COLLOQUIUM MATHEMATICUM

VOL. 147 2017 NO. 1

WHEN THE FLAT AND GORENSTEIN FLAT DIMENSIONS COINCIDE?

BY

SAMIR BOUCHIBA (Meknes)

Abstract. It is well known that, given a ring R, if M is an R-module such that $\mathrm{pd}_R(M) < \infty$, then $\mathrm{Gid}_R(M) = \mathrm{id}_R(M)$ (Holm, 2004). This shows in particular that if R is a Noetherian ring such that $\mathrm{Gid}(R) < \infty$, then R is Gorenstein. Dually, if M is an R-module such that $\mathrm{id}_R(M) < \infty$, then $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ (Holm, 2004). Regarding the Gorenstein flat dimension, there have been no appropriate analogs of these two theorems. The unique result, in this vein, states, under the strong hypothesis of R being a left and right coherent ring with finite right finitistic projective dimension, that $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$ for any R-module M such that $\mathrm{id}_R(M) < \infty$ (Holm, 2004).

We give the appropriate analogs of the above two formulas for the Gorenstein flat dimension. Actually, in the general setting, we prove that if M is an R-module admitting a short flat resolution $0 \to K \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$ such that K is Gorenstein flat and $\mathrm{fd}_R(M^+) < \infty$, then K is flat and $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$, where A^+ stands for the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ of a module A. This implies, in particular, that if R is a left GF-closed ring, then $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$ for any R-module M such that $\mathrm{fd}_R(M^+) < \infty$. Dually, we prove that if R is left GF-closed, then $\mathrm{Gfd}_R(N^+) = \mathrm{fd}_R(N^+)$ for any R-module N such that $\mathrm{fd}_R(N) < \infty$.

1. Introduction. Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R-modules.

Recall that Gorenstein projective (resp., Gorenstein injective, Gorenstein flat) modules originate from the classical notions of projective (resp., injective, flat) modules, being images and kernels of the differentials of complete projective (resp., injective, flat) resolutions. More precisely, a module M is said to be *Gorenstein projective* if there exists an exact sequence of projective modules, called a *complete projective resolution*,

$$\mathbf{P} := \cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

such that $\mathbf{P}$ remains exact after applying the functor $\mathrm{Hom}_R(-, P)$ for each

2010 *Mathematics Subject Classification*: 13D02, 13D05, 13D07, 16E05, 16E10.
Key words and phrases: coherent ring, flat dimension, GF-closed ring, Gorenstein flat dimension, Gorenstein global dimension, Gorenstein weak global dimension.
Received 9 December 2015; revised 13 March 2016.
Published online 9 December 2016.

DOI: 10.4064/cm6833-3-2016

projective module P and $M := \mathrm{Im}(P_0 \to P_{-1})$. The *Gorenstein injective* modules are defined dually. Also, a module M is said to be *Gorenstein flat* if there exists an exact sequence of flat modules

$$\mathbf{F} := \cdots \to F_2 \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$$

such that $\mathbf{F}$ remains exact after applying the functor $I \otimes_R -$ for each injective right R-module I and $M := \mathrm{Im}(F_0 \to F_{-1})$. These new concepts allowed Enochs and Jenda [14] to introduce new (Gorenstein homological) dimensions $\mathrm{Gpd}_R(M)$ and $\mathrm{Gid}_R(M)$, extending the G-dimension defined by Auslander and Bridger [1, 2]. It turns out, in particular, that these Gorenstein homological dimensions are refinements of the classical dimensions of a module M, in the sense that $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$, $\mathrm{Gid}_R(M) \leq \mathrm{id}_R(M)$ and $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$, with equality each time the corresponding classical homological dimension is finite. The reader is referred to [3, 8, 10–16, 19, 21, 22] for basics on classical homological algebra and recent investigations on Gorenstein homology.

Further, it is known that if M is an R-module such that $\mathrm{pd}_R(M) < \infty$, then $\mathrm{Gid}_R(M) = \mathrm{id}_R(M)$ [20, Theorem 2.1]. Dually, Holm shows that if M is an R-module such that $\mathrm{id}_R(M) < \infty$, then $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ [20, Theorem 2.2]. This duality of the (Gorenstein) projective dimension and the (Gorenstein) injective dimension also appears in the expression of the Gorenstein global dimension of R in terms of the cohomological invariants introduced by Gedrich and Gruenberg [18], $\text{l-silp}(R) := \sup\{\mathrm{id}_R(P) : P$ is a projective left R-module$\}$ and $\text{l-spli}(R) := \sup\{\mathrm{pd}_R(I) : I$ is an injective left R-module$\}$. Effectively, we proved in [7, Theorem 3.3], that

$$\text{l-G-gldim}(R) = \sup\{\text{l-silp}(R), \text{l-spli}(R)\}.$$

It is worth noting that till now there have been no analogs of the above two theorems for the Gorenstein flat dimension. The single effort in this direction is a result of Holm [20, Theorem 2.6] showing that under the strong hypothesis of R being a left and right coherent ring such that right $\mathrm{FPD}(R) < \infty$, if M is an R-module with $\mathrm{id}_R(M) < \infty$, then $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$. In a recent result in this vein, in [7] we gave a formula for the Gorenstein weak global dimension in terms of the cohomological invariants $\text{l-sfli}(R) := \sup\{\mathrm{fd}_R(I) : I$ is an injective left R-module$\}$ and $\text{r-sfli}(R) := \sup\{\mathrm{fd}_R(I) : I$ is an injective right R-module$\}$. In fact, in [8, Theorem 6] we proved that if R is a GF-closed ring, then the Gorenstein weak global dimension is left-right symmetric and

$$\text{G-wgldim}(R) = \sup\{\text{l-sfli}(R), \text{r-sfli}(R)\}.$$

This indicates, as the apparent correlation between the above results [20, Theorems 2.1 and 2.2] and the Gorenstein global dimension formula [7, Theorem 3.3] may hint, that any candidate formulas for the Gorenstein flat dimension has to incorporate finiteness of the flat dimension of appropriate left modules as well as of appropriate right modules. In this paper, we mainly give appropriate versions of the above-cited two theorems for the Gorenstein flat dimension.

2. Main results. In this section, we give two formulas establishing the equality of the Gorenstein flat dimension and flat dimension as counterparts for those ensuring equality of the projective and Gorenstein projective dimensions, on the one hand, and the injective and Gorenstein injective dimensions, on the other.

For the convenience of the reader, we begin by giving a brief account of resolving classes of modules. Recall that a class of R-modules Γ is called *projectively resolving* if Γ includes all projective modules and for any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \Gamma$, have $X \in \Gamma$ if and only if $X' \in \Gamma$. Similarly, Γ is called *injectively resolving* if Γ includes all injective modules and for any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \Gamma$, we have $X \in \Gamma$ if and only if $X'' \in \Gamma$. In this context, recall that the category of Gorenstein projective (resp., injective) modules, denoted by $\mathcal{GP}(R)$ (resp., $\mathcal{GI}(R)$), is projectively (resp., injectively) resolving. As to the category of Gorenstein flat modules $\mathcal{GF}(R)$, it is still an open problem whether it is projectively resolving or not. A ring is said to be *left GF-closed* if $\mathcal{GF}(R)$ is projectively resolving. Note that the class of left GF-closed rings encompasses all properly right coherent rings and all rings with finite weak global dimension. Also, it is worth reminding the reader of the useful adjointness isomorphism for derived functors

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_n^R(A,B),\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}_R^n(A,\mathrm{Hom}_{\mathbb{Z}}(B,\mathbb{Q}/\mathbb{Z}))$$

for any left R-module B, any right R-module A and any integer $n \geq 0$. For any R-module M, we denote by $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ the *Pontryagin dual* (or *character module*) of M.

Our first main theorem provides, in the general setting, conditions guaranteeing equality of the Gorenstein flat dimension and the flat dimension. Obviously, given a module M, if $\mathrm{Gfd}_R(M) = \infty$, then $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M) = \infty$. This is the case, in particular, if the copure flat dimension of M is infinite, that is, $\mathrm{cfd}_R(M) := \sup\{n \geq 1 : \mathrm{Tor}_n^R(I,M) \neq 0$ for some injective right module $I\} = \infty$ implies $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M) = \infty$. Therefore our focus next is on modules of finite Gorenstein flat dimension, precisely on modules possessing a flat resolution with a certain Gorenstein flat yoke.

Also, it is worth noticing, as is pointed out in [12, Lemma 3.3], that if G is a Gorenstein flat R-module and H is a right R-module with $n := \mathrm{fd}_R(H) < \infty$, then $\mathrm{Tor}_1^R(H,G) = 0$ since G is the nth yoke of a flat resolution of some module K.

THEOREM 2.1. *Let R be a ring. Let M be an R-module admitting an exact sequence $0 \to K \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$ such that $F_0, F_1, \dots, F_{n-1}$ are flat R-modules and K is a Gorenstein flat R-module. If $\mathrm{fd}_R(M^+) < \infty$, then K is flat and $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$.*

Proof. As K is Gorenstein flat, there exists an exact sequence

$$0 \to K \to E_0 \to E_1 \to \cdots \to E_{n-1} \to G' \to 0$$

where the E_i are flat modules, G' is Gorenstein flat and each $\mathrm{Im}(E_i \to E_{i+1})$ is Gorenstein flat, for $i = 0, \dots, n-2$. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the above sequence yields an exact sequence of right modules

$$0 \to G'^+ \to E_{n-1}^+ \to \cdots \to E_0^+ \to K^+ \to 0$$

such that the E_j^+ are injective right R-modules, G'^+ and K^+ are Gorenstein injective right modules, and each $\mathrm{Ker}(E_{i+1}^+ \to E_i^+) = \mathrm{Im}(E_i \to E_{i+1})^+$ is a Gorenstein injective right module for $i = 0, 2, \dots, n-2$. Thus the functor $\mathrm{Hom}_R(Q, -)$ leaves exact the sequence $0 \to G'^+ \to E_{n-1}^+ \to \cdots \to E_0^+ \to K^+ \to 0$ for each injective right R-module Q. Therefore we get a commutative diagram

$$\begin{array}{ccccccccccccc} 0 \to & M^+ & \to & F_0^+ & \to & F_1^+ & \to \cdots \to & F_{n-1}^+ & \to & K^+ & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\ 0 \to & G'^+ & \to & E_{n-1}^+ & \to & E_{n-2}^+ & \to \cdots \to & E_0^+ & \to & K^+ & \to 0 \end{array}$$

This diagram gives a chain map of complexes

$$\begin{array}{ccccccccccc} 0 \to & M^+ & \to & F_0^+ & \to & F_1^+ & \to \cdots \to & F_{n-1}^+ & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & G'^+ & \to & E_{n-1}^+ & \to & E_{n-2}^+ & \to \cdots \to & E_0^+ & \to 0 \end{array}$$

which induces isomorphisms in homology. Its mapping cone

$$0 \to M^+ \to G^+ \to E_{n-1}^+ \oplus F_1^+ \to \cdots \to E_0^+ \to 0$$

is exact with $G := G' \oplus F_0$ Gorenstein flat, so that $G^+ = G'^+ \oplus F_0^+$ is (right) Gorenstein injective and all the remaining modules are (right) injective. Let $H := \mathrm{Im}(G^+ \to E_{n-1}^+ \oplus F_1^+)$. It follows that $0 \to M^+ \to G^+ \to H \to 0$ is an exact sequence and $\mathrm{id}_R(H) \leq n-1$.

Now, let $0 \to G \to F \to D \to 0$ be an exact sequence of R-modules such that F is flat and D is Gorenstein flat over R, and consider the pullback diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & D^+ & \to & E & \to & M^+ & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & D^+ & \to & F^+ & \to & G^+ & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & H & = & H & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Hence $0 \to D^+ \to E \to M^+ \to 0$ is an exact sequence of right R-modules with $\mathrm{id}_R(E) \le n$ and D is Gorenstein flat (thus D^+ is Gorenstein injective). Therefore, via the above adjointness isomorphism for derived functors we have

$$\mathrm{Ext}_R^1(M^+, D^+) \cong \mathrm{Tor}_1^R(M^+, D)^+ = 0 \qquad [12, \text{Lemma } 3.3]$$

as $\mathrm{fd}_R(M^+) < \infty$ and D is Gorenstein flat. It follows that the exact sequence $0 \to D^+ \to E \to M^+ \to 0$ splits, and thus

$$\mathrm{fd}_R(M) = \mathrm{id}_R(M^+) \le \mathrm{id}_R(E) \le n.$$

Consequently, $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$ and K is flat, completing the proof of the theorem. ∎

Recall that Stenström [24] introduced FP-injective modules as an extension of the notion of injective module in the following way: A module M is said to be *FP-injective* if $\mathrm{Ext}_R^1(A, M) = 0$ for any finitely presented module A. Obviously, any injective module is FP-injective. The *FP-injective dimension* of a module M is the smallest positive integer n such that $\mathrm{Ext}_R^{n+1}(A, M) = 0$ for any finitely presented left R-module A. If no such n exists, we set $\mathrm{FP\text{-}id}_R(M) = \infty$. Note that $\mathrm{FP\text{-}id}_R(M) \le \mathrm{id}_R(M)$ for any module M. The following corollary specifies the conditions given in Theorem 2.1 in the setting of a left coherent ring.

Corollary 2.2. *Let R be a left coherent ring. Let M be an R-module admitting an exact sequence $0 \to K \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$ such that $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules and K is a Gorenstein flat R-module. If $\mathrm{FP\text{-}id}_R(M) < \infty$ (in particular, if $\mathrm{id}_R(M) < \infty$), then K is flat and $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$.*

Proof. This is straightforward from Theorem 2.1 since when R is left coherent, $\mathrm{fd}_R(M^+) = \mathrm{FP\text{-}id}_R(M)$ for any R-module M [17, Theorem 2.2]. ∎

As a consequence of Theorem 2.1, we get the following result which presents a sufficient condition for a Gorenstein flat module to be flat.

Corollary 2.3. *Let R be a ring and M be a Gorenstein flat R-module.*

(1) *If* $\mathrm{fd}_R(M^+) < \infty$, *then M is flat.*
(2) *Moreover, if R is left coherent and* $\text{FP-id}_R(M) < \infty$ (*in particular, if* $\mathrm{id}_R(M) < \infty$), *then M is flat.*

Proof. (1) As M is Gorenstein flat there exists a short exact sequence $0 \to N \to F \to M \to 0$ such that F is a flat module and N is Gorenstein flat. As $\mathrm{fd}_R(M^+) < \infty$, by Theorem 2.1, $\mathrm{fd}_R(M) = \mathrm{Gfd}_R(M) = 0$, as desired.

(2) Follows directly from (1). ∎

Now, we give the analog for the Gorenstein flat dimension of [20, Theorem 2.2] which states that if $\mathrm{id}_R(M) < \infty$, then $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$. Note that the following corollary generalizes [20, Theorem 2.6].

Corollary 2.4. *Let R be a left GF-closed ring and M an R-module.*

(1) *If* $\mathrm{fd}_R(M^+) < \infty$, *then* $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$.
(2) *Moreover, if R is left coherent and* $\text{FP-id}_R(M) < \infty$, *then* $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$.

Proof. (1) Assume that $\mathrm{fd}_R(M^+) < \infty$. If $\mathrm{Gfd}_R(M) = \infty$, then, as $\mathrm{Gfd}_R(M) \le \mathrm{fd}_R(M)$, we get $\mathrm{fd}_R(M) = \infty$. Let now $\mathrm{Gfd}_R(M) = n < \infty$ and let $0 \to K \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$ be an exact sequence such that $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules. As R is left GF-closed, it follows, by [4, Theorem 1.8], that K is Gorenstein flat. Therefore, Theorem 2.1 shows that K is flat and $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$, as desired.

(2) follows directly from (1). ∎

Recall that Benson and Goodearl [6, Theorem 2.5] proved that if M is a periodic projective module which is flat, then M is projective. Subsequently, Simson [23] improved this theorem by studying *pure-periodic modules*, that is, modules M such that there exists a pure-exact sequence $0 \to M \to P_1 \to \cdots \to P_n \to M \to 0$ with the P_i pure projective modules. He mainly proved that any pure-periodic module M is pure projective [23, Theorem 4.4]. The following proposition sheds light on the impact of the finiteness of the flat dimension of the Pontryagin dual of modules on periodic exact sequences.

Proposition 2.5. *Let M be a periodic projective module. If* $\mathrm{fd}_R(M^+) < \infty$, *then M is projective.*

Proof. Let M be a periodic projective module. Then there exists an exact sequence $0 \to M \to P_1 \to \cdots \to P_n \to M \to 0$ of R-modules such that $P_1, \ldots, P_n$ are projective modules. Assume that $\mathrm{fd}_R(M^+) < \infty$. Let $K := \mathrm{Im}(P_1 \to P_2)$. Then the sequence $0 \to M \to P_1 \to K \to 0$ is exact and there exist projective modules P'_i such that $0 \to K \to P'_1 \to P'_2 \to \cdots$ is exact. Consider the exact sequences of character modules $0 \to K^+ \to P_1^+ \to M^+ \to 0$ and $\cdots \to P_2'^+ \to P_1'^+ \to K^+ \to 0$. As each $P_i'^+$ is injective, the last sequence shows that K^+ is the nth cosyzygy module of an injective resolution of the character module J_n^+ of some module J_n for any integer $n \geq 1$. Hence

$$\mathrm{Ext}_R^1(M^+, K^+) \cong \mathrm{Ext}_R^{n+1}(M^+, J_n^+) \cong \mathrm{Tor}_{n+1}^R(M^+, J_n)^+$$

for any positive integer $n \geq 1$. As $\mathrm{fd}_R(M^+) < \infty$, it follows that $\mathrm{Ext}_R^1(M^+, K^+) = 0$, and thus the exact sequence $0 \to K^+ \to P_1^+ \to M^+ \to 0$ splits. This means that, being a direct summand of the injective module P_1^+, M^+ is injective, and thus M is a flat module. Now, by [6, Theorem 2.5], M is projective, completing the proof. ∎

Corollary 2.6. *Let R be a left coherent ring (resp., Noetherian ring). Let M be a periodic projective module. If* FP-$\mathrm{id}_R(M) < \infty$ *(resp.,* $\mathrm{id}_R(M) < \infty$*), then M is a projective R-module.*

Our next theorem provides an analog for the Gorenstein flat dimension of [20, Theorem 2.1] which asserts that if $\mathrm{pd}_R(M) < \infty$, then $\mathrm{Gid}_R(M) = \mathrm{id}_R(M)$. Recall that Bennis proved a result for the Gorenstein flat dimension which guarantees a sort of a Gorenstein flat precover of a given module M: Let R be a left GF-closed ring and let M be an R-module. If $\mathrm{Gfd}_R(M) < \infty$, then there exists an exact sequence $0 \to M \to E \to G \to 0$ with $\mathrm{fd}_R(E) = \mathrm{Gfd}_R(M) < \infty$ and G Gorenstein flat [5, Lemma 2.2].

Theorem 2.7. *Let R be a left GF-closed ring and M an R-module. If* $\mathrm{fd}_R(M) < \infty$*, then* $\mathrm{Gfd}_R(M^+) = \mathrm{fd}_R(M^+)$.

In light of this theorem, character modules show nice behavior vis-à-vis the above-cited Holm theorem [20, Theorem 2.6] without its restrictive conditions, that is, if R is left GF-closed and M is an R-module, then

$$\mathrm{id}_R(M^+) < \infty \;\Rightarrow\; \mathrm{Gfd}_R(M^+) = \mathrm{fd}_R(M^+).$$

Proof. Assume $\mathrm{fd}_R(M) < \infty$. The equality $\mathrm{Gfd}_R(M^+) = \mathrm{fd}_R(M^+)$ clearly holds if $\mathrm{Gfd}_R(M^+) = \infty$. Now suppose $\mathrm{Gfd}_R(M^+) < \infty$. By [5, Lemma 2.2], there exists an exact sequence of right modules $0 \to M^+ \to E \to G \to 0$ such that $\mathrm{fd}_R(E) = \mathrm{Gfd}_R(M^+) < \infty$ and G is Gorenstein flat. Observe that $\mathrm{Ext}_R^1(G, M^+) \cong \mathrm{Tor}_1^R(G, M)^+ = 0$ [12, Lemma 3.3] since $\mathrm{fd}_R(M) < \infty$ and G is a Gorenstein flat right module. Hence the exact sequence $0 \to M^+ \to E \to G \to 0$ splits, implying that M^+ is a direct summand of E and thus $\mathrm{fd}_R(M^+) < \infty$. It follows that $\mathrm{Gfd}_R(M^+) = \mathrm{fd}_R(M^+)$, as desired. ∎

Next, we combine the formulas of Corollary 2.4 and Theorem 2.7 into the following one.

Corollary 2.8. *Let R be a left GF-closed ring and M an R-module. If* $\min(\mathrm{fd}_R(M), \mathrm{fd}_R(M^+)) < \infty$*, then* $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$ *and* $\mathrm{Gfd}_R(M^+) = \mathrm{fd}_R(M^+)$.

We finish the paper by the following useful result.

COROLLARY 2.9. *Let R be a left GF-closed ring. Let M be an R-module. Then the following assertions are equivalent:*

(1) $\mathrm{fd}_R(M^+) < \infty$ *and* $\mathrm{Gfd}_R(M) < \infty$*;*
(2) $\mathrm{Gfd}_R(M^+) < \infty$ *and* $\mathrm{fd}_R(M) < \infty$*.*

Proof. Apply Corollary 2.4 and Theorem 2.7. ■

Acknowledgments. The author would like to thank the referee for careful reading of the paper and valuable comments and suggestions.

REFERENCES

[1] M. Auslander, *Anneaux de Gorenstein et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'algèbre commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro, École Normale Supérieure de Jeunes Filles.
[2] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. 94 (1969).
[3] L. L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) 85 (2002), 393–440.
[4] D. Bennis, *Rings over which the class of Gorenstein flat modules is closed under extensions*, Comm. Algebra 37 (2009), 855–868.
[5] D. Bennis, *Weak Gorenstein global dimension*, Int. Electron. J. Algebra 8 (2010), 140–152.
[6] D. J. Benson and K. R. Goodearl, *Periodic flat modules, and flat modules for finite groups*, Pacific J. Math. 196 (2000), 45–67.
[7] S. Bouchiba, *Finiteness aspects of Gorenstein homological dimensions*, Colloq. Math. 131 (2013), 171–194.
[8] S. Bouchiba, *On Gorenstein flat dimension*, J. Algebra Appl. 14 (2015), 1–18.
[9] S. Bouchiba, *A variant theory for the Gorenstein flat dimension*, Colloq. Math. 140 (2015), 183–204.
[10] S. Bouchiba and M. Khaloui, *Stability of Gorenstein flat modules*, Glasgow Math. J. 54 (2012), 169-175.
[11] L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Math. 1747, Springer, Berlin, 2000.
[12] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra 302 (2006), 231–279.
[13] I. Emmanouil, *On the finiteness of Gorenstein homological dimensions*, J. Algebra 372 (2012), 376–396.
[14] E. Enochs and O. Jenda, *On Gorenstein injective modules*, Comm. Algebra 21 (1993), 3489–3501.
[15] E. Enochs and O. Jenda, *Gorenstein injective and projective modules*, Math. Z. 220 (1995), 611–633.

[16] E. Enochs, O. Jenda, and J. Xu, *Foxby duality and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. 348 (1996), 3223–3234.
[17] D. J. Fieldhouse, *Character modules, dimension and purity*, Glasgow J. Math. 13 (1972), 144–146.
[18] T. V. Gedrich and K. W. Gruenberg, *Complete cohomological functor on groups*, Topology Appl. 25 (1987), 203–223.
[19] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra 189 (2004), 167–193.
[20] H. Holm, *Rings with finite Gorenstein injective dimension*, Proc. Amer. Math. Soc. 132 (2004), 1279–1283.
[21] D. Simson, *On projective resolutions of flat modules*, Colloq. Math. 29 (1974), 209–218.
[22] D. Simson, *A remark on projective dimension of flat modules*, Math. Ann. 209 (1974), 181–182.
[23] D. Simson, *Pure-periodic modules and a structure of pure-projective resolutions*, Pacific J. Math. 207 (2002), 235–256.
[24] B. Stenström, *Coherent rings and FP-injectivity*, J. London Math. Soc. (2) 2 (1970), 323–329.

Samir Bouchiba
Department of Mathematics
Faculty of Sciences
University Moulay Ismail
50000 Meknes, Morocco
E-mail: bouchibasamir@gmail.com