

Dedekind Sums, Mean Square Value of L -Functions at $s = 1$ and Upper Bounds on Relative Class Numbers

by

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Summary. Explicit formulas for the quadratic mean value at $s = 1$ of the Dirichlet L -functions associated with the set X_f^- of the $\phi(f)/2$ odd Dirichlet characters mod f are known. They have been used to obtain explicit upper bounds for relative class numbers of cyclotomic number fields. Here we present a generalization of these results: we show that explicit formulas for quadratic mean values at $s = 1$ of Dirichlet L -functions associated with subsets of X_f^- can be obtained. As an application we use them to obtain explicit upper bounds for relative class numbers of imaginary subfields of cyclotomic number fields.

1. Introduction. We refer the reader to [Was, Chapters 3, 4 and 11] for more background details. Let $f > 2$ be an integer. Let X_f be the multiplicative group of the $\phi(f)$ Dirichlet characters mod f . Let $X_f^- = \{\chi \in X_f; \chi(-1) = -1\}$ be the set of the $\phi(f)/2$ odd Dirichlet characters mod f . If H is a subgroup of the multiplicative group $(\mathbb{Z}/f\mathbb{Z})^*$, we set

$$X_f^-(H) = \{\chi \in X_f^-; \chi|_H = 1\}.$$

Let $L(s, \chi)$ be the Dirichlet L -function associated with $\chi \in X_f$. The mean square value $M(f, H)$ of $L(1, \chi)$ as χ ranges in $X_f^-(H)$ is defined by

$$(1) \quad M(f, H) := \frac{1}{\#X_f^-(H)} \sum_{\chi \in X_f^-(H)} |L(1, \chi)|^2.$$

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We have (see [Lou94, Proposition 1])

$$(2) \quad L(1, \chi) = \frac{\pi}{2f} \sum_{a=1}^{f-1} \chi(a) \cot\left(\frac{\pi a}{f}\right) \quad (\chi \in X_f^-).$$

Let K be an imaginary abelian number field of degree $m = 2n$ and prime conductor $p \geq 3$, i.e. an imaginary subfield of the cyclotomic number field $\mathbb{Q}(\zeta_p)$ (Kronecker–Weber’s theorem). The Galois group $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is canonically isomorphic to the multiplicative cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$, and $H := \text{Gal}(\mathbb{Q}(\zeta_p)/K)$ is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of index m and order $(p-1)/m$. Notice that K being imaginary, we have $-1 \notin H$ and the set

$$X_K^- := X_p^-(H) := \{\chi \in X_p^-; \chi|_H = 1\}$$

is of cardinality n . Let K^+ be the maximal real subfield of K of degree n fixed by the complex conjugation. The class number h_{K^+} of K^+ divides the class number h_K of K . The *relative class number* of K is defined by $h_K^- = h_K/h_{K^+}$. Let w_K be the number of complex roots of unity of K . Let d_K and d_{K^+} be the absolute values of the discriminants of K and K^+ . Hence, $d_K/d_{K^+} = p^n$. Using the arithmetic-geometric mean inequality, we have

$$(3) \quad h_K^- = \frac{w_K}{(2\pi)^n} \sqrt{\frac{d_K}{d_{K^+}}} \prod_{\chi \in X_K^-} L(1, \chi) \leq w_K \left(\frac{pM(p, H)}{4\pi^2} \right)^{n/2}.$$

Up to now this approach has only been used for cyclotomic number fields, i.e. with the choice $H = \{1\}$. Using (2) and the orthogonality relations for characters we obtain (see [Wal])

$$(4) \quad M(p, \{1\}) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right) \quad (p \geq 3).$$

Hence, for $K = \mathbb{Q}(\zeta_p)$ the cyclotomic number field of prime conductor $p \geq 3$ we have

$$(5) \quad h_{\mathbb{Q}(\zeta_p)}^- \leq 2p \left(\frac{pM(p, \{1\})}{4\pi^2} \right)^{(p-1)/4} \leq 2p \left(\frac{p}{24} \right)^{(p-1)/4}$$

(see also [Lou94], [Met]).

The aim of the present paper is to develop this approach for proper imaginary subfields K of $\mathbb{Q}(\zeta_p)$ of degree $2 \leq 2n < p-1$. Since $w_K = 2$ and the mean square value of $L(1, \chi)$, $\chi \in X_p^-$, is asymptotic to $\pi^2/6$ by (4), we might expect to have bounds close to

$$(6) \quad M(p, H) \leq \pi^2/6 \quad \text{and} \quad h_K^- \leq 2(p/24)^{n/2}.$$

However, it is hopeless to expect such a universal mean square upper bound. Indeed, it is likely that there are infinitely many imaginary abelian number

fields of a given degree $m = 2n$ and prime conductors p for which

$$M(p, H) = \frac{1}{n} \sum_{\chi \in X_K^-} |L(1, \chi)|^2 \geq \left(\prod_{\chi \in X_K^-} L(1, \chi) \right)^{2/n} \gg (\log \log p)^2$$

(see e.g. [CK] and [MW]). Nevertheless, in Theorems 1 and 5, we manage to prove (6) for some non-cyclotomic number fields:

THEOREM 1. *Let $p \equiv 1 \pmod{6}$ be a prime integer. Let K be the imaginary subfield K of degree $(p-1)/3$ of the cyclotomic number field $\mathbb{Q}(\zeta_p)$. Let H be the only subgroup of order 3 of the multiplicative cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$. Then $\#X_p^-(H) = (p-1)/6$ and we have the mean square value formula (compare with (4))*

$$(7) \quad M(p, H) := \frac{6}{p-1} \sum_{\chi \in X_p^-(H)} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left(1 - \frac{1}{p}\right).$$

Hence, the expected bounds (6) hold true, i.e. $M(p, H) \leq \pi^2/6$ and we have the upper bound (compare with (5))

$$(8) \quad h_K^- \leq 2(p/24)^{(p-1)/6}.$$

2. Dedekind and Dedekind–Rademacher sums. For $c \in \mathbb{Z}$ and $d > 1$ an integer such that $\gcd(c, d) = 1$, the *Dedekind sum* is defined by

$$s(c, d) = \frac{1}{4d} \sum_{n=1}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right)$$

(see [Apo, Chapter 3, Exercise 11] or [RG, (26)]). It depends only on $c \pmod{d}$. We also set $s(c, 1) = 0$ for $c \in \mathbb{Z}$. Notice that $s(-c, d) = -s(c, d)$ and that $s(c^*, d) = s(c, d)$ whenever $cc^* \equiv 1 \pmod{d}$ (make the change of variables $n \mapsto nc$ in $s(c^*, d)$). Recall that

$$(9) \quad s(1, d) = \frac{(d-1)(d-2)}{12d} \quad (d \geq 1)$$

(see e.g. [Lou94, Lemma (a)(i)]) and that we have a reciprocity law for Dedekind sums (see e.g. [Apo, Theorem 3.7], [RG, (4)] or [Lou15, (7) and (9)])

$$(10) \quad s(c, d) + s(d, c) = \frac{c^2 + d^2 - 3cd + 1}{12cd} \quad (c, d \geq 1, \gcd(c, d) = 1).$$

For $b, c \in \mathbb{Z}$ and $d > 1$ such that $\gcd(b, d) = \gcd(c, d) = 1$, the *Dedekind–Rademacher sum* is defined by

$$s(b, c, d) = \frac{1}{4d} \sum_{n=1}^{d-1} \cot\left(\frac{\pi nb}{d}\right) \cot\left(\frac{\pi nc}{d}\right).$$

We also set $s(b, c, 1) = 0$. Notice that $s(c, d) = s(1, c, d) = s(c, 1, d)$ and that $s(b, c, d) = s(ab, ac, d)$ for any $a \in \mathbb{Z}$ with $\gcd(a, d) = 1$. For $b, c, d \geq 1$ and $\gcd(b, c) = \gcd(c, d) = \gcd(d, b) = 1$ we have a reciprocity law for Dedekind–Rademacher sums (see e.g. [Rad] or [BR])

$$(11) \quad s(b, c, d) + s(d, b, c) + s(c, d, b) = \frac{b^2 + c^2 + d^2 - 3bcd}{12bcd}.$$

3. A mean square value formula for L -functions. We have the following mean square value formula:

$$(12) \quad M(f, \{1\}) = \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} |L(1, \chi)|^2 = \frac{\pi^2}{6} \frac{\phi(f)}{f} \left(\prod_{p|f} \left(1 + \frac{1}{p} \right) - \frac{3}{f} \right)$$

(see [Lou94, Th. 2 and 3], [Lou11, Prop. 5], [Qi] and [Lou14] for generalizations). Here we prove a general result which when applied with the trivial subgroup $H = \{1\}$ gives $S_{\{1\}}(\delta) = 0$ for any divisor δ of f and (12):

THEOREM 2. *Let H be a subgroup of order d of the multiplicative group $(\mathbb{Z}/f\mathbb{Z})^*$, $f > 2$. Assume that $-1 \notin H$, which is the case if d is odd. Set*

$$S_H(\delta) := \sum_{1 \neq h \in H} s(h, \delta).$$

Then $\#X_f^-(H) = \phi(f)/(2d)$ and we have the mean square value formula

$$\begin{aligned} M(f, H) &:= \frac{2d}{\phi(f)} \sum_{\chi \in X_f^-(H)} |L(1, \chi)|^2 \\ &= \frac{\pi^2}{6} \frac{\phi(f)}{f} \left(\prod_{p|f} \left(1 + \frac{1}{p} \right) - \frac{3}{f} \right) + \frac{2\pi^2}{f} \sum_{\substack{\delta|f \\ \delta \neq f}} \frac{\mu(\delta)}{\delta} S_H(f/\delta). \end{aligned}$$

Proof. Let $H^+ = \langle -1, H \rangle$ be the subgroup of order $2d$ generated by -1 and H . For any abelian group G of order m its group of characters \hat{G} is also of order m and

$$\frac{1}{m} \sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} 1 & \text{if } g = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

Applying this to the quotient group $G = (\mathbb{Z}/f\mathbb{Z})^*/H$ of order $\phi(f)/d$ for which $\hat{G} = \{\chi \in X_f; \chi|_H = 1\}$ and to the quotient group $G^+ := (\mathbb{Z}/f\mathbb{Z})^*/H^+$ of order $\phi(f)/(2d)$ for which $\hat{G}^+ = \{\chi \in X_f; \chi|_{H^+} = 1\}$, we find that the set $X_f^-(H)$, which is equal to $\hat{G} \setminus \hat{G}^+$, is of cardinality

$\phi(f)/(2d)$, and

$$\frac{2d}{\phi(f)} \sum_{\chi \in X_{\bar{f}}(H)} \chi(x) = \begin{cases} +1 & \text{if } x \in H, \\ -1 & \text{if } -x \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \epsilon(a, b) &:= \frac{2d}{\phi(f)} \sum_{\chi \in X_{\bar{f}}(H)} \chi(a)\overline{\chi(b)} \\ &= \begin{cases} +1 & \text{if } \gcd(a, f) = \gcd(b, f) = 1 \text{ and } a = bh \text{ with } h \in H, \\ -1 & \text{if } \gcd(a, f) = \gcd(b, f) = 1 \text{ and } a = -bh \text{ with } h \in H, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $c > 1$ and $d \in \mathbb{Z}$ with $\gcd(c, d) = 1$, set

$$\tilde{s}(c, d) := \frac{1}{4d} \sum_{\substack{n=1 \\ \gcd(n,d)=1}}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right) = \sum_{\delta|d} \frac{\mu(\delta)}{\delta} s(c, d/\delta),$$

by the Möbius inversion formula and the convention that $s(c, 1) = 0$ for $c \in \mathbb{Z}$. Using (2), we obtain

$$\begin{aligned} M(f, H) &= \frac{\pi^2}{4f^2} \sum_{a=1}^{f-1} \sum_{b=1}^{f-1} \epsilon(a, b) \cot\left(\frac{\pi a}{f}\right) \cot\left(\frac{\pi b}{f}\right) \\ &= \frac{\pi^2}{2f^2} \sum_{h \in H} \sum_{\substack{b=1 \\ \gcd(b,f)=1}}^{f-1} \cot\left(\frac{\pi b}{f}\right) \cot\left(\frac{\pi hb}{f}\right) \\ &= \frac{2\pi^2}{f} \sum_{h \in H} \tilde{s}(h, f) \\ &= \frac{2\pi^2}{f} \sum_{h \in H} \sum_{\delta|f} \frac{\mu(\delta)}{\delta} s(h, f/\delta) \\ &= \frac{2\pi^2}{f} \sum_{\delta|f} \frac{\mu(\delta)}{\delta} s(1, f/\delta) + \frac{2\pi^2}{f} \sum_{1 \neq h \in H} \sum_{\delta|f} \frac{\mu(\delta)}{\delta} s(h, f/\delta). \end{aligned}$$

Since $s(h, 1) = 0$ for any h , we can restrict $\delta|f$ to $f \neq \delta|f$ in the second term of the last line. By (9) we have $s(1, d) = (d - 3 + 2/d)/12$ for $d \geq 1$ and

$$\frac{2\pi^2}{f} \sum_{\delta|f} \frac{\mu(\delta)}{\delta} s(1, f/\delta) = \frac{\pi^2}{6} \left(\sum_{\delta|f} \frac{\mu(\delta)}{\delta^2} - \frac{3}{f} \sum_{\delta|f} \frac{\mu(\delta)}{\delta} + \frac{2}{f^2} \sum_{\delta|f} \mu(\delta) \right).$$

Finally, since $\sum_{\delta|f} \mu(\delta)/\delta^2 = \prod_{p|f} (1 - 1/p^2)$, $\sum_{\delta|f} \mu(\delta)/\delta = \prod_{p|f} (1 - 1/p)$ and $\sum_{\delta|f} \mu(\delta) = 0$, the desired result follows. ■

COROLLARY 3. *Let d be an odd divisor of $p - 1$, where $p \geq 3$ is an odd prime number. Let $H = \langle h_0 \rangle$ be the only cyclic subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of order d , where h_0 is any element of order d in the multiplicative cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$. Then $\#X_p^-(H) = (p - 1)/(2d)$ and we have the mean square value formula*

$$\begin{aligned} M(p, H) &:= \frac{2d}{p - 1} \sum_{\chi \in X_p^-(H)} |L(1, \chi)|^2 \\ &= \frac{\pi^2}{6} \left(1 + \frac{24 \sum_{k=1}^{(d-1)/2} s(h_0^k, p) - 3}{p} + \frac{2}{p^2} \right). \end{aligned}$$

Proof. Notice that $\delta = 1$ is the only $\delta \neq p$ dividing p , and

$$S_H(p) = \sum_{1 \neq h \in H} s(h, p) = \sum_{k=1}^{d-1} s(h_0^k, p) = 2 \sum_{k=1}^{(d-1)/2} s(h_0^k, p),$$

using $h_0^{d-k} = h_0^{-k}$ and $s(c^*, p) = s(c, p)$ whenever $cc^* \equiv 1 \pmod{p}$. ■

4. Proof of Theorem 1. Let h_0 of order 3 be a generator of H . Corollary 3 applied with $d = 3$ gives

$$M(p, H) = \frac{\pi^2}{6} \left(1 + \frac{24s(h_0, p) - 3}{p} + \frac{2}{p^2} \right).$$

Then the lemma below gives the desired result (we thank Abdelmejid Bayad for having sent us its present proof using Dedekind–Rademacher sums):

LEMMA 4. *Assume that $f = a^2 + ab + b^2 > 3$, where $a \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a, b) = 1$. Then $\gcd(a, f) = \gcd(b, f) = 1$, $h_0 = a/b$ is of order 3 in the multiplicative group $(\mathbb{Z}/f\mathbb{Z})^*$ and*

$$s(h_0, f) = s(h_0^2, f) = \frac{f - 1}{12f}.$$

Consequently, if $p \equiv 1 \pmod{6}$ is a prime integer then for any $h \in (\mathbb{Z}/p\mathbb{Z})^$ of order 3 in this group we have $s(h, p) = \frac{p-1}{12p}$.*

Proof. Assume that $a \geq 1$. Then $f \equiv b^2 \pmod{a}$ yields $s(b, f, a) = s(b, b^2, a) = s(1, b, a) = s(b, a)$, and $f \equiv a^2 \pmod{b}$ gives $s(f, a, b) = s(a^2, a, b) = s(a, 1, b) = s(a, b)$. The reciprocity law for Dedekind sums yields

$$s(b, f, a) + s(f, a, b) = s(b, a) + s(a, b) = \frac{a^2 + b^2 - 3ab + 1}{12ab}$$

and the reciprocity for the Dedekind–Rademacher sums implies

$$s(a, b, f) + s(b, f, a) + s(f, a, b) = \frac{a^2 + b^2 + f^2 - 3abf}{12abf}.$$

Hence, we finally obtain

$$\begin{aligned} s(h_0, f) = s(a, b, f) &= \frac{a^2 + b^2 + f^2 - 3abf}{12abf} - \frac{a^2 + b^2 - 3ab + 1}{12ab} \\ &= \frac{f - 1}{12f}. \end{aligned}$$

Now, assume that $a \leq -1$. Then

$$s(b, f, -a) + s(f, -a, b) = s(b, -a) + s(-a, b) = \frac{a^2 + b^2 + 3ab + 1}{-12ab}$$

and

$$s(-a, b, f) + s(b, f, -a) + s(f, -a, b) = \frac{a^2 + b^2 + f^2 + 3abf}{-12abf}.$$

Hence,

$$s(a, b, f) = -s(-a, b, f) = -\left(\frac{a^2 + b^2 + f^2 + 3abf}{-12abf} - \frac{a^2 + b^2 + 3ab + 1}{-12ab} \right)$$

and $s(h_0, f) = s(a, b, f) = \frac{f-1}{12f}$.

Since $h_0^2 = h_0^*$, i.e. $h_0^2 h_0 = h_0^3 \equiv 1 \pmod{f}$, we have $s(h_0^2, f) = s(h_0, f)$.

Finally, let $p \equiv 1 \pmod{6}$ be a prime integer. Then p splits completely in the imaginary quadratic number field $L = \mathbb{Q}(\sqrt{-3})$. Since the class number of L is equal to 1, there is an algebraic integer $\alpha = a + b(1 + \sqrt{-3})/2$ in L , $a, b \in \mathbb{Z}$, such that $p = N_{L/\mathbb{Q}}(\alpha) = a^2 + ab + b^2$. Clearly, $\gcd(a, b) = 1$ and we may assume that $b \geq 1$. Then $h_0 = a/b$ and h_0^2 are the only elements of order 3 in the multiplicative cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$. ■

5. Some imaginary abelian number fields of conductor $p \equiv 1 \pmod{5}$ and degree $(p-1)/5$. We conclude this paper by explaining why it is probably hopeless to find a statement as neat as Theorem 1 for subgroups H of $(\mathbb{Z}/p\mathbb{Z})^*$ of odd order d greater than 3. Even for $d = 5$ and $p \equiv 1 \pmod{10}$ we could not find a generalization of Theorem 1, i.e. a formula for $M(p, H)$, where H is the only subgroup of order 5 of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$. These prime integers $p \equiv 1 \pmod{10}$ are the prime integers that split completely in the cyclotomic number field $\mathbb{Q}(\zeta_5)$ of class number one, therefore they are the prime integers $p \neq 5$ of the form

$$p = P(a, b, c, d) := N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(a - b\zeta_5 - c\zeta_5^2 - d\zeta_5^3), \quad a, b, c, d \in \mathbb{Z}.$$

The first problem is that we could not find a closed formula for an element h_0 of order 5. Even for the prime integers $p \neq 5$ of the form

$$p = N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(a - b\zeta_5) = a^4 + a^3b + a^2b^2 + ab^3 + b^4,$$

for which $h_0 = a/b$ is of order 5 in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$, we could not find a generalization of Lemma 4, i.e. closed formulas for $s(a, b, f)$ and $s(a^2, b^2, f)$. At least, for $b = 1$, i.e. for the prime integers $p \neq 5$ of the form $p = N_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(a - \zeta_5) = a^4 + a^3 + a^2 + a + 1$ we have:

THEOREM 5. *Let $p \neq 5$ be a prime of the form*

$$p = a^4 + a^3 + a^2 + a + 1,$$

e.g. $(a, p) \in \{(-2, 11), (2, 31), (-3, 61), (-5, 521), (7, 2801), (-10, 9091)\}$. Then $p \equiv 1 \pmod{10}$. Let H be the only subgroup of order 5 of the multiplicative cyclic group $(\mathbb{Z}/p\mathbb{Z})^$ and let K be the imaginary subfield of degree $(p-1)/5$ of the cyclotomic number field $\mathbb{Q}(\zeta_p)$. Then $\#X_p^-(H) = (p-1)/10$ and we have the mean square value formula (compare with (4) and (7))*

$$(13) \quad M(p, H) := \frac{10}{p-1} \sum_{\chi \in X_p^-(H)} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left(1 + \frac{2a(a+1)^2 - 1}{p} \right).$$

Hence, for $a \leq -2$ the expected bounds (6) hold true, i.e. $M(p, H) \leq \pi^2/6$ and (compare with (5) and (8)) $h_K^- \leq 2(p/24)^{(p-1)/10}$.

Proof. Applying Corollary 3 with $d = 5$ and $h_0 = a$, we have

$$M(p, H) = \frac{\pi^2}{6} \left(1 + \frac{24(s(a, p) + s(a^2, p)) - 3}{p} + \frac{2}{p^2} \right).$$

By Lemma 6, the desired result follows. ■

LEMMA 6. *Let $q \geq 3$ be an odd prime. For $a \in \mathbb{Z}$, $a \neq -1, 0, 1$, set $f := (a^q - 1)/(a - 1) = a^{q-1} + a^{q-2} + \dots + a + 1 > 1$, an odd integer. Assume that $f \neq q$. Then a is of order q in the multiplicative group $(\mathbb{Z}/f\mathbb{Z})^*$, and*

$$s(a, f) = \frac{(f-1)(f-a^2-1)}{12af} \quad \text{and} \quad s(a^2, f) = \frac{(f-1)^2 + a^2(a^2-f)}{12a^2f}.$$

Proof. Using (10) we get

$$s(a, f) = \frac{a^2 + f^2 - 3af + 1}{12af} - s(f, a).$$

Using $f \equiv 1 \pmod{a}$ and (9) we get $s(f, a) = s(1, a) = (a-1)(a-2)/(12a)$. The desired result for $s(a, f)$ follows.

In the same way, $a^2 \equiv 1 \pmod{a+1}$ implies

$$s(a^2, a+1) = s(1, a+1) = \frac{a(a-1)}{12(a+1)},$$

and $f \equiv a + 1 \pmod{a^2}$ and (10) yield

$$s(f, a^2) = s(a + 1, a^2) = \frac{(a + 1)^2 + a^4 - 3(a + 1)a^2 + 1}{12(a + 1)a^2} - s(a^2, a + 1),$$

$s(f, a^2) = -(a^2 - 1)/(6a^2)$ and

$$s(a^2, f) = \frac{a^4 + f^2 - 3a^2f + 1}{12a^2f} - s(f, a^2).$$

The desired result for $s(a^2, f)$ follows. ■

6. Conclusion: two open problems

1. For the cyclotomic fields of prime conductors we have the following improvement on (5):

$$(14) \quad h_{\mathbb{Q}(\zeta_p)}^- \leq 2p(p/32)^{(p-1)/4}$$

(see [Feng] and [Lou11]). It would be worth proving a similar improvement on (8), i.e. proving that if K is an imaginary subfield of degree $(p - 1)/3$ of the cyclotomic number field $\mathbb{Q}(\zeta_p)$, where $p \equiv 1 \pmod{6}$, then

$$h_K^- \leq 2(p/32)^{(p-1)/6}.$$

2. As explained in Section 5, it seems hopeless for a given odd $d > 3$ to obtain an explicit formula for $M(p, H)$ that would hold true for all the prime integers $p \equiv 1 \pmod{2d}$, where H is the only subgroup of order d of the group $(\mathbb{Z}/p\mathbb{Z})^*$. Nevertheless, it would be worth obtaining an asymptotic $M(p, H) = (\pi^2/6)(1 + o(1))$.

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