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HYPERPLANE SECTIONS OF CYLINDERS

BY

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Abstract. We provide a formula to compute the volume of the intersection of a generalized cylinder with a hyperplane. Then we prove an integral inequality involving Bessel functions similar to Keith Ball's well-known inequality. Using this inequality we obtain upper bounds for the section volume. For large radii of the cylinder we determine the maximal section.

1. Introduction. The study of sections of convex bodies has a long history. The first formula for sections of the cube with a hyperplane dates back to Laplace in 1812. The first results on bounds for the volume were found by Hensley [7] and Ball [2]. The upper bound for the cube leads to a simple counterexample to the Busemann–Petty problem [3]. Many different convex bodies have been investigated, for example, ℓ_p-balls in [15] and [14], complex cubes in [18], simplices in [5], also non-central sections in [16] as well as measures other than Lebesgue measure in [10]. In this paper we deal with generalized cylinders.

Throughout this paper we use the following notation. The Euclidean norm is denoted by $\|x\|$, the standard scalar product by $\langle x, y\rangle$. For $a \in \mathbb{R}^n$ with $\|a\| = 1$ and $t \in \mathbb{R}$, let $H_a^t := \{x \in \mathbb{R}^n \mid \langle a, x\rangle = t\} = H_a + t \cdot a$ be a translated hyperplane, in particular $H_a := H_a^0$. If H is a k-dimensional (affine) subspace and $A \subset H$, the k-dimensional volume of A is the standard induced Lebesgue volume on the subspace, denoted by $\mathrm{vol}_k(A)$. The characteristic function of a set A is denoted by χ_A.

The *normalized* Bessel function of order $\nu \in \mathbb{R}$ is given by

$$j_\nu(s) := 2^\nu \Gamma(\nu+1) J_\nu(s)/s^\nu \quad \text{for } s > 0 \quad \text{and} \quad j_\nu(0) := 1,$$

where J_ν is the Bessel function of order ν. Note that the normalized Bessel function j_ν is continuous at 0. A classical introduction to Bessel functions is [20].

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We consider *generalized cylinders*. Let

$$Z := \tfrac{1}{2}B_\infty^n \times rB_2^m \subset \mathbb{R}^{n+m}$$

for $r > 0$, $n, m \in \mathbb{N}$, where $B_\infty^n := [-1,1]^n$ and $B_2^m := \{x \in \mathbb{R}^m \mid \|x\|_2 \le 1\}$. We are interested in the volume of *central sections*, i.e. in the quantity

$$\mathrm{vol}_{n+m-1}(H_a \cap Z)$$

for $a \in \mathbb{R}^{n+m}$ with $\|a\| = 1$. We may assume that $a = (a_1, \ldots, a_n, a_{n+1}, 0, \ldots, 0)$ with $a_1, \ldots, a_{n+1} \ge 0$, since Z is rotationally symmetric with respect to the coordinates $n+1, \ldots, n+m$ and symmetric with respect to the origin.

Our first result, proved in Section 2 by the classical Fourier analytic method, is a volume formula.

THEOREM 1. *For the cylinder $Z \subset \mathbb{R}^{n+m}$ with $m, n \in \mathbb{N}$, $r > 0$, and a normal vector $a \in \mathbb{R}^{n+m}$, the volume of the hyperplane section $H_a \cap Z$ is given by*

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) = r^m \frac{\pi^{m/2-1}}{\Gamma(m/2+1)} \int_0^\infty \prod_{j=1}^n \frac{\sin(a_j s/2)}{a_j s/2} \cdot j_{m/2}(a_{n+1} r s)\, ds.$$

Note that $j_{1/2}(s) = (\sin s)/s$, so for $m = 1$ we get the formula for the cube.

Using Hölder's inequality in order to get an upper bound on the section volume is also a classical method. In Section 3 we follow this approach and find estimates on the volume.

THEOREM 2. *Let $n, m > 1$ and $r > 0$. Then for all $a \in \mathbb{R}^{n+m}$ with $\|a\| = 1$,*

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) \le \begin{cases} r^m \frac{\pi^{m/2}}{\Gamma(m/2+1)} \cdot \sqrt{2}, & r \ge \frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)} \frac{1}{\sqrt{\pi}}, \\ r^{m-1} \frac{\pi^{(m-1)/2}}{\Gamma((m-1)/2+1)} \cdot \sqrt{2}, & r < \frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)} \frac{1}{\sqrt{\pi}}. \end{cases} \tag{1}$$

For $r \ge \frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)} \frac{1}{\sqrt{\pi}}$, the bound is attained for $a = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots, 0)$.

For the three-dimensional case ($n = 1$, $m = 2$) real calculus suffices to characterize the maximal section. Note that the intersecting hyperplane can be described by one variable.

THEOREM 3. *Let $Z := [-1/2, 1/2] \times rB_2^2$. Depending on r we have:*

(i) *For $r > 1/(2\sqrt{3})$ a section orthogonal to $(\sqrt{1-\alpha^2}, \alpha, 0)$ for some $\alpha \in (\sqrt{1/(4r^2+1)}, 1)$ is maximal. So the maximal section is a truncated ellipse.*

(ii) *For $r \le 1/(2\sqrt{3})$ the section orthogonal to $(0,1,0)$ is maximal. So the maximal section is a rectangle.*

In Section 4 we prove the main integral inequality, which is also of independent interest. For the proof we use three slightly different approaches, depending on m. The inequality is stated in the following theorem.

THEOREM 4. *For all $m \in \mathbb{N}$, $m \geq 2$, and $p \in \mathbb{R}$, $p \geq 2$, we have*

$$\mathcal{J}_m(p) := \sqrt{p}\int_0^\infty |j_{m/2}(s)|^p\,ds \leq \sqrt{\pi}\sqrt{m/2+1}$$

and $\lim_{p\to\infty}\mathcal{J}_m(p) = \sqrt{\pi}\sqrt{m/2+1}$.

Recall Ball's inequality [2]: for $p \geq 2$,

$$\mathcal{J}_1(p) = \sqrt{p}\int_0^\infty \left|\frac{\sin(u)}{u}\right|^p du \leq \frac{\pi}{\sqrt{2}}, \tag{2}$$

and $\lim_{p\to\infty}\mathcal{J}_1(p) = \sqrt{3\pi/2} < \pi/\sqrt{2}$.

2. Volume formula. We apply the standard method.

Proof of Theorem 1. Define $A(a,t) := \mathrm{vol}_{n+m-1}(H_a^t \cap Z)$ for $a \in \mathbb{R}^{n+m}$ with $\|a\| = 1$ and $t \geq 0$; in particular $A(a) := A(a,0)$. We apply the Fourier transform and the inversion formula to the function $t \mapsto A(a,t)$. By Fubini's theorem and the well-known integrals

$$\int_{B_2^n} \exp(-is\langle x,a\rangle)\,dx = \frac{\pi^{n/2}}{\Gamma(n/2+1)} j_{n/2}(s\|a\|) \qquad \text{for } s>0,\ a\in\mathbb{R}^n,$$

$$\int_{[-1/2,1/2]^n} \exp(-is\langle x,a\rangle)\,dx = \prod_{j=1}^n \frac{\sin(a_j s/2)}{a_j s/2} \qquad \text{for } s\in\mathbb{R},\ a\in\mathbb{R}^n,$$

we have

$$\begin{aligned}
(2\pi)^{1/2}\hat{A}(a,s) &= \int_{\mathbb{R}} A(a,t)\exp(-ist)\,dt \\
&= \int_{\mathbb{R}}\int_{\langle x,a\rangle = t} \chi_{[-1/2,1/2]^n}((x_1,\dots,x_n))\chi_{rB_2^m}((x_{n+1},\dots,x_{n+m}))\exp(-ist)\,dx\,dt \\
&= \int_{\mathbb{R}^{n+m}} \chi_{[-1/2,1/2]^n}((x_1,\dots,x_n))\chi_{rB_2^m}((x_{n+1},\dots,x_{n+m}))\exp(-is\langle x,a\rangle)\,dx \\
&= \int_{[-1/2,1/2]^n} \exp\Big(-is\sum_{j=1}^n a_jx_j\Big)\,d(x_1,\dots,x_n)\int_{rB_2^m}\exp\Big(-is\sum_{j=n+1}^{n+m} a_jx_j\Big)\,dx \\
&= \prod_{j=1}^n \frac{\sin(a_js/2)}{a_js/2}\cdot r^m\frac{\pi^{m/2}}{\Gamma(m/2+1)} j_{m/2}(rsa_{n+1}).
\end{aligned}$$

Finally, by the Fourier inversion formula we get the formula stated in Theorem 1. ∎

For the three-dimensional cylinder, i.e. $n = 1$ and $m = 2$, using an equation from [6, 6.693 (4), p. 720] we get

LEMMA 2.1. *Let Z be the three-dimensional cylinder with radius $r > 0$. For $\alpha \in [0,1]$ let $a = (\sqrt{1-\alpha^2}, \alpha, 0)$. Then the volume, i.e. the area, of the section $H_a \cap Z$ is given by the function $A\colon [0,1] \to \mathbb{R}$,*

$$A(\alpha) = \begin{cases} \pi r \dfrac{r}{\sqrt{1-\alpha^2}} & \text{for } 0 \le \alpha \le \dfrac{1}{\sqrt{1+4r^2}}, \\ \dfrac{r}{\alpha}\sqrt{1-\dfrac{1-\alpha^2}{4\alpha^2 r^2}} + \dfrac{2r^2}{\sqrt{1-\alpha^2}} \arcsin\left(\dfrac{\sqrt{1-\alpha^2}}{2\alpha r}\right) & \text{for } \dfrac{1}{\sqrt{1+4r^2}} < \alpha < 1, \\ 2r & \text{for } \alpha = 1. \end{cases}$$

The three cases correspond to the geometric shape of the section, namely an ellipse resp. a disk, a truncated ellipse and a rectangle. Clearly, this formula can also be obtained by elementary geometric considerations.

3. Volume estimates

3.1. Three-dimensional case. The three-dimensional case can be treated by real calculus.

Proof of Theorem 3. The function A from Lemma 2.1 defined on the closed interval $[0,1]$ is differentiable. Let $\alpha^* := 1/\sqrt{1+4r^2}$. For $0 < \alpha < \alpha^*$ we have $A'(\alpha) = \pi r^2 \alpha/(1-\alpha^2)^{3/2}$. This is larger than 0 for all $r > 0$. For the left derivative in α^* we get $A'_-(\alpha^*) = \pi(1+4r^2)/(8r)$.

For $\alpha^* < \alpha < 1$ we find

$$A'(\alpha) = \frac{1}{4r\alpha^4}\frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}} - \frac{r}{\alpha^2}\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}} + \frac{2\alpha r^2}{(1-\alpha^2)^{3/2}}\arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right) - \frac{r}{\alpha^2(1-\alpha^2)}\frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}}. \tag{3}$$

Compute the limit of (3) as $\alpha \to \alpha^*$, $\alpha > \alpha^*$. Note that $\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}} = 0$ for $\alpha = \alpha^*$. The sum of the first and the last summands of (3) tends to 0 by L'Hôpital's rule. The second summand tends to 0 as well. The third summand tends to $\pi(1+4r^2)/(8r)$, which coincides with the left derivative at α^*. So A is differentiable in $(0,1)$ with $A'(\alpha^*) = \pi(1+4r^2)/(8r) > 0$ for all $r > 0$.

In particular, A' is positive on $(0, \alpha^*]$. So A is maximal for some $\alpha \in (\alpha^*, 1]$. The maximum is attained for some $\alpha < 1$ if and only if $A'(\alpha)$ has a zero in $(\alpha^*, 1)$. Otherwise A is increasing from 0 to 1 and attains its maximum for $\alpha = 1$.

For $\alpha \in (\alpha^*, 1)$ the equation $A'(\alpha) = 0$ is equivalent to

$$\begin{aligned}\frac{2\alpha r^2}{(1-\alpha^2)^{3/2}} \arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right) = {}& \frac{r}{\alpha^2(1-\alpha^2)} \frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}} \\ & + \frac{r}{\alpha^2}\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}} \\ & - \frac{1}{4r\alpha^4}\frac{1}{\sqrt{1+\frac{1}{4r^2}-\frac{1}{4r^2\alpha^2}}}.\end{aligned}$$

Multiplying this by $(1-\alpha^2)/r$ and adding the first and third summands on the right-hand side simplifies this to

$$\frac{\arcsin\left(\frac{\sqrt{1-\alpha^2}}{2\alpha r}\right)}{\frac{\sqrt{1-\alpha^2}}{2\alpha r}} = \frac{2-\alpha^2}{\alpha^3}\sqrt{\alpha^2 + \frac{\alpha^2}{4r^2} - \frac{1}{4r^2}}. \tag{4}$$

Set $x := \frac{\sqrt{1-\alpha^2}}{2\alpha r}$; then $\alpha = \frac{1}{\sqrt{1+4r^2x^2}}$. Equation (4) then reads

$$\frac{\arcsin(x)}{x} = (1+8r^2x^2)\sqrt{1-x^2}. \tag{5}$$

So $A'(\alpha) = 0$ for some $\alpha \in (\alpha^*, 1)$ is equivalent to (5) for some $x \in (0,1)$. Estimating both sides of (5) using Taylor's theorem we find that A' has a zero smaller than 1 if and only if $r > 1/(2\sqrt{3})$. ■

3.2. General dimension. The first step is the application of Hölder's inequality.

LEMMA 3.1. *Let $a \in \mathbb{R}^{n+m}$ be a normal vector. Then*

$$\operatorname{vol}_{n+m-1}(H_a \cap Z) \le r^m \frac{\pi^{m/2-1}}{\Gamma(m/2+1)} \prod_{j=1}^{n} \left(2\mathcal{J}_1\left(\frac{1}{a_j^2}\right)\right)^{a_j^2} \left(\frac{1}{r}\mathcal{J}_m\left(\frac{1}{a_{n+1}^2}\right)\right)^{a_{n+1}^2},$$

where

$$\mathcal{J}_m(p) := \sqrt{p}\Big(\int_0^\infty |j_{m/2}(u)|^p\, du\Big).$$

Proof. We apply Hölder's inequality to the formula from Theorem 1 and then substitute $u = a_j s/2$ resp. $u = a_{n+1} rs$:

$$\begin{aligned}
&\mathrm{vol}_{n+m-1}(H_a \cap Z)\\
&\le r^m \frac{\pi^{m/2-1}}{\Gamma(m/2+1)} \prod_{j=1}^{n} \left(\frac{2}{a_j} \int_0^\infty \left| \frac{\sin u}{u} \right|^{1/a_j^2} du \right)^{a_j^2} \left(\frac{1}{r a_{n+1}} \int_0^\infty |j_{m/2}(u)|^{1/a_{n+1}^2}\, du \right)^{a_{n+1}^2}\\
&= r^m \frac{\pi^{m/2-1}}{\Gamma(m/2+1)} \prod_{j=1}^{n} \left(2\mathcal{J}_1\left(\frac{1}{a_j^2}\right)\right)^{a_j^2} \left(\frac{1}{r} \mathcal{J}_m\left(\frac{1}{a_{n+1}^2}\right)\right)^{a_{n+1}^2}. \ \blacksquare
\end{aligned}$$

Proof of Theorem 2. The integral inequality from Theorem 4 and also Ball's inequality (2) may only be used if all coordinates of a are smaller than $1/\sqrt{2}$. If there is a coordinate larger than $1/\sqrt{2}$, we use a different estimate that is also used in Ball's proof (see [2] for example).

Case 1: $|a_j| \le 1/\sqrt{2}$ *for all* $j = 1, \ldots, n+1$. We apply the integral inequality (2) and the one from Theorem 4 to Lemma 3.1. For the third inequality, note that $\frac{1}{r}\sqrt{\pi}\sqrt{m/2+1} < \sqrt{2}\,\pi$ if and only if $r > \frac{\sqrt{m/2+1}}{\sqrt{2\pi}}$, so

$$\begin{aligned}
\mathrm{vol}_{n+m-1}(H_a \cap Z) &\le r^m \frac{\pi^{m/2-1}}{\Gamma(m/2+1)} \prod_{j=1}^{n} (\sqrt{2}\,\pi)^{a_j^2} \cdot \left(\frac{1}{r} \mathcal{J}_m\left(\frac{1}{a_{n+1}^2}\right)\right)^{a_{n+1}^2}\\
&\le r^m \frac{\pi^{m/2-1}}{\Gamma(m/2+1)} (\sqrt{2}\,\pi)^{\sum_{j=1}^n a_j^2} \left(\frac{1}{r}\sqrt{\pi}\sqrt{m/2+1}\right)^{a_{n+1}^2}\\
&\le \begin{cases} r^m \dfrac{\pi^{m/2}}{\Gamma(m/2+1)}\sqrt{2}, & r > \dfrac{\sqrt{m/2+1}}{\sqrt{2\pi}},\\[2ex] r^{m-1} \dfrac{\pi^{(m-1)/2}}{\Gamma(m/2+1)}\sqrt{m/2+1}, & r \le \dfrac{\sqrt{m/2+1}}{\sqrt{2\pi}}. \end{cases}
\end{aligned}$$

Case 2: $|a_j| > 1/\sqrt{2}$ *for some* $j = 1, \ldots, n$. Let P be the orthogonal projection onto the hyperplane $\{x_j = 0\}$. Since $P(H \cap Z) \subset P(Z)$, we have $\mathrm{vol}(P(H \cap Z)) \le \mathrm{vol}(P(Z))$. The projected cylinder $P(Z)$ is isomorphic to $\frac{1}{2} B_\infty^{n-1} \times r B_2^m$, so the volume can be computed elementarily. Furthermore,

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) = \frac{1}{|a_j|} \mathrm{vol}_{n+m-1}(P(H_a \cap Z)).$$

Therefore

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) < \sqrt{2}\,\mathrm{vol}_{n+m-1}(P(Z)) = \sqrt{2}\, r^m \frac{\pi^{m/2}}{\Gamma(m/2+1)}.$$

Case 3: $|a_{n+1}| > 1/\sqrt{2}$. We consider the orthogonal projection onto $\{x_{n+1} = 0\}$. Now $P(Z)$ is isomorphic to $\frac{1}{2} B_\infty^n \times B_2^{m-1}$. By the same argument as in Case 2,

$$\mathrm{vol}_{n+m-1}(H_a \cap Z) < \sqrt{2}\, r^{m-1} \frac{\pi^{(m-1)/2}}{\Gamma((m-1)/2+1)}.$$

We summarize the estimates. Note that by Lemma 4.3 for $m \geq 2$,

$$\frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)} \frac{1}{\sqrt{\pi}} > \frac{\sqrt{m/2+1}}{\sqrt{2\pi}}. \tag{6}$$

Let $r \geq \frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)} \frac{1}{\sqrt{\pi}}$. Due to (6), also $r > \frac{\sqrt{m/2+1}}{\sqrt{2\pi}}$. So in all three cases, we have $\mathrm{vol}_{n+m-1}(H_a \cap Z) \leq r^m \frac{\pi^{m/2}}{\Gamma(m/2+1)} \sqrt{2}$. This bound is attained for the normal vector $a = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots, 0)$.

If $r < \frac{\Gamma(m/2+1)}{\Gamma(m/2+1/2)} \frac{1}{\sqrt{\pi}}$, then the bound from Case 3 is the largest. ■

Remarks. (i) We have not touched upon the question if the distinction of the cases in (1) is sharp. In Theorem 3 the distinction of the cases is sharp. In this theorem, for $n = 1$ and $m = 2$ the critical radius would be $4/\pi^2$, which is much larger than the critical radius $1/(2\sqrt{3})$ from Theorem 3.

(ii) For the three-dimensional cylinder we found that a *truncated ellipse* gives maximal volume for large r. For the generalized cylinder there is a different behavior. The volume-maximal section of the cylinder is the Cartesian product of the maximal section of the cube and a ball of dimension m. For example, for a 4-dimensional cylinder, i.e. $n = 2 = m$, for large r the maximal section is a three-dimensional cylinder of height $\sqrt{2}$ and radius r.

(iii) We conjecture that if r is sufficiently small, the section orthogonal to $a = (0, \ldots, 0, 1, 0, \ldots, 0)$ is maximal, where the $(n+1)$th coordinate of a is 1. The volume of this section is equal to

$$\mathrm{vol}_n\left(\frac{1}{2} B_\infty^n\right) \mathrm{vol}_{m-1}(r B_2^{m-1}) = r^{m-1} \frac{\pi^{(m-1)/2}}{\Gamma((m-1)/2+1)}.$$

Comparing this with the bound from (1), there is an error of $\sqrt{2}$. Numerical experiments suggest that for medium sized r, some non-standard direction is maximal.

(iv) Ball's inequality and ours have a different behavior. This indicates why Theorem 2 is not always sharp. Note that $\mathcal{J}_1(2) > \lim_{p\to\infty} \mathcal{J}_1(p)$ in contrast to $\mathcal{J}_m(2) \leq \lim_{p\to\infty} \mathcal{J}_m(p)$ for $m \geq 2$. So for $m = 1$, equality holds for $p = 2$ in contrast to $m \geq 2$, where equality holds for $p = \infty$.

(v) As Theorem 3 shows, there is a critical value of the radius that originates in the geometry of the cylinder. For generalized cylinders an additional distinction comes from the method, and this does not give the sharp geometric distinction as in Theorem 3.

4. Integral inequality. Integral inequalities similar to Theorem 4 and (2) were established for complex cubes and for generalized cubes (see [18] and [4]). If we identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, hyperplane sections of the complex cube have real dimension $2n-2$. The integral inequality needed for this case is

$$\sqrt{p}\int_0^\infty |j_1(s)|^p s\,ds \le \frac{4}{p} \tag{7}$$

for $p \ge 2$. Compared to the integral in (2) there is an additional factor s in front of ds. For generalized cubes one has to consider a similar integral with some higher power of s in front of ds.

We prove Theorem 4 by applying the following lemma due to Nazarov and Podkorytov [17]. They used this lemma to simplify Ball's proof of inequality (2). The oscillating behavior of the function $\sin(s)/s$ is a main difficulty. By the Nazarov–Podkorytov lemma one avoids the oscillations by considering distribution functions, which are decreasing.

For a function $f : X \to \mathbb{R}_{\ge 0}$ on a measure space (X,μ), define the cumulative distribution function $F : \mathbb{R}_{>0} \to \mathbb{R}_{\ge 0}$ by

$$F(y) := \mu(\{x \in X \mid f(x) > y\}).$$

LEMMA 4.1 (Nazarov–Podkorytov). *Let h, g be non-negative measurable functions on a measure space (X,μ). Let H, G be their distribution functions. Assume that $H(y)$, $G(y)$ are finite for all $y > 0$. Also assume that*

- (N1) *there is some $y_0 > 0$ such that $G(y) \le H(y)$ for all $y < y_0$ and $G(y) \ge H(y)$ for all $y > y_0$, i.e. the difference $G - H$ changes its sign exactly once from $-$ to $+$;*
- (N2) *for some $p_0 > 0$, $\int_X h^{p_0}\,d\mu = \int_X g^{p_0}\,d\mu$.*

Then

$$\int_X h^p\,d\mu \le \int_X g^p\,d\mu$$

for all $p > p_0$ as long as the integrals exist.

4.1. Technical estimates. The proof of the integral inequality uses some technical estimates that we state here. A main tool is Stirling's formula, used to approximate the gamma function. We refer to [13, Section 18.3].

LEMMA 4.2 (Stirling's formula). *Let $x > 0$. Then there exists some $\mu(x) \in (0, 1/(12x))$ such that*

$$\Gamma(x) = \sqrt{2\pi}\,x^{x-1/2}\exp(-x)\exp(\mu(x)).$$

LEMMA 4.3. *For $x \ge 2$ we have*

$$\frac{\Gamma(x)}{\Gamma(x-1/2)} > \sqrt{\frac{x}{2}}.$$

Proof. We estimate the gamma function by using Lemma 4.2:

$$\frac{\Gamma(x)}{\Gamma(x-1/2)} \geq \left(\frac{x}{x-1/2}\right)^x \frac{x-1/2}{\sqrt{x}} \frac{1}{\exp(1/2)} \frac{1}{\exp(1/24)}$$
$$\geq \frac{x-1/2}{\sqrt{x}} \frac{1}{\exp(1/24)} = \left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right) \exp\left(-\frac{1}{24}\right).$$

Note that $\left(\frac{x}{x-1/2}\right)^x$ strictly decreases to $\exp(1/2)$. Additionally the function $\left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)\exp(-1/24)$ increases faster than $\sqrt{x/2}$ and the inequality holds for $x = 2$. ∎

Lemma 4.4. *Let $m \geq 5$. Then*

$$\frac{\Gamma(m/2+1)^2\Gamma(m)}{\Gamma(m/2+1/2)^2\Gamma(m+1/2)} \leq \frac{m+2}{m+1} \frac{\sqrt{m}}{2}.$$

Proof. Using Legendre's duplication formula, we find

$$\frac{\Gamma(m/2+1)^2\Gamma(m)}{\Gamma(m/2+1/2)^2\Gamma(m+1/2)} = \frac{(m/2)^2\Gamma(m/2)^2\Gamma(m/2)^2\Gamma(m)\Gamma(m)}{[\Gamma(m/2+1/2)^2\Gamma(m/2)^2]\,[\Gamma(m+1/2)\Gamma(m)]}$$
$$= \frac{m^2}{4} \frac{\Gamma(m/2)^4\Gamma(m)^2}{\left[\frac{\sqrt{\pi}}{2^{m-1}}\Gamma(m)\right]^2 \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$
$$= 2^{4m-5}\frac{m^2}{\pi^{3/2}} \frac{\Gamma(m/2)^4}{\Gamma(2m)}.$$

So we need to show

$$q(m) := \frac{2^{4m-4}}{\pi^{3/2}} \frac{m^{3/2}(m+1)}{m+2} \frac{\Gamma(m/2)^4}{\Gamma(2m)} \leq 1.$$

By application of Lemma 4.2 we find

$$q(m) \leq \frac{2^{4m-4}}{\pi^{3/2}} \frac{m^{3/2}(m+1)}{m+2} \frac{\left(\sqrt{2\pi}\,(m/2)^{m/2-1/2}\exp(-m/2)\exp\left(\frac{1}{6m}\right)\right)^4}{\sqrt{2\pi}(2m)^{2m-1/2}\exp(-2m)}$$
$$= \frac{m+1}{m+2}\exp\left(\frac{2}{3m}\right) =: \tilde{q}(m).$$

As a function on $\mathbb{R}_{\geq 0}$, the derivative of $\tilde{q}(m)$ only has a zero at $m = 3 + \sqrt{13} > 6$. Note that $\tilde{q}(5) = \frac{6}{7}\exp(2/15) < 1$ and $\tilde{q}(6) = \frac{7}{8}\exp(2/18) < \tilde{q}(5)$. Obviously $\tilde{q}(m) \to 1$ for $m \to \infty$. Therefore $\tilde{q}(m)$ is increasing for $m \geq 7$. This proves $\tilde{q}(m) < 1$ for all $m \geq 5$. ∎

A bound for the absolute value of Bessel functions follows from [6, (8.479)]. For the normalized Bessel functions this reads:

Lemma 4.5. *Let $m \in \mathbb{N}$ and $s > m/2$. Then*

$$|j_{m/2}(s)| \leq \frac{2^{(m+1)/2}\Gamma(m/2+1)}{\sqrt{\pi}} \frac{1}{(s^2-m^2/4)^{1/4}} \frac{1}{s^{m/2}}.$$

More elaborate estimates were used in several contexts. We collect a few results that we need later.

LEMMA 4.6. *Let $m \geq 2$ and $s \in [0, m/2+3]$ resp. let $m = 1$ and $s \in [0, 3.38]$. Then*

$$|j_{m/2}(s)| \leq \exp\left(-\frac{s^2}{2m+4} - \frac{s^4}{4(m^2+4m+4)(m+4)}\right).$$

Proof. This can be proven by a modification of the proof in [11, Prop. 12, p. 131], as already noted in the preprint version [12, p. 19]. ■

LEMMA 4.7. *Let $m \geq 5$ and $s \in [0, m]$. Then*

$$|j_{m/2}(s)| \leq \exp\left(-\frac{s^2}{2m+4}\right).$$

Proof. Let $m = 5$ or $m = 6$. Then the inequality follows directly from Lemma 4.6, since $m/2 + 3 \geq m$. The same lemma shows the inequality for $m \geq 7$ and $s \in [0, m/2+3]$. In [4, Lemma 3.17] it is proved that for all $m \geq 7$ and $s \in [m/2+3, m]$ the claimed inequality also holds. Brzezinski's proof uses the estimate from Lemma 4.5. ■

LEMMA 4.8. *Let $m \in \mathbb{N}$ and $s \geq m/2 + 3$. Then*

$$|j_{m/2}(s)| \leq 2^{(m+1)/2} \frac{\Gamma(m/2+1)}{\sqrt{\pi}} \frac{\sqrt{m+6}}{\sqrt[4]{12m+36}} \frac{1}{s^{(m+1)/2}}.$$

Proof. For $s \geq m/2 + 3$ we have

$$\left(s^2 - \frac{m^2}{4}\right)^{-1/4} s^{-m/2} = \left(1 - \frac{m^2}{4s^2}\right)^{-1/4} s^{-(m+1)/2} \leq \frac{\sqrt{m+6}}{\sqrt[4]{12m+36}} s^{-(m+1)/2}.$$

The estimate follows from Lemma 4.5. ■

We also need a lower bound on $|j_{m/2}(\cdot)|$.

LEMMA 4.9. *For all $m \in \mathbb{N}$ and $s \in [0,1]$ we have*

$$|j_{m/2}(s)| \geq \exp\left(-\frac{s^2}{2m+4} - s^4\right).$$

Proof. This is found in [4, Lemma 3.5, part 2]. The estimate there is even stronger. ■

LEMMA 4.10. *For $p > 0$ and $m \in \mathbb{N}$ we have*

$$\int_0^\infty \exp\left(-\frac{ps^2}{2m+4} - \frac{ps^4}{4(m+2)^2(m+4)}\right) ds$$
$$\leq \frac{1}{\sqrt{p}} \sqrt{m/2+1}\sqrt{\pi}\left(1 - \frac{3}{4}\,\frac{1}{p(m+4)} + \frac{105}{16}\,\frac{1}{2p^2(m+4)^2}\right).$$

Proof. By substituting $u := \frac{ps^2}{2m+4}$ we get

$$\int_0^\infty \exp\left(-\frac{ps^2}{2m+4} - \frac{ps^4}{4(m+2)^2(m+4)}\right) ds$$
$$= \frac{1}{2}\sqrt{\frac{2m+4}{p}} \int_0^\infty \exp(-u) \exp\left(-\frac{u^2}{p(m+4)}\right) u^{-1/2}\, du.$$

Then we estimate the exponential function $\exp\left(-\frac{u^2}{p(m+4)}\right)$ by the first three summands of its series expansion. ∎

4.2. The limit of the integral. We prove the asymptotic result of the integral inequality from Theorem 4. Using Lemmas 4.6 and 4.8, we estimate

$$\sqrt{p}\int_0^\infty |j_{m/2}(s)|^p\, ds$$
$$\leq \sqrt{p}\int_0^{m/2+3} \exp\left(-\frac{s^2}{2m+4}\right)^p ds$$
$$+ \sqrt{p}\left(2^{(m+1)/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\right)^p \left(\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \int_{m/2+3}^\infty s^{-p(m+1)/2}\, ds$$
$$\leq \sqrt{p}\int_0^\infty \exp\left(-\frac{ps^2}{2m+4}\right) ds$$
$$+ \sqrt{p}\left(2^{m+1/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\right)^p \left(\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p$$
$$\times \frac{1}{p(m+1)/2-1}\left(\frac{m}{2}+3\right)^{-p(m+1)/2+1}.$$

For $p \to \infty$ and fixed m, the first summand is equal to $\sqrt{\pi}\sqrt{m/2+1}$ since $\int_0^\infty \exp(-x^2/K)\, dx = \sqrt{K\pi}/2$ for $K > 0$. The second summand tends to 0 for $p \to \infty$ and fixed m.

On the other hand, using Lemma 4.9, by the substitution $u = \sqrt{p}\, s$ and by the series expansion of the exponential function we have

$$\sqrt{p}\int_0^\infty |j_{m/2}(s)|^p\, ds \geq \sqrt{p}\int_0^1 \exp\left(-\frac{ps^2}{2m+4} - ps^4\right) ds$$
$$= \int_0^{\sqrt{p}} \exp\left(-\frac{u^2}{2m+4} - \frac{u^4}{p}\right) du$$
$$\geq \int_0^{\sqrt{p}} \exp\left(-\frac{u^2}{2m+4}\right)\left(1 - \frac{u^4}{p}\right) du$$

$$\geq \int_0^{\sqrt{p}} \exp\Big(-\frac{u^2}{2m+4}\Big)\,du - \frac{1}{p}\int_0^{\sqrt{p}} u^4 \exp\Big(-\frac{u^2}{2m+4}\Big)\,du$$
$$= \int_0^{\sqrt{p}} \exp\Big(-\frac{u^2}{2m+4}\Big)\,du - \frac{1}{2p}\int_0^{p} u^{3/2} \exp\Big(-\frac{u}{2m+4}\Big)\,du.$$

For $p \to \infty$, we observe that the first summand again tends to $\sqrt{\pi}\sqrt{m/2+1}$, and the second summand vanishes since $\int_0^\infty x^{3/2}\exp(-x)\,dx < \infty$. By the sandwich lemma we have found the limit as claimed in Theorem 4.

4.3. The case $m = 2$. For $m = 2$ the integral inequality from Theorem 4 is similar to Oleszkiewicz and Pełczyński's inequality estimating the section volume of complex cubes (see (7)). They used a different technique than we do. We use the Nazarov–Podkorytov lemma. This proof is a modification of an unpublished proof of Oleszkiewicz's and Pełczyński's inequality by König [9].

We apply the Nazarov–Podkorytov Lemma 4.1 to the functions

$$h(s) := |j_1(s)| = |2J_1(s)/s| \quad \text{and} \quad g(s) := \exp(-s^2/8).$$

By H resp. G we denote their distribution functions with respect to the Lebesgue measure λ on $\mathbb{R}_{\geq 0}$. We check the two conditions of Lemma 4.1.

4.3.1. *Condition* (N2). Independently of p we have

$$\sqrt{p}\int_0^\infty g(s)^p\,ds = \sqrt{2\pi}.$$

For $p = 2$, we evaluate the other integral explicitly, using [20, p. 403]:

$$\sqrt{2}\int_0^\infty h(s)^2\,ds = \frac{16\sqrt{2}}{3\pi} < \sqrt{2\pi}.$$

By [1, (9.2.1)] we know the asymptotic behavior of Bessel functions:

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}}\cos\Big(s - \Big(\frac{1}{2}\nu - \frac{1}{4}\Big)\pi\Big) + O(s^{-3/2}).$$

So $\sqrt{p}\int_0^\infty h(s)^p\,ds$ diverges as $p \to 2/3$. By the intermediate value theorem, there is $p_0 \in (2/3, 2)$ such that

$$\sqrt{p_0}\int_0^\infty h(s)^{p_0}\,ds = \sqrt{p_0}\int_0^\infty g(s)^{p_0}\,ds. \tag{8}$$

4.3.2. *Condition* (N1). We investigate the two distribution functions H and G.

The distribution function G is given by the inverse of g, since g is a decreasing and bijective function $[0,\infty) \to (0,1]$. So for $y \geq 1$, $G(y) = 0$

and for $s \in (0,1)$ we write explicitly

$$G(y) = \lambda(\{s \mid y < \exp(-s^2/8)\}) = \lambda(\{s \mid s < \sqrt{8\ln(1/y)}\}) = \sqrt{8\ln(1/y)}.$$

Its derivative is

$$G'(y) = -\frac{\sqrt{2}}{y\sqrt{\ln(1/y)}}. \tag{9}$$

Later, we need that $1/|G'(y)|$ is increasing for $0 \le y \le 1/\sqrt{e}$.

Now we investigate H. The function h is oscillating. Denote the kth local maximum of h by $y_k := \max\{h(s) \mid s \in (s_k, s_{k+1})\}$, with s_k the kth zero of the Bessel function J_1 and $s_0 = 0$. The approximation of the first zeros is taken from [20, p. 748, Table VII]: $s_1 = 3.832$, $s_2 = 7.016$, $s_3 = 10.173$.

Step (i): *There is at least one intersection of G and H.* From Lemma 4.6 we know $h(s) = |j_1(s)| \le \exp(-s^2/8) = g(s)$ for $s \in [0,4]$. So for $y \ge y_1$,

$$\begin{aligned} H(y) &= \lambda(\{x \in [0,\infty) \mid h(x) > y\}) = \lambda(\{x \in [0,s_1] \mid h(x) > y\}) \\ &\le \lambda(\{x \in [0,s_1] \mid g(x) > y\}) = G(y). \end{aligned}$$

So $G - H \ge 0$ for $y \in (y_1, \infty)$. Consider (8) and observe that by Fubini and substitution

$$\begin{aligned} 0 &= \int_0^\infty (g(s)^{p_0} - h(s)^{p_0})\, ds = \int_0^\infty (G(y^{1/p_0}) - H(y^{1/p_0}))\, dy \\ &= p_0 \int_0^\infty y^{p_0-1}(G(y) - H(y))\, dy. \end{aligned}$$

So $G - H$ has to change its sign at least once.

Step (ii): *There is at most one intersection of G and H.* If we prove that $G - H$ is increasing on $(0, y_1)$, this implies $G - H$ changes its sign only once. We show this by proving that for each interval (y_{k+1}, y_k), the quotient $|H'|/|G'|$ is strictly larger than 1. The distribution functions are decreasing, so their derivatives are negative (or 0). So $|H'|/|G'| > 1$ implies $H' < G'$ and therefore $G - H$ is increasing.

Step (iii): *Estimating the local maxima of H.* From [19, p. 116] we know the approximate position of the zeros of the Bessel function J_1:

$$s_k \in (k\pi, (k+1/4)\pi). \tag{10}$$

In [8, p. 32] it is noted that the successive maxima of $|\sqrt{\pi/2}\,\sqrt{s}\,J_1(s)|$ are decreasing to 1. This implies

$$2\sqrt{\frac{2}{\pi}}\frac{1}{s_{k+1}^{3/2}} \le y_k.$$

In particular, together with (10), we get

$$\frac{2\sqrt{2}}{\pi^2}\frac{1}{(k+5/4)^{3/2}} \le y_k. \tag{11}$$

STEP (iv): *Computing H.* For $y \neq y_k$ we claim that

$$|H'(y)| = \sum_{s>0,\, h(s)=y} \frac{1}{|h'(s)|}.$$

To see this, note that for a bijective function f, the distribution function F is given by $F = f^{-1}$ and $F' = 1/f' \circ f^{-1}$. Now H can be decomposed into the sum of the bijective parts of h, where $H(y)$ is the length of the intervals on the real line [17, p. 6]. The equation $h(s) = y$ has one root in $(0, s_1)$ and two roots in each interval (s_k, s_{k+1}) for $1 \le k \le K$, with some $K \in \mathbb{N}$ depending on y.

STEP (v): *Estimating h'.* We estimate $h'(s)$ at these roots. By the recurrence relation for Bessel functions, we have $|h'(s)| = (|2J_1(s)|/s)' = 2|J_2(s)|/s$. We approximate J_2 with [6, (8.479)] and find

$$\left|\frac{2J_2(s)}{s}\right| \le 2\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{s^2-4}}\frac{1}{s} \quad \text{for } s \ge 2.$$

Additionally for $s \ge 3$,

$$\frac{1}{\sqrt[4]{s^2-4}}\frac{1}{s} \le \frac{1}{s^{3/2}}\sqrt{\frac{\pi}{2}}.$$

So for $s \ge 3$ we estimate

$$|h'(s)| \le 2\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{s^2-4}}\frac{1}{s} \le 2\frac{1}{s^{3/2}}.$$

This holds in particular for $s \in (s_k, s_{k+1})$, $k \ge 1$, since $s_1 \ge 3$. Therefore

$$|h'(s)| \le 2\frac{1}{s_k^{3/2}} \le 2\frac{1}{(\pi k)^{3/2}}.$$

For $s \in [0, s_1)$, a rough estimate is sufficient:

$$|h'(s)| \le 0.4.$$

STEP (vi): *Estimating H'/G'.* Fix $k \ge 1$ and let $y \in (y_{k+1}, y_k)$. Then

$$|H'(y)| \ge \frac{5}{2} + 2\cdot\frac{1}{2}\pi^{3/2}\sum_{l=1}^{k} l^{3/2}.$$

Numerically, we find $y_1 \le 0.132 < 1/\sqrt{e}$. Since $y_k \le y_1 < 1/\sqrt{e}$, we may use (9) and (11) to estimate

$$\frac{1}{|G'(y)|} \ge \frac{1}{|G'(y_{k+1})|} \ge \frac{1}{\left|G'\left(\frac{2\sqrt{2}}{\pi^2}(k+9/4)^{-3/2}\right)\right|}.$$

For the quotient we get

$$|H'(y)|/|G'(y)|$$
$$\geq \left(\frac{5}{2}+\pi^{3/2}\sum_{l=1}^{k} l^{3/2}\right)\frac{2}{\pi^2(k+9/4)^{3/2}}\sqrt{\ln\left(\frac{\pi^2}{2\sqrt{2}}\left(k+\frac{9}{4}\right)^{3/2}\right)} \quad (=: Q(k))$$
$$\geq \left(\frac{5}{2}+\pi^{3/2}\frac{2}{5}k^{5/2}\right)\frac{2}{\pi^2(k+9/4)^{3/2}}\sqrt{\ln\left(\frac{\pi^2}{2\sqrt{2}}\left(k+\frac{9}{4}\right)^{3/2}\right)} \quad (=: \tilde{Q}(k)).$$

We have used the estimate $\sum_{l=1}^{k} l^{3/2} \geq \int_0^k l^{3/2}\,dl = \frac{2}{5}k^{5/2}$. Note that $\tilde{Q}(k)$ is increasing in k. By evaluation, $Q(2) > 1$ and $\tilde{Q}(3) > 1$, so $|H'(y)|/|G'(y)| > 1$ for all $y \in (y_{k+1}, y_k)$ and $k \geq 2$.

Since $Q(1) < 1$, the estimate needs to be sharper for $k = 1$. Let $y \in (y_2, y_1)$. To estimate $H'(y)$ from below is is enough to consider $y = y_2$. Numerically we find that $y_2 > 0.064$, thus $1/|G'(y)| \geq 1/|G'(y_2)| \geq 0.075$. Solving numerically the equation $h(s) = 0.064$ we get the bounds for its three consecutive roots $\sigma_1 < \sigma_2 < \sigma_3$ as follows: $\sigma_1 < 3.56$, $\sigma_2 > 4.18$ and $\sigma_3 < 6.33$. This yields $|H'(y)| > 1/|h'(3.56)| + 1/|h'(4.18)| + 1/|h'(6.33)| > 21.35$. Therefore $|H'(y)|/|G'(y)| > 21.35 \cdot 0.075 > 1$.

Now we have shown $|H'(y)|/|G'(y)| > 1$ for each interval (y_{k+1}, y_k).

This finishes the proof of condition (N1) and therefore the proof of Theorem 4 for $m = 2$. ∎

4.4. The case $m \geq 5$. The previous proof relied on the approximate knowledge of the zeros of the Bessel function. Here we use a different approach. The idea is due to [4]. The aim is to simplify $j_{m/2}$, use its rapid decay, and get rid of the oscillating behavior. Due to the rougher estimates this only works for $m \geq 5$. We define

$$\tilde{j}_{m/2}(s) := \begin{cases} |j_{m/2}(s)|, & s \in [0, m), \\ 2^{(m+1)/2}\dfrac{\Gamma(m/2+1)}{\sqrt{\pi}}(s^2 - m^2/4)^{-1/4}s^{-m/2}, & s \in [m, \infty). \end{cases} \tag{12}$$

For this simplification, by Lemma 4.5 it is true that for all $s \geq 0$,

$$j_{m/2}(s) \leq \tilde{j}_{m/2}(s). \tag{13}$$

So it is sufficient to prove the inequality for this simplification of $j_{m/2}$. We apply the Nazarov–Podkorytov Lemma 4.1.

4.4.1. *Condition* (N1). We compare $\tilde{\jmath}_{m/2}(s)$ and $g(s) := \exp\bigl(-\frac{s^2}{2m+4}\bigr)$. We claim

$$\tilde{\jmath}_{m/2}(s) < g(s), \quad s \in [0, m], \tag{14}$$

$$\tilde{\jmath}_{m/2}(s) > g(s), \quad s \in (m+2, \infty), \tag{15}$$

$$\tilde{\jmath}_{m/2}(s) = g(s) \quad \text{for exactly one } s \in (m, m+2). \tag{16}$$

Inequality (14) corresponds to Lemma 4.7. Inequality (15) is [4, Lemma 3.19]; note that the lemma there is also true for $m = 5$ by exactly the same argument. Property (16) is from [4, Lemma 3.18]; this does not include $m = 5$ and $m = 6$, but one can easily check the statement by hand with analogous arguments.

Since g and $\tilde{\jmath}_{m/2}$ are bounded by 1, for $y \geq 1$ we have $G(y) = 0 = \tilde{J}_{m/2}(y)$, where $\tilde{J}_{m/2}$ is the distribution function of $\tilde{\jmath}_{m/2}$. The functions g and $\tilde{\jmath}_{m/2}$ intersect exactly once, so the difference of the cumulative distribution functions changes its sign exactly once as well. This shows (N1).

4.4.2. *Condition* (N2). We will show

$$\sqrt{p}\int_0^\infty \tilde{\jmath}_{m/2}(s)^p \, ds \to \infty \quad \text{as } p \to \frac{2}{m+1}, \tag{17}$$

$$\sqrt{2}\int_0^\infty \tilde{\jmath}_{m/2}(s)^2 \, ds < \sqrt{\pi}\,\sqrt{m/2+1}, \tag{18}$$

$$\exists p_0 \in \left(\frac{2}{m+1}, 2\right] : \sqrt{p_0}\int_0^\infty \tilde{\jmath}_{m/2}(s)^{p_0} \, ds = \sqrt{p_0}\int_0^\infty g(s)^{p_0} \, ds = \sqrt{\pi}\,\sqrt{m/2+1}. \tag{19}$$

For large s the function $\tilde{\jmath}_{m/2}$ is asymptotically equal to $\frac{2^{(m+1)/2}}{\sqrt{\pi}}\Gamma(m/2+1)s^{-(m+1)/2}$. Therefore $\tilde{\jmath}_{m/2}(\cdot)^p$ is integrable for $p > 2/(m+1)$, and $\int_0^\infty \tilde{\jmath}_{m/2}(s)^p \, ds$ diverges as $p \to 2/(m+1)$; this is (17).

For inequality (18) evaluate the integral. We have

$$\begin{aligned}
&\sqrt{2}\int_0^\infty \tilde{\jmath}_{m/2}(s)^2 \, ds \\
&= \sqrt{2}\int_0^m |j_{m/2}(s)|^2 \, ds + \sqrt{2}\int_m^\infty \left(2^{(m+1)/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\left(s^2 - \frac{m^2}{4}\right)^{-1/4} s^{-m/2}\right)^2 ds \\
&\leq \sqrt{2}\int_0^\infty |j_{m/2}(s)|^2 \, ds + \sqrt{2}\int_m^\infty \left(2^{(m+1)/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\left(s^2 - \frac{m^2}{4}\right)^{-1/4} s^{-m/2}\right)^2 ds
\end{aligned}$$

$$= \sqrt{2}\int_0^\infty 2^m \Gamma\left(\frac{m}{2}+1\right)^2 \frac{J_{m/2}(s)^2}{s^m}\,ds$$
$$+ 2^{m+3/2}\frac{\Gamma(m/2+1)^2}{\pi}\int_m^\infty \left(s^2-\frac{m^2}{4}\right)^{-1/2} s^{-m}\,ds.$$

The first integral is evaluated by [6, 6.575 (2)] and then estimated by Lemma 4.4:

$$\sqrt{2}\int_0^\infty 2^m \Gamma(m/2+1)^2 \frac{J_{m/2}(s)^2}{s^m}\,ds = \sqrt{2}\,\sqrt{\pi}\frac{\Gamma(m/2+1)^2\Gamma(m)}{\Gamma(m+1/2)\Gamma(m/2+1/2)^2}$$
$$\le \sqrt{\pi}\frac{m+2}{m+1}\,\frac{\sqrt{m}}{\sqrt{2}}.$$

For the second summand, we estimate the integrand by $(s^2-\frac{m^2}{4})^{-1/2}s^{-m} \le \sqrt{4/3}\,s^{-m-1}$, which is true for $s \ge m$. Then use again Stirling's formula for $\Gamma(m/2+1)^2$ to get

$$2^{m+3/2}\frac{\Gamma(m/2+1)^2}{\pi}\int_m^\infty (s^2-m^2/4)^{-1/2}s^{-m}\,ds$$
$$\le 2^{m+3/2}\frac{\Gamma(m/2+1)^2}{\pi}\sqrt{\frac{4}{3}}\int_m^\infty s^{-m-1}\,ds$$
$$= 2^{m+3/2}\sqrt{\frac{4}{3}}\,\frac{\Gamma(m/2+1)^2}{\pi}m^{-(m+1)} \le \frac{\sqrt{2}^5}{\sqrt{3}}\exp\left(\frac{1}{3m}\right)\exp(-m).$$

It remains to show

$$\sqrt{\pi}\frac{m+2}{m+1}\,\frac{\sqrt{m}}{\sqrt{2}} + \frac{\sqrt{2}^5}{\sqrt{3}}\exp\left(\frac{1}{3m}\right)\exp(-m) \le \sqrt{\pi}\sqrt{m/2+1}. \tag{20}$$

This follows if we prove the stronger inequality

$$3.65\exp(-m) \le \sqrt{\pi}\left(\sqrt{m/2+1} - \frac{m+2}{m+1}\,\frac{\sqrt{m}}{\sqrt{2}}\right). \tag{21}$$

Note that $\exp(m)\big(\sqrt{m+2}-\frac{m+2}{m+1}\sqrt{m}\big) \ge \exp(m)\frac{1}{3}m^{-3/2}$, and $\exp(m)\frac{1}{3}m^{-3/2}$ is increasing in m. For $m=5$, inequality (21) is true, and so it is true for all $m \ge 5$. This proves (18).

Now (19) follows by the intermediate value theorem.

Thus we proved (N1) and (N2), so the Nazarov–Podkorytov Lemma gives the desired result. ∎

4.5. The case $m \in \{3,4\}$. The estimates made above by the simplification of $j_{m/2}$ in (12) are too rough for $m < 5$, since $j_{m/2}$ decreases too slowly for them to work. So we need a different approach here that involves numerical estimates. Therefore one has to treat the cases $m \in \{3,4\}$ separately. The idea is basically given in [4], and it is a generalization of [18].

With Lemma 4.10, we prove the original integral inequality from Theorem 4 for $m \in \{3,4\}$. Split the integral into two parts and estimate them separately:

$$\int_0^\infty |j_{m/2}(s)|^p\,ds = \int_0^{m/2+3} |j_{m/2}(s)|^p\,ds + \int_{m/2+3}^\infty |j_{m/2}(s)|^p\,ds.$$

For the first integral, we use the pointwise estimate Lemma 4.6 and then Lemma 4.10:

$$\begin{aligned}(22)\qquad &\int_0^{m/2+3} |j_{m/2}(s)|^p\,ds \\ &\qquad \le \frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{m/2+1}\left(1-\frac{3}{4}\,\frac{1}{p(m+4)}+\frac{105}{16}\,\frac{1}{2p^2(m+4)^2}\right).\end{aligned}$$

For the second integral, we estimate the integrand pointwise by Lemma 4.8. This gives

$$\begin{aligned}(23)\qquad &\int_{m/2+3}^\infty |j_{m/2}(s)|^p\,ds \\ &\le \left(2^{(m+1)/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\,\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \int_{m/2+3}^\infty s^{-(m+1)p/2}\,ds \\ &= \left(2^{(m+1)/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\,\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \frac{2}{(m+1)p-2}(m/2+3)^{1-(m+1)p/2}.\end{aligned}$$

With the estimates (22) and (23) of the two parts of the integral, it remains to prove the following inequality for $p \ge 2$ and $m \in \{3,4\}$:

$$\begin{aligned}&\frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{m/2+1}\left(1-\frac{3}{4}\,\frac{1}{p(m+4)}+\frac{105}{16}\,\frac{1}{2p^2(m+4)^2}\right) \\ &+\left(2^{(m+1)/2}\frac{\Gamma(m/2+1)}{\sqrt{\pi}}\,\frac{\sqrt{m+6}}{\sqrt[4]{12m+36}}\right)^p \frac{2}{(m+1)p-2}(m/2+3)^{1-(m+1)p/2} \\ &\le \frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{m/2+1}.\end{aligned}$$

Subtract $\frac{\sqrt{\pi}}{\sqrt{p}}\sqrt{m/2+1}$ from both sides. For $m=3$ this reads

$$\frac{\sqrt{5\pi/2}}{\sqrt{p}}\left(\frac{15}{224p^2}-\frac{3}{28p}\right)+\left(\frac{9}{\sqrt[4]{2}\sqrt{6}}\right)^p \frac{2}{4p-2}\left(\frac{9}{2}\right)^{1-2p} \le 0.$$

After multiplying by $p^{5/2}(4p-2)$ and simplifying, this reduces to showing

$$-p^2\frac{3}{7}\sqrt{\frac{5}{2}\pi}+p\frac{27}{56}\sqrt{\frac{5}{2}\pi}-\frac{30}{224}\sqrt{\frac{5}{2}\pi}+\left(\frac{4}{9\sqrt[4]{2}\sqrt{6}}\right)^p 9p^{5/2}\le 0. \tag{24}$$

The last summand of the left-hand side of (24) is decreasing in p for $p \ge 2$ and its value for $p = 2$ equals $32/27$. So we estimate the left-hand side of (24) by a quadratic function and get

$$-p^2\frac{3}{7}\sqrt{\frac{5}{2}\pi}+p\frac{27}{56}\sqrt{\frac{5}{2}\pi}-\frac{30}{224}\sqrt{\frac{5}{2}\pi}+\left(\frac{4}{9\sqrt[4]{2}\sqrt{6}}\right)^p 9p^{5/2}$$
$$\le -p^2\frac{3}{7}\sqrt{\frac{5}{2}\pi}+p\frac{27}{56}\sqrt{\frac{5}{2}\pi}-\frac{30}{224}\sqrt{\frac{5}{2}\pi}+\frac{32}{27}.$$

This function has its maximum at $p = 9/16$, so it is decreasing for $p \ge 2$. For $p = 2$ the value is $-\frac{99}{224}\sqrt{10}\sqrt{\pi}+32/27<0$. This proves the inequality.

For $m = 4$ this argument works analogously. ∎

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