathcal{B}_1 \subset \mathcal{B}_2$, where $\mathcal{B}_2$ is a submodule-closed subcategory of finite type in $\mathcal{C}$. Proceeding this way, if the process does not stop after finitely many steps, we would obtain an infinite sequence of proper inclusions of submodule-closed subcategories of finite type in C,

$$\mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_k \subset \cdots .$$

Set $\mathcal{B}' = \bigcup_{k=1}^{\infty} \mathcal{B}_k$. By Theorem 2.1, $\mathcal{B}'$ has a strong preinjective partition, in particular $\mathcal{B}'$ has a finite cogenerating set $\{M_1, \dots, M_k\}$ of indecomposable modules. Each M_i is contained in some $\mathcal{B}_j$, so there is a positive integer m such that $\{M_1, \ldots, M_k\} \subseteq \mathcal{B}_m$. Then each module in $\mathcal{B}_{m+1}$ is embedded as a submodule of a finite direct sum of modules in the set $\{M_1, \ldots, M_k\}$, and since $\mathcal{B}_m$ is submodule-closed, it follows that $\mathcal{B}_{m+1} \subseteq \mathcal{B}_m$, which is a contradiction. Therefore, the process will have to stop after a finite number of steps, i.e. $\mathcal{B}_0 = \mathcal{B}$ is contained in a submodule-closed subcategory $\mathcal{A}$ of finite type in $\mathcal{C}$, and $\mathcal{A}$ is maximal with these properties, so (b) holds. It is clear that $\mathcal{A}$ satisfies condition (a) above.

Next, we show that $\mathcal{A}$ satisfies (c). Let $\mathcal{D}$ be the intersection of all submodule-closed subcategories of infinite type in R-mod that contain $\mathcal{A}$. Since $\mathcal{C}$ is one of these subcategories, this set is not empty. Clearly $\mathcal{D}$ is a submodule-closed subcategory in $\mathcal{C}$, and $\mathcal{D}$ contains $\mathcal{A}$. If $\mathcal{D}$ is of finite type, then $\mathcal{D} = \mathcal{A}$ because of the maximality of $\mathcal{A}$, proving (c). Therefore, we assume that $\mathcal{D}$ is of infinite type. By Theorem 2.1, $\mathcal{D}$ has a strong preinjective partition, hence a minimal finite cogenerating set $\{X_1, \ldots, X_k\}$ of indecomposable modules, which are also splitting injective in $\mathcal{D}$ by Lemma 2.2. If all modules $X_1, \ldots, X_k$ belong to $\mathcal{A}$, then because $\mathcal{A}$ is submodule-closed, each module finitely cogenerated by the family $\{X_1, \ldots, X_k\}$ also belongs to $\mathcal{A}$. But this would imply that $\mathcal{D}$ is contained in $\mathcal{A}$, hence $\mathcal{D} = \mathcal{A}$, contradicting our assumption. Thus, there is some module in $\{X_1, \ldots, X_k\}$, say X_1, that does not belong to $\mathcal{A}$. Let $\mathcal{A}'$ be the subcategory of R-mod consisting of all finite direct sums of modules in $\{\mathrm{ind}(\mathcal{D}) \setminus X_1\}$. We show that $\mathcal{A}'$ is a submodule-closed subcategory of R-mod. Indeed, for any module $A \in \mathcal{A}'$ and any indecomposable submodule $X \subseteq A$, we have $X \in \mathrm{ind}(\mathcal{D})$ because $\mathcal{D}$ is submodule-closed. If $X \cong X_1$, then since X_1 is splitting injective in $\mathcal{D}$ by Lemma 2.2, X_1 is isomorphic to an indecomposable direct summand of A. This is obviously a contradiction, showing that $X \in \mathcal{A}'$, hence any submodule of A also belongs to $\mathcal{A}'$. We thus have the following proper inclusions of submodule-closed subcategories: $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{D}$, where $\mathcal{A}$ is of finite type, and $\mathcal{A}', \mathcal{D}$ are of infinite type, contrary to the construction of $\mathcal{D}$. The proof of Theorem 3.1 is complete. ■

We obtain the following immediate consequences.

COROLLARY 3.2. *Let R be a left pure semisimple ring. Then R is of finite representation type if and only if for every indecomposable module M in* R-mod *there are only finitely many non-isomorphic indecomposable modules N in* R-mod *with Gabriel–Roiter inclusions $f : M \to N$.*

Proof. The "only if" part is obvious. For the "if" part, apply Theorem 3.1 for $\mathcal{C} = R$-mod. ■

COROLLARY 3.3. *Let R be a left pure semisimple ring. There exist maximal submodule-closed subcategories of finite type in* R-mod. *Moreover, each* *submodule-closed subcategory of finite type in* R-mod *is contained in a maximal one.*

Proof. The first statement follows directly from Theorem 3.1 when $\mathcal{C} = R$-mod. The proof of Theorem 3.1 actually yields a more general fact, that if $\mathcal{C}$ is any submodule-closed subcategory of infinite type in R-mod, then any submodule-closed subcategory $\mathcal{B}$ of finite type in $\mathcal{C}$ is contained in a maximal submodule-closed subcategory $\mathcal{A}$ of finite type in $\mathcal{C}$, and the second statement follows by setting $\mathcal{C} = R$-mod when R is of infinite representation type. The result is obvious if R is of finite representation type. ∎

The proof of Theorem 3.1 uses the Gabriel–Roiter measure. Our next result, providing a different construction of submodule-closed subcategories of finite type with similar properties, will be based on the concept of a strong preinjective partition [29]. We will use an adapted version of the proof of [29, Proposition 7] (which in turn was inspired by ideas due to Auslander and Smalø [10]).

PROPOSITION 3.4. *Let* R *be a left pure semisimple ring and* $\mathcal{C}$ *a submodule-closed subcategory of infinite type in* R-mod*. Then* $\mathcal{C}$ *has a strong preinjective partition* $\mathrm{ind}(\mathcal{C}) = \bigcup_{\alpha<\rho} \mathcal{I}_\alpha$*, where* ρ *is not a limit ordinal. Let* $\rho = \beta + n$*, where* β *is a limit ordinal and* $n \geq 1$ *is finite. Set* $\mathcal{F} = \mathcal{I}_\beta \cup \cdots \cup \mathcal{I}_{\beta+n-1}$ *and* $\mathcal{B} = \mathrm{add}(\mathcal{F})$*. Then* $\mathcal{B}$ *is a submodule-closed subcategory of finite type in* $\mathcal{C}$*, and* $\mathcal{B}$ *contains an indecomposable module which is not the source of a left almost split morphism in* R-mod*.*

Proof. We know by Theorem 2.1 that $\mathcal{C}$ has a strong preinjective partition $\mathrm{ind}(\mathcal{C}) = \bigcup_{\alpha<\rho} \mathcal{I}_\alpha$ for some ordinal ρ. Assume that ρ is a limit ordinal. Let $X_1, \ldots, X_k$ be a complete set of all simple modules in $\mathcal{C}$. There is an ordinal $\theta < \rho$ such that $\{X_1, \ldots, X_k\} \subseteq \bigcup_{\alpha\leq\theta} \mathcal{I}_\alpha$. Since $\mathcal{C}$ is submodule-closed, any module in $\mathcal{I}_{\theta+1}$ must contain a simple submodule from $\{X_1, \ldots, X_k\}$, so we get a contradiction since all modules in $\bigcup_{\alpha\leq\theta} \mathcal{I}_\alpha$ are splitting injective in $\mathrm{add}(\bigcup_{\alpha>\theta} \mathcal{I}_\alpha)$ by Lemma 2.2. Thus ρ is not a limit ordinal, and hence $\rho = \beta + n$, where β is a limit ordinal, and $n \geq 1$ is finite.

Let $\mathcal{F} = \mathcal{I}_\beta \cup \cdots \cup \mathcal{I}_{\beta+n-1}$ and let $\mathcal{B} = \mathrm{add}(\mathcal{F})$. Since all modules in $\bigcup_{\alpha<\beta} \mathcal{I}_\alpha$ are splitting injective in $\mathcal{B}$ by Lemma 2.2, there are no monomorphisms from modules in $\mathrm{ind}(\mathcal{C}) \setminus \mathcal{F}$ to modules in $\mathcal{B}$, and because $\mathcal{C}$ is submodule-closed in R-mod, it follows that $\mathcal{B}$ is submodule-closed. Moreover, $\mathcal{B}$ is of finite type because $\mathrm{ind}(\mathcal{B}) = \mathcal{F}$ is a finite set of non-isomorphic indecomposable modules.

Next, we show that each module in $\mathcal{I}_\beta \subseteq \mathcal{B}$ is not the source of a left almost split morphism in R-mod. Take $X \in \mathcal{I}_\beta$, and assume that there is a left almost split morphism $f : X \to Y$ in R-mod. Let A_α be the *reject* of $\mathcal{I}_\alpha$ on Y for any $\alpha < \beta$ (i.e., $A_\alpha = \bigcap \mathrm{Ker}(h)$, $h \in \mathrm{Hom}(Y, Z)$, $Z \in \mathcal{I}_\alpha$).

Observe that if $\alpha_1 < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$, because $\mathcal{I}_{\alpha_1}$ cogenerates $\mathcal{I}_{\alpha_2}$. Then the ascending chain $\{A_\alpha \mid \alpha < \beta\}$ of submodules of Y must become stationary for some $\gamma < \beta$. Note that Y/A_γ is cogenerated by $\mathcal{I}_\gamma$, i.e. Y/A_γ is embedded as a submodule of a module in $\mathrm{add}(\mathcal{I}_\gamma)$. Since $\mathcal{C}$ is submodule-closed, we have $Y/A_\gamma \in \mathcal{C}$. Moreover, each indecomposable summand of Y/A_γ is cogenerated by $\mathcal{I}_\alpha$ for $\gamma < \alpha < \beta$. It follows that Y/A_γ belongs to $\mathrm{add}(\bigcup_{\alpha\geq\beta}\mathcal{I}_\alpha)$.

For any ordinal $\alpha < \beta$, X is cogenerated by $\mathcal{I}_\alpha$, hence there is a non-split monomorphism $f_\alpha : X \to M_\alpha$, where $M_\alpha \in \mathrm{add}(\mathcal{I}_\alpha)$. Since $f : X \to Y$ is a left almost split morphism, there is a homomorphism $g_\alpha : Y \to M_\alpha$ such that $g_\alpha \circ f = f_\alpha$ for each $\alpha < \beta$. The homomorphism $f' : X \to Y/A_\gamma$ induced by the composition $X \xrightarrow{f} Y \to Y/A_\gamma$ is indeed a monomorphism. This is because, for a non-zero $x \in X$, if $f(x) \in A_\gamma$ then for $\beta > \alpha > \gamma$ we have $A_\gamma \subseteq \mathrm{Ker}(g_\alpha)$, and so $f_\alpha(x) = g_\alpha f(x) \in g_\alpha(A_\gamma) = 0$. On the other hand, $f_\alpha(x) \neq 0$ because f_α is a monomorphism, a contradiction. Thus $f' : X \to Y/A_\gamma$ is a monomorphism, hence it splits since $X \in \mathcal{I}_\beta$, and $Y/A_\gamma \in \mathrm{add}(\bigcup_{\alpha\geq\beta}\mathcal{I}_\alpha)$. But then $f : X \to Y$ must split, which is a contradiction, completing the proof. ∎

An important special case of submodule-closed subcategories over a left pure semisimple ring R is given by a *splitting torsion pair* in R-ind, i.e. a partition $(\mathcal{A}, \mathcal{B})$ of R-ind such that $\mathrm{Hom}_R(X, Y) = 0$ whenever $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ (see [21]; cf. [6, p. 188]). The first part of the next proposition is inspired by a result due to Carlson and Happel [11], in the context of artin algebras. Recall that a homomorphism $f : M \to N$ between indecomposable modules in R-mod is called *irreducible* if f is not an isomorphism, and if $f = h \circ g$ for $g : M \to X$ and $h : X \to N$ with $X \in R$-mod, then g is a split monomorphism or h is a split epimorphism. If R is a hereditary left artinian ring, an indecomposable module M in R-mod is said to be *preprojective* if $\mathrm{Hom}_R(X, M) \neq 0$ for only finitely many non-isomorphic indecomposable modules X in R-mod.

Proposition 3.5. *Let R be a left pure semisimple ring and $\mathcal{C}$ a submodule-closed subcategory in* R-mod. *Then the following conditions are equivalent:*

(a) $(R\text{-ind} \setminus \mathrm{ind}(\mathcal{C}), \mathrm{ind}(\mathcal{C}))$ *is a splitting torsion pair in* R-ind.
(b) *For any irreducible epimorphism $f : X \to Y$ in* R-ind, *if Y belongs to $\mathcal{C}$, then so does X.*

Assume, in addition, that R is indecomposable hereditary, and $\mathcal{C}$ satisfies the equivalent conditions (a) *and* (b). *Then either $\mathcal{C}$ is of infinite type and contains all preprojective modules, or $\mathcal{C}$ is of finite type and each indecomposable module in $\mathcal{C}$ is preprojective.*

Proof. (a)⇒(b). This is clear, since (a) means that there are no non-zero homomorphisms from indecomposable modules not in $\mathcal{C}$ to modules in $\mathrm{ind}(\mathcal{C})$.

(b)⇒(a). Assume on the contrary that (a) does not hold, i.e. there are an indecomposable module $M \notin \mathcal{C}$ and a non-zero homomorphism from M to a module in $\mathcal{C}$. Note that, by Lemma 2.4, $\mathcal{C}$ is covariantly finite in R-mod. Since each module in $\mathcal{C}$ is pure-injective, by [33, Theorem 2.3], M has a non-zero minimal left $\mathcal{C}$-approximation $f : M \to X$ with $X \in \mathcal{C}$. Let $X = X_1 \oplus \cdots \oplus X_n$ be an indecomposable decomposition of X. For each X_k, it follows by Auslander's theorem (see [8, II.5.1]) that there is an almost split sequence $0 \to Z_k \to A_k \xrightarrow{g_k} X_k \to 0$ in R-mod. For each indecomposable summand L of A_k, the restriction of g_k on L gives an irreducible homomorphism $h_k : L \to X_k$, which is either a monomorphism or an epimorphism (see, e.g., [6, Theorem 1.13]). If h_k is a monomorphism, since $\mathcal{C}$ is submodule-closed, $L \in \mathcal{C}$. If h_k is an epimorphism, then by hypothesis, L also belongs to $\mathcal{C}$. Thus $A_k \in \mathcal{C}$ for each $k = 1, \ldots, n$. Set $A = A_1 \oplus \cdots \oplus A_n$; then $A \in \mathcal{C}$ and the homomorphisms $g_k : A_k \to X_k$, $k = 1, \ldots, n$, induce a homomorphism $g = (g_1, \ldots, g_n) : A \to X$ such that for any indecomposable module $X' \in R$-mod, any non-split homomorphism $g' : X' \to X$ factors through g. In particular, there is a homomorphism $\beta : M \to A$ such that $g \circ \beta = f$. Since $f : M \to X$ is a left $\mathcal{C}$-approximation, there is a homomorphism $\alpha : X \to A$ such that $\beta = \alpha \circ f$. Set $\mu = g \circ \alpha$. Then $\mu \circ f = g \circ \alpha \circ f = g \circ \beta = f$. But f is a minimal left $\mathcal{C}$-approximation, hence μ is an automorphism of X, which implies that $g : A \to X$ is a splitting epimorphism, a contradiction because each $g_k : A_k \to X_k$ is not split. This shows that $(R\text{-ind} \setminus \mathrm{ind}(\mathcal{C}), \mathrm{ind}(\mathcal{C}))$ is a splitting torsion pair in R-ind.

Now, assume that R is indecomposable hereditary left pure semisimple, and $\mathcal{C}$ satisfies the equivalent conditions (a) and (b) above. By [20, Theorem 1.1], R-ind has a unique Ext-injective partition $R\text{-ind} = \bigcup_{\alpha \leq \delta} \mathcal{U}_\alpha$ with $\delta = \beta + m$, where β is an infinite limit ordinal and m is a non-negative integer. Moreover, by [20, Theorem 1.1(2)], the finite set $\mathcal{U}_\beta \cup \cdots \cup \mathcal{U}_{\beta+m}$ consists of all preprojective modules in R-mod. Let $\mathcal{C}'$ be the union of $\mathrm{ind}(\mathcal{C})$ and the set of all non-isomorphic indecomposable projective left R-modules, and set $\mathcal{D}' = R\text{-ind} \setminus \mathcal{C}'$. Then, since any homomorphism from a module in $\mathcal{D}'$ to an indecomposable projective module splits, it is clear that $(\mathcal{D}', \mathcal{C}')$ is a splitting torsion pair of R-ind and $\mathcal{C}'$ contains all indecomposable projective left R-modules. By [21, Theorem 3.3], there exist an ordinal $\alpha \leq \delta$ and an integer $n \geq 0$ such that all the sets $\mathcal{U}_\gamma$ with $\gamma < \alpha$ have an empty intersection with $\mathcal{C}'$, and all the sets $\mathcal{U}_\gamma$ with $\gamma \geq \alpha + n$ are contained in $\mathcal{C}'$. If $\alpha < \beta$, then since β is an infinite limit ordinal, we have $\alpha + n < \beta$, hence $\mathcal{C}$ contains an infinite number of the sets $\mathcal{U}_\gamma$ with $\alpha + n \leq \gamma < \beta$, so $\mathcal{C}$ is of infinite type. Moreover, $\mathcal{C}$ then contains all modules in $\mathcal{U}_\gamma$ with $\gamma \geq \beta$, hence all preprojective modules in R-mod. If $\alpha \geq \beta$, then clearly $\mathcal{C}$ is of finite type and every indecomposable module in $\mathcal{C}$ is preprojective. ∎

It was shown by Ringel [38, Examples 1 and 2] that if R is an indecomposable hereditary artin algebra of infinite representation type, then the preprojective component of R-mod is a minimal submodule-closed subcategory of infinite type in R-mod, and if R is also of tame representation type, then this is the only minimal one. Our next result will show a relationship between a maximal submodule-closed subcategory of finite type of R-mod and the preprojective left R-modules, when R is a left pure semisimple hereditary ring. Note that, by results of Herzog [28], the pure semisimplicity conjecture is true if and only if it is true for all hereditary rings. It was proven by Simson [42] (cf. [43]) that the pure semisimplicity conjecture is true for all hereditary rings if and only if it is true for all triangular matrix rings $R = \left(\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right)$ where F, G are division rings and B is a G-F-bimodule. Up to isomorphism, these rings are precisely the left pure semisimple hereditary indecomposable basic rings with two simple modules.

Proposition 3.6. *Let R be a left pure semisimple hereditary indecomposable ring of infinite representation type, and $\mathcal{C}$ a maximal submodule-closed subcategory of finite type in R-mod. Then $\operatorname{ind}(\mathcal{C})$ properly contains all preprojective left R-modules. If, furthermore, R has only two simple modules, then there is a unique maximal submodule-closed subcategory of finite type in R-mod whose indecomposable modules consist of all preprojective modules and the unique simple injective left R-module.*

Proof. First, we show that if M is an indecomposable module in R-mod, and $M \notin \mathcal{C}$, then there exist an infinite number of indecomposable modules X in R-mod such that $\operatorname{Hom}(X, M) \neq 0$ (the hereditary condition on R is not required here). Suppose, on the contrary, that there are only a finite number of such modules. Let $\mathcal{D}$ be the submodule-closed subcategory of R-mod cogenerated by the modules in the finite set $\{M\} \cup \operatorname{ind}(\mathcal{C})$. We show that $\mathcal{D}$ is of finite type. Let $N \in \mathcal{D}$, and consider an indecomposable submodule $Y \subseteq N$. We can assume that $N = N_1 \oplus N_2$, where $N_1 \in \operatorname{add}(M)$ and $N_2 \in \mathcal{C}$. If $\operatorname{Hom}(Y, N_1) \neq 0$, then $\operatorname{Hom}(Y, M) \neq 0$, and Y must belong to a finite set of indecomposable modules by the assumption above. If $\operatorname{Hom}(Y, N_1) = 0$, then Y is isomorphic to a submodule of N_2, and then $Y \in \mathcal{C}$ since $\mathcal{C}$ is submodule-closed, and so Y also belongs to a finite set $\operatorname{ind}(\mathcal{C})$ because $\mathcal{C}$ is of finite type. This shows that $\mathcal{D}$ is of finite type. Thus, $\mathcal{D}$ is a submodule-closed subcategory of finite type that properly contains $\mathcal{C}$, which contradicts the maximality of $\mathcal{C}$, proving our claim.

Now, assume in addition that R is hereditary indecomposable. If an indecomposable module M in R-mod is preprojective, then there are only finitely many indecomposable modules X in R-mod such that $\mathrm{Hom}(X, M) \neq 0$. Thus, by the above, since $\mathcal{C}$ is a maximal submodule-closed subcategory of finite type, $\mathcal{C}$ must contain all preprojective left R-modules. Next, we show that if $\mathcal{F}$ is any submodule-closed subcategory of finite type in R-mod, and $\mathcal{P}$ is the set of all finite direct sums of preprojective left R-modules, then $\mathcal{F} \cup \mathcal{P}$ cogenerates a submodule-closed subcategory of finite type. Let $M = M_1 \oplus M_2$, where $M_1 \in \mathcal{F}$ and $M_2 \in \mathcal{P}$, and X be any indecomposable submodule of M. If $\mathrm{Hom}(X, M_2) = 0$, then X is isomorphic to a submodule of M_1, hence X belongs to the finite set $\mathrm{ind}(\mathcal{F})$. If $\mathrm{Hom}(X, M_2) \neq 0$, then because M_2 is a direct sum of preprojective modules, and there is only a finite number of non-isomorphic indecomposable modules that have non-zero homomorphisms to preprojective modules, X must belong to a finite set of indecomposable modules. We conclude that $\mathcal{F} \cup \mathcal{P}$ cogenerates a submodule-closed subcategory of finite type. Take, in particular, $\mathcal{F}$ to be the subcategory of all finitely generated semisimple left R-modules. Then clearly $\mathcal{F}$ is a submodule-closed subcategory of finite type. Moreover, since R is hereditary, there is a simple injective left R-module E, and E is not preprojective (see [20, Theorem 1.1]). Therefore the submodule-closed subcategory cogenerated by $\mathcal{F} \cup \mathcal{P}$ must properly contain $\mathcal{P}$.

Assume additionally that R has only two simple modules. Then by [21, Theorem 3.8] there is a strong preinjective partition $R\text{-ind} = \bigcup_{\alpha \leq \sigma} \mathcal{A}_\alpha$, where $\mathcal{A}_0$ consists of the two indecomposable injective modules (E_0, E_1), with E_0 simple, and for all $\alpha \geq 1$, $\mathcal{A}_\alpha$ consists of a single module $\mathcal{A}_\alpha = \{X_\alpha\}$. Let $\mathcal{C}$ be a maximal submodule-closed subcategory of finite type in R-mod. By the above, $\mathcal{C}$ must contain all preprojective modules. If $\mathcal{C}$ contains some non-preprojective module X_α with $\alpha \geq 1$, then since X_α cogenerates all modules X_γ with $\gamma \geq \alpha$, this would imply that $\mathcal{C}$ is not of finite type, a contradiction. Assume now that $\mathcal{C}$ contains the indecomposable injective module E_1. By [21, Theorem 3.8], there is an almost split sequence $0 \to X_1 \to E_1^m \to E_0 \to 0$ for some positive integer m. Hence, X_1 is cogenerated by E_1, and since X_1 cogenerates all modules X_α with $\alpha \geq 1$, it follows that E_1 cogenerates all modules X_α with $\alpha \geq 1$, which is a contradiction because $\mathcal{C}$ is of finite type. Therefore $\mathcal{C}$ can contain the simple injective module E_0 as the only non-preprojective module, and by the above, the submodule-closed subcategory whose indecomposable modules consist of all preprojective modules and the simple injective module E_0 is of finite type. ■

Recall that, for a left R-module N with $S = \mathrm{End}({}_RN)$, the *local dual* of N is defined as the right R-module $D(N) = \mathrm{Hom}_S(N_S, C_S)$, where C_S is a minimal injective cogenerator of Mod-S. As usual, $\mathrm{Tr}(M)$ will denote the transpose of a finitely presented module M (see, e.g., [1, 6]). In the next result, we will provide sufficient conditions for a submodule-closed subcategory over a general left artinian ring to be of finite type. Following [13], a left R-module M is called *endofinite* if M is of finite length as a module over its endomorphism ring. A module M is said to be *product-complete* if $\mathrm{Add}(M)$ is closed under products [33], and M is Σ-pure-injective if any direct sum of copies of M is pure-injective.

PROPOSITION 3.7. *Let R be a left artinian ring and $\mathcal{C}$ a submodule-closed subcategory in R-mod. Assume that each indecomposable module X in $\mathcal{C}$ is Σ-pure-injective, and $D(X)$ is a finitely presented right R-module, where $D(X)$ is the local dual of X. Then $\mathcal{C}$ is of finite type if and only if the direct sum of all non-isomorphic indecomposable modules in $\mathcal{C}$ is pure-injective. In this case, each module in $\mathcal{C}$ is endofinite.*

Proof. Suppose first that $\mathcal{C}$ is of finite type. Then clearly the direct sum of all non-isomorphic indecomposable modules in $\mathcal{C}$ is pure-injective. Conversely, let $\mathrm{ind}(\mathcal{C}) = \{M_i \mid i \in I\}$, and suppose $M = \bigoplus_{i\in I} M_i$ is pure-injective. We first show that M must be Σ-pure-injective (this is true for an arbitrary ring R). Note that this fact also follows by Prest [35, Example 4.4.18]. Since no explicit proof of this result was given in [35], for the reader's convenience we provide a direct module-theoretic argument.

Consider the right functor ring T of R, where T is a ring with enough idempotents, and a fully faithful functor $F : R\text{-Mod} \to T\text{-Mod}$ such that X is pure-injective in R-Mod if and only if $F(X)$ is injective in T-Mod (see, e.g., [17] for relevant background). Set $L_i = F(M_i)$ for all $i \in I$. Then each L_i is Σ-injective, and $L = \bigoplus_{i\in I} L_i$ is injective, as left T-modules. We aim to prove that L is Σ-injective, and this is equivalent to showing that $L^{(\mathbb{N})}$ is injective. Suppose that $L^{(\mathbb{N})}$ is not injective. Let $L^{(\mathbb{N})} = (\bigoplus_{i\in I} L_i)^{(\mathbb{N})} = \bigoplus_{i\in I}(\bigoplus_{k=1}^{\infty} L_{i,k})$, where $L_{i,k} \cong L_{i,l} \cong L_i$ for each $i \in I$ and all $k \neq l$. There are a finitely generated left T-module A, a submodule $B \subseteq A$, and a morphism $f : B \to L^{(\mathbb{N})}$ such that f cannot be extended to a morphism $h : A \to L^{(\mathbb{N})}$. Suppose that $f(B)$ is contained in a finite subsum $\bigoplus_{i\in J} N_i$, where $N_i = \bigoplus_{k=1}^{\infty} L_{i,k}$. Then because $\bigoplus_{i\in J} N_i$ is injective, f can be extended to a morphism $h : A \to \bigoplus_{i\in J} N_i$, which is a contradiction. Hence there are a countable set of distinct indices $\{i_1, i_2, \dots\} \subseteq I$ and a sequence of positive integers $\{m_1, m_2, \dots\}$ such that $p_{i_1,m_1}(f(B)) \neq 0$, $\dots, p_{i_n,m_n}(f(B)) \neq 0, \dots$, where $p_{i,j} : L^{(\mathbb{N})} \to L_{i,j}$ is the canonical projection. Set $N = \bigoplus_{t=1}^{\infty} L_{i_t,m_t}$. Then N is a direct summand of $L = \bigoplus_{i\in I} L_i$, hence N is injective. Let $p : L^{(\mathbb{N})} \to N$ be the canonical projection. Then the composition $p \circ f : B \to N$ can be extended to a morphism $h : A \to N$. Because A is finitely generated, $h(A)$ is contained in a finite subsum of $\bigoplus_{t=1}^{\infty} L_{i_t,m_t}$, hence $(p\circ f)(B)$ is contained in a finite subsum of $\bigoplus_{t=1}^{\infty} L_{i_t,m_t}$, which contradicts the choice of the modules L_{i_t,m_t}. This shows that L is Σ-injective, thus $M = \bigoplus_{i\in I} M_i$ is Σ-pure-injective, proving our claim.

To continue the proof of the proposition, if $S = \mathrm{End}_R(M)$, then because M is Σ-pure-injective, ${}_SM$ is *coperfect*, i.e. it satisfies the DCC on finitely generated modules (see, e.g., [31]). Since $M = \bigoplus_{i\in I} M_i$ is a direct sum of finitely presented modules with local endomorphism rings, and $S = \mathrm{End}_R(M)$ is coperfect, it follows by Angeleri Hügel's result [2, Theorem 4.4] that $\mathrm{Add}(M)$ is a *covering class*, i.e. every left R-module has an $\mathrm{Add}(M)$-cover. Because $\mathcal{C}$ is a submodule-closed subcategory in R-mod, $\mathcal{C}$ is covariantly finite in R-mod (see [10, Proposition 4.8]), and $\mathrm{Add}(\mathcal{C})$ is a covering class, it follows by [2, Theorem 5.1] that $M = \bigoplus_{i\in I} M_i$ is a product-complete module. Now applying [18, Proposition 3.10], since $M = \bigoplus_{i\in I} M_i$ is a direct sum of finitely presented modules, each local dual $D(M_i)$ is finitely presented as a right R-module, and M is product-complete, we deduce that M contains only finitely many non-isomorphic indecomposable direct summands, i.e. $\mathcal{C}$ is a subcategory of finite type. Because each M_i is finitely presented Σ-pure-injective and its local dual $D(M_i)$ is finitely presented in Mod-R, we know by [17, Corollary 4.5] that M_i is endofinite, implying that each module in $\mathcal{C}$ is endofinite. ■

Our next corollary sharpens a result due to Huisgen-Zimmermann [29, Corollary E], where the Σ-pure-injectivity condition was required instead of our pure-injectivity condition, and the proof was different.

Corollary 3.8 (cf. [29, Corollary E]). *Let R be an artin algebra. A submodule-closed subcategory $\mathcal{C}$ of R-*mod *is of finite type if and only if the direct sum of all indecomposable modules in $\mathcal{C}$ is pure-injective.*

Proof. This follows from Proposition 3.7 if we keep in mind that, over an artin algebra, each finitely generated indecomposable module is endofinite (hence Σ-pure-injective), and the local duality coincides with the Morita self-duality between finitely generated left and right modules. ■

We conclude the paper with some observations.

Remark 3.9. (a) Let R be a left pure semisimple ring. It would be interesting to characterize submodule-closed subcategories of finite type in R-mod. Since each submodule-closed subcategory of finite type is contained in a maximal one (Corollary 3.3), the question is reduced to characterizing maximal submodule-closed subcategories of finite type in R-mod. Indeed, the pure semisimplicity conjecture is equivalent to the statement that, for any left pure semisimple ring R, maximal submodule-closed subcategories of finite type in R-mod coincide with R-mod.

(b) Let R be a left pure semisimple hereditary indecomposable ring of infinite representation type. In [3] Angeleri Hügel introduced and studied a finitely generated product-complete left R-module W, called a *key module* over R, induced by the family of all preinjective left R-modules. It was shown in [3] (see also [4, 19]) that W stores important information about R, in particular, no indecomposable summands of W are sources of left almost split morphisms in R-mod. Let $\mathcal{A}$ be a maximal submodule-closed subcategory of finite type in R-mod. Then $\mathcal{A}$ also shares some useful properties similar to those of the key module W. Namely, the finite direct sum of all indecomposable modules in $\mathcal{A}$ is product-complete, and (by Proposition 3.6) $\mathcal{A}$ contains in particular all preprojective left R-modules that are not sources of left almost split morphisms in R-mod.

This last class of modules in $\mathcal{A}$ can be used to clarify and strengthen a result due to Herzog [28] in the hereditary case. It was shown in [28] (see also [14]) that if R is any left pure semisimple ring, then R has a finite number m of preprojective left R-modules, a finite number n of (finitely presented indecomposable) preinjective right R-modules, moreover $m \geq n$, and $m = n$ if and only if R is of finite representation type. Assume now that R is also hereditary indecomposable. If Y is a preinjective right R-module, then $D(Y)$ is a preprojective left R-module which is the source of a left almost split morphism in R-mod (see [28, Corollary 4.5]). If X is a preprojective left R-module and X is the source of a left almost split morphism in R-mod, then $D(X)$ is a finitely presented right R-module (see [16, Proposition 2.5]), and $\mathrm{Hom}(D(X), Y) \neq 0$ for only finitely many non-isomorphic finitely presented indecomposable right R-modules (see, e.g., [5, Lemma 3.5]). Thus clearly $D(X)$ is a preinjective right R-module (in the sense of [28]), and moreover $D(D(X)) \cong X$ (see [16, Proposition 2.5]). Therefore, there is a one-to-one correspondence (through the local duality) between the isomorphism classes of preinjective right R-modules and preprojective left R-modules that are sources of left almost split morphisms in R-mod. On the other hand, if R is of infinite representation type, the direct sum of all non-isomorphic preprojective modules that are not sources of left almost split morphisms in R-mod is a tilting module (see [20, Theorem 1.1]), hence the number of these modules is exactly the number k of simple R-modules (see, e.g., [12, Corollary 3.7.5]). It follows that $m - n = k$, which is a stronger form of Herzog's result [28] in the hereditary case.

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