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CELL STRUCTURES AND COMPLETELY METRIZABLE SPACES AND THEIR MAPPINGS

BY

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Dedicated to Professor Jerzy Mioduszewski

Abstract. A (combinatorial) graph is a discrete set of vertices together with a set of edges. We define cell structures as inverse sequences of graphs with mild convergence conditions and we define cell mappings between cell structures. These cell structures yield completely metrizable spaces as perfect images of closed subsets of countable products of discrete spaces. Cell mappings between cell structures define the continuous mappings between the corresponding spaces. In this way we can envision a continuous mapping between metric spaces as the limit of a sequence of discrete approximations. Thus, cell structures provide a kind of bridge between discrete and continuous mathematics.

1. Introduction. Properties of topological spaces depend on both geometry and set theory. We develop completely metrizable spaces from a geometric point of view. Every complete metric space X admits a rapidly decreasing defining sequence of locally finite open covers. The sequence of 0-skeletons of the nerves of these covers naturally forms an inverse sequence with bonding maps defined by inclusions. We call this inverse sequence a (complete) cell structure. The inverse limit of this cell structure is a complete 0-dimensional metric space G_∞. Using the 1-skeletons of the nerves of these covers we define an upper semicontinuous equivalence relation with compact equivalence classes on G_∞ whose quotient space is homeomorphic to the original space X. Mappings between complete metric spaces correspond to families of cell mappings between the corresponding cell structures. We give examples of easily computable cell structures as well as some classes of compacta given by particular classes of cell structures.

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The idea of using inverse sequences to describe compact metric spaces goes back to P. S. Alexandroff [1] (see also Lefschetz [9]). Freudenthal [7] considered mappings of inverse sequences of polyhedra, and showed that if X is a compact metric space then it is homeomorphic to the inverse limit of a sequence of polyhedra whose dimensions are bounded by the dimension of X. Eilenberg and Steenrod [4] gave a detailed exposition of inverse limits. In the 1950's Mardešić [10] and Pasynkov [18] showed independently that Freudenthal's theorem fails for compact Hausdorff spaces.

Mioduszewski [17] (see also Brown [2]) showed that mappings between limits of inverse sequences of compact polyhedra are induced by sequences of mappings between the coordinate spaces in the inverse sequences. These mappings between coordinate spaces in general commute only approximately with bonding mappings. For non-compact spaces inverse systems do not work well at all.

Mardešić [11] introduced resolutions to deal with mappings of arbitrary spaces. Mardešić and Watanabe [16] introduced approximate resolutions of arbitrary spaces and mappings.

In a different direction Kopperman and Wilson [8] showed how to obtain compact Hausdorff spaces as the Hausdorff reflections of inverse limits of inverse systems of finite T_0 spaces. Above we have very thinly sketched the development of spectral representation of spaces and mappings. The history is huge. The interested reader may want to look at [3], [12], [13], [14], [15], [19], [20] and [21] for a more detailed picture.

We describe completely metrizable spaces and mappings between such spaces by methods which lie somewhere in between the two approaches mentioned above. Interestingly, we use only some of the 1-dimensional information given by inverse systems. We restrict ourselves in this paper to completely metrizable spaces because here relatively simple inverse sequences suffice and this is also likely the level of generality required by non-experts in topology. We shall present the extensions of this work to the compact Hausdorff and other more general cases elsewhere.

When we refer to dimension, we shall always mean covering dimension.

The theory of cell structures orginated in the late 1980's at the Chair of Topology directed by Professor Jerzy Mioduszewski at the Institute of Mathematics in Katowice, University of Silesia.

2. Preliminaries. By a *completely metrizable space* we shall mean a space that admits an equivalent complete metric. By a *graph* we mean an ordered pair (G, r) where G is a discrete set and r is a symmetric and reflexive relation on G. The non-singleton elements of r denote the edges of G. The points of G will be called *cells*. Cells a and b will be said to be *adjacent* if $(a, b) \in r$. Let

$$\mathrm{st}_r(a) = \{b \in G \mid (a,b) \in r\} \quad \text{and} \quad \mathrm{st}_r^2(a) = \bigcup\{\mathrm{st}_r(b) \mid b \in \mathrm{st}_r(a)\}$$

for $a \in G$. Let $\mathbb{N}$ denote the set of natural numbers $\{0, 1, \ldots\}$.

We first define an *inverse sequence* $\{(G_i, g_i^j)\}$ of graphs. Let $\{(G_i, r_i)\}_{i\in\mathbb{N}}$ be a sequence of graphs and $\{g_i^j\}_{j\geq i}$ be a family of continuous functions $g_i^j : G_j \to G_i$ satisfying

(i) g_i^i is the identity on G_i,
(ii) $g_i^k = g_i^j \circ g_j^k$ for $i < j < k$, and
(iii) if $(a,b) \in r_{i+1}$ then $(g_i^{i+1}(a), g_i^{i+1}(b)) \in r_i$.

Condition (iii) is the continuity condition. The points $a \in G_i$ are called the *cells of the ith generation* or *cells of the ith degree*, and we write $\deg(a) = i$. If $a \in G_{i+1}$ and $g_i^{i+1}(a) = b$ then a is said to be a *direct subcell* of b.

Let $\prod_{i\in\mathbb{N}} G_i$ be the topological product, where each G_i has the discrete topology. An element $x = (x(0), x(1), \ldots) \in \prod_{i\in\mathbb{N}} G_i$ such that $x(i) = g_i^{i+1}(x(i+1))$ for each i is called a *thread*. The set of all threads is denoted by

$$G_\infty \equiv \varprojlim(G_i, g_i^j)$$

and is the inverse limit of the inverse sequence

$$(*) \qquad G_0 \xleftarrow{g_0^1} G_1 \xleftarrow{g_1^2} G_2 \leftarrow \cdots .$$

For each i let $g_i : G_\infty \to G_i$ be the ith coordinate projection. For $a \in G_i$ let

$$\langle a \rangle = \{x \in G_\infty \mid x(i) = a\} = g_i^{-1}(a).$$

For $A \subset G_i$ let

$$\langle A \rangle = \bigcup\{\langle a \rangle \mid a \in A\}.$$

The family

$$\{\langle a(i) \rangle \mid a \in G_\infty,\ i \in \mathbb{N}\}$$

is a basis of closed and open sets for G_∞.

Our first two propositions are well-known facts about inverse limits.

PROPOSITION 2.1. *G_∞ is a closed subset of $\prod_{i\in\mathbb{N}} G_i$ where each G_i has the discrete topology and $\prod_{i\in\mathbb{N}} G_i$ has the product topology. Hence, G_∞ is a 0-dimensional, completely metrizable space.*

Proof. See [6, Proposition 2.5.1 and Theorem 4.3.12] and [5, Theorem 4.1.25]. ∎

COROLLARY 2.2. *If each G_i is countable then G_∞ is homeomorphic to a closed subspace of the irrational numbers.*

Proof. The space of irrational numbers is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. ∎

We define a relation $\sim$ on G_∞ by setting $x \sim y$ in G_∞ if and only if $(x(i), y(i)) \in r_i$ for each i. Then $\sim$ is reflexive and symmetric. We check that $\sim$ has a closed graph. The relation $\sim$ is equal to $\bigcap_{i \in N} R_i$, where $R_i = \{(x, y) \in G_\infty \times G_\infty \mid (x(i), y(i)) \in r_i\}$. Now R_i is closed since r_i is closed in the discrete space $G_i \times G_i$.

Hence, if $\sim$ is also transitive, we can define the natural quotient map $\pi : G_\infty \to G^*$ where $G^* = G_\infty/{\sim}$ has the quotient topology. In this case π is a continuous function by definition. For x in G_∞ let

$$[x] = \{y \in G_\infty \mid x \sim y\} = \pi^{-1}(\pi(x)).$$

3. Cell structures. We carry over the notation of Section 2. In order to prove theorems about the space G^* induced by the relation $\sim$ defined on G_∞ we need to impose additional conditions on the graphs G_i and on the bonding maps g_i^j. We begin with some definitions.

Cells $a \in G_m$ and $b \in G_n$ are said to be *close* if $(g_k^m(a), g_k^n(b)) \in r_k$ for $k = \min\{m, n\}$. A sequence $u = \{u(j)\}$ of cells in $\bigcup G_i$ is said to be a *Cauchy sequence* if

(α) $\lim_{j\to\infty} \deg(u(j)) = \infty$, and
(β) $u(i)$ and $u(j)$ are close for all i and j sufficiently large.

The sequence $\{u(i)\}$ is said to *converge* to a thread $x \in G_\infty$ if $\lim_{i\to\infty} \deg(u(i)) = \infty$ and $x(i)$ and $u(j)$ are close for all i and for all sufficiently large j. Note that $\{u(i)\}$ may converge to different threads x and y. We introduce condition (iv) below to ensure in this case that $x \sim y$ and that $\sim$ is an equivalence relation.

Consider the following conditions on the inverse sequence $(*)$ of graphs:

(iv) for each thread x and each natural number i there exists an integer j greater than or equal to i such that $g_i^j(\mathrm{st}_{r_j}^2(x(j))) \subset \mathrm{st}_{r_i}(x(i))$,
(v) for each thread x and each natural number i there exists an integer j greater than or equal to i such that $g_i^j(\mathrm{st}_{r_j}(x(j)))$ is finite,
(vi) each Cauchy sequence of cells converges.

An inverse sequence of graphs satisfying (iv)–(v) is called a *cell structure*. If a cell structure also satisfies (vi), it is called a *complete cell structure*. We note that proofs in this section do not require property (vi). It only becomes important in Section 4.

LEMMA 3.1. *If $(*)$ is a cell structure then $\sim$ is an equivalence relation on G_∞.*

Proof. Symmetry and reflexivity of $\sim$ have already been noted. Now suppose x, y and z are threads such that $x \sim y \sim z$. Let $i \in \mathbb{N}$ be fixed. Then $(x(j), y(j)) \in r_j$ and $(y(j), z(j)) \in r_j$ for each j. So $z(j) \in \mathrm{st}_{r_j}^2(x(j))$ for each j. By (iv) there exists j large enough that $z(i) = g_i^j(z(j)) \in \mathrm{st}_{r_i}(x(i))$. So $(x(i), z(i)) \in r_i$, and hence $x \sim z$. We have proved that $\sim$ is transitive. ∎

A similar argument shows that if a Cauchy sequence converges to different threads x and y then $x \sim y$.

If $(*)$ is a cell structure, we let G^* denote the quotient space $G_\infty/{\sim}$ with the quotient topology. We shall call G^* the space *determined by the cell structure.*

LEMMA 3.2. *Let*

$$A_0 \stackrel{g_0^1}{\longleftarrow} A_1 \stackrel{g_1^2}{\longleftarrow} A_2 \leftarrow \cdots$$

be an inverse sequence of non-empty discrete spaces. If for each i, $g_i^j(A_j)$ is finite for sufficiently large j then A_∞ is a non-empty, compact, metrizable space.

Proof. Let $i \in \mathbb{N}$. We have $A_i \supset g_i^{i+1}(A_{i+1}) \supset g_i^{i+2}(A_{i+2}) \supset \cdots$. Since $g_i^j(A_j)$ is finite for each i and each sufficiently large j, the above sequence stabilizes and there is a finite set B_i in A_i such that $g_i^j(A_j) = B_i$ for each sufficiently large j. Then $A_\infty = B_\infty$ is a non-empty (by [6, Exercise 2.5.A(a)]), compact, metrizable space. ∎

LEMMA 3.3. *If $(*)$ is a cell structure then for each x in G_∞, $[x]$ is a non-empty, compact, metrizable space.*

Proof. By (v) the result follows immediately from Lemma 3.2. ∎

Recall that a mapping is *perfect* if it is closed and has compact fibers.

PROPOSITION 3.4. *Each perfect image of a closed subset of a countable product of discrete spaces is a completely metrizable space.*

Proof. By [6, Theorem 4.3.12] a countable product of discrete spaces is completely metrizable. A closed subset of a completely metrizable space is completely metrizable by [6, Theorem 4.3.11].

By [6, Theorem 4.4.15] the perfect image of a metrizable space is metrizable, and by [6, Theorem 3.9.10] the perfect image of a Čech complete space is Čech complete. By [6, Theorem 4.3.26] the notions "topologically complete" and "Čech complete" are equivalent for metrizable spaces. ∎

REMARK 3.5. In the second part of this section and in Section 4 we shall prove the converse of Proposition 3.4, i.e. each completely metrizable space is a perfect image of a closed subset of a countable product of discrete spaces.

THEOREM 3.6. *If $(*)$ is a cell structure then $\pi : G_\infty \to G^*$ is a perfect mapping and G^* is a completely metrizable space.*

Proof. If $x \in G_\infty$ then $\pi^{-1}(\pi(x)) = [x]$, which is compact by Lemma 3.3. Let A be a closed set in G_∞ and let $x \in \mathrm{cl}(\pi^{-1}(\pi(A)))$. For each i,

$\langle x(i)\rangle$ is a basic neighbourhood of x, so $\pi^{-1}(\pi(A))\cap\langle x(i)\rangle\neq\emptyset$. For each i let $A_i=\{a\in \mathrm{st}_{r_i}(x(i)) \mid \langle a\rangle\cap A\neq\emptyset\}$. By (v) the sequence

$$A_0 \xleftarrow{g_0^1} A_1 \xleftarrow{g_1^2} A_2 \leftarrow \cdots$$

satisfies the hypothesis of Lemma 3.2, so A_∞ is non-empty. Let $y\in A_\infty$. Then $y(i)\in A_i$ and $\langle y(i)\rangle\cap A\neq\emptyset$ for each i. Hence, $y\in A$ since A is closed and since A meets every basic neighbourhood of y. Since $y(i)\in \mathrm{st}_{r_i}(x(i))$ for each i, we have $x\sim y$, and so $x\in\pi^{-1}(\pi(A))$. Thus, $\pi^{-1}(\pi(A))$ is a closed set. By [6, Corollary 2.4.10], π is a closed mapping.

Since π is a closed mapping and has compact fibres, it is perfect. By Proposition 3.4, the perfect image G^* of the completely metrizable space G_∞ is completely metrizable. ■

Corollary 3.7. *If $(*)$ is a cell structure and each G_i is at most countable then G^* is a separable and completely metrizable space.*

We shall need the following two facts:

Proposition 3.8. *If $(*)$ is a cell structure then*

$$\{G^*\setminus\pi(G_\infty\setminus\langle A\rangle) \mid A\subset G_i,\ i\in\mathbb{N}\}$$

is a basis of open sets for the topology on G^*.

Proof. Let $p\in G^*$ and let U be an open neighbourhood of p in G^*. Now, $\pi^{-1}(U)$ is open in G_∞ and contains the compact set $\pi^{-1}(p)$. For each x in $\pi^{-1}(p)$ there exists a positive integer i_x such that $\langle x(i_x)\rangle\subset\pi^{-1}(U)$. By compactness of $\pi^{-1}(p)$ there exists a finite subcover $\mathcal{U}=\{\langle x_1(i_{x_1})\rangle,\ldots,\langle x_n(i_{x_n})\rangle\}$ of $\pi^{-1}(p)$. Let $k=\max\{i_{x_1},\ldots,i_{x_n}\}$. Now

$$\mathcal{V}=\{\langle a\rangle \mid a\in G_k,\ \langle a\rangle\subset\pi^{-1}(U) \text{ and } \langle a\rangle\cap\pi^{-1}(p)\neq\emptyset\}$$

is a cover of $\pi^{-1}(p)$ by pairwise disjoint closed and open sets. Note that $\mathcal{V}$ refines $\mathcal{U}$. Since $\pi^{-1}(p)$ is compact and $\mathcal{V}$ is pairwise disjoint, $\mathcal{V}$ is finite. Let $V=\bigcup\mathcal{V}$. Further, $G_\infty\setminus V$ is both open and closed in G_∞, covers $G_\infty\setminus\pi^{-1}(U)$ and misses $\pi^{-1}(p)$. Hence, $\pi(G_\infty\setminus\langle A\rangle)$ misses p in G^*, is closed in G^* and contains $G^*\setminus U$. So $p\in G^*\setminus\pi(G_\infty\setminus V)\subset U$ and $G^*\setminus\pi(G_\infty\setminus V)$ is open. ■

Lemma 3.9. *Let $(*)$ be a cell structure. A Cauchy sequence $u=(u_j)$ of cells in $\bigcup G_i$ is convergent if and only if for each i there exists $k\geq i$ such that*

$$A_i=\{g_i^{\deg(u(j))}(u(j)) \mid \deg(u(j))\geq k\}$$

is a finite set in G_i.

Proof. First assume that each A_i is finite. Without loss of generality, it may be assumed that $u(i)$ and $u(j)$ are close for all i and j. For each i, $g_i^{i+1}(A_{i+1})\subset A_i$, so $A_\infty=\varprojlim(A_i,g_i^j)$ is not empty. Let $x\in A_\infty$. For each i, any two elements of A_i are adjacent since bonding maps preserve adjacency. Hence, for each i, we have $u(i) \in \mathrm{st}^2_{r_{\deg(u(i))}}(x(\deg(u(i))))$. By (iv), $u(i) \in \mathrm{st}_{r_{\deg(u(i))}}(x(\deg(u(i))))$. So u converges to x.

If the Cauchy sequence u converges to $x \in G_\infty$ then each A_i is finite by (v) and property (α) of a Cauchy sequence.

Clearly, if $(*)$ is an inverse sequence of graphs satisfying (iv) and (v) and such that each element of G_i for each i has only finitely many adjacent elements, then $(*)$ also satisfies (vi), and hence is a complete cell structure. ∎

4. Completely metrizable spaces from cell structures. In Theorem 3.6 it was shown that a cell structure determines a completely metrizable space G^*. In this section we prove the converse: every topologically complete metrizable space is obtainable via Theorem 3.6 from a complete cell structure. In particular, we show that a defining sequence of closed (resp. open) locally finite covers with meshes converging to zero of a complete metric space X determines a complete cell structure $(*)$ such that G^* is homeomorphic to X.

Let $\mathcal{F}_i$ be closed locally finite covers of a topological space X such that $\mathcal{F}_{i+1}$ is refinement of $\mathcal{F}_i$ and each member of $\mathcal{F}_i$ is non-empty for each i. The pairs $(\mathcal{F}_i, r_i)$ are graphs where

$$r_i = \{(F,G) \in \mathcal{F}_i \times \mathcal{F}_i \mid F \cap G \neq \emptyset\}$$

and each $\mathcal{F}_i$ has the discrete topology. Define functions $g_i^{i+1} : \mathcal{F}_{i+1} \to \mathcal{F}_i$ such that $F \subset g_i^{i+1}(F)$ for $F \in \mathcal{F}_{i+1}$. Define $g_i^j = g_i^{i+1} \circ \cdots \circ g_{j-1}^j$ for $j > i$ and let g_i^i be the identity function on $\mathcal{F}_i$. Then we have an inverse sequence $(*)$ as before.

Theorem 4.1. *Let X be a T_1 space and let $\{\mathcal{F}_i\}$ be a sequence of closed locally finite covers of X such that $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$, the intersection of the elements of each thread in $\mathcal{F}_\infty$ is a non-empty set in X and for each open set $U \subset X$ and $p \in U$ there exists i such that each element of $\mathcal{F}_i$ containing p is contained in U. Then using the relations r_i and mappings g_i^j defined as above one can form a cell structure which determines the space X. In particular, X is completely metrizable.*

Proof. Let $x \in \mathcal{F}_\infty$. Then $\bigcap_{i\ge1} x(i) \neq \emptyset$. Let $p \in \bigcap_{i\ge1} x(i)$. Since $\mathcal{F}_i$ is locally finite, there is an open set U in X such that $p \in U$ and $U \subset \bigcup\{F \in \mathcal{F}_i \mid p \in F\}$. Then there exists $j \ge i$ such that each element of $\mathcal{F}_j$ to which p belongs is contained in U. Hence, each element $F \in \mathrm{st}_{r_j}(x(j))$ has non-empty intersection with U. Consequently, $g_i^j(F) \cap U \neq \emptyset$. This means that the set $g_i^j(\mathrm{st}_{r_j}(x(j)))$ is finite. Hence, (v) is satisfied.

Let $x \in \mathcal{F}_\infty$ and let $p \in \bigcap_{i\ge1} x(i)$. Fix $i \in \mathbb{N}$. Then $G = \bigcup\{F \in \mathcal{F}_i \mid p \notin F\}$ is a closed set which does not contain p as $\mathcal{F}_i$ is a closed locally finite cover. Hence, there exist $k \ge i$ such that each set $F \in \mathcal{F}_k$ containing p is contained in $X \setminus G$. This means that G cannot be joined to p by a single element $F \in \mathcal{F}_k$. Now let $H = \bigcup\{F \in \mathcal{F}_k \mid p \notin F\}$ and let $l \geq k$ be such that the set H cannot be joined to p by a single element from $\mathcal{F}_l$. So G cannot be reached from p by a chain of two elements from $\mathcal{F}_l$. Similarly, there exist $j \geq l$ such that the set G cannot be joined to p by a chain of three elements from $\mathcal{F}_j$. Consequently, for $F \in \mathrm{st}^2_{r_j}(x(j))$ we have $F \cap G = \emptyset$. So $F \subset g^j_i(F) \not\subset G$, which implies that $p \in g^j_i(F)$. Hence, $g^j_i(\mathrm{st}^2_{r_j}(x(j))) \subset X \setminus G \subset \mathrm{st}_{r_i}(x(i))$. We have shown that (iv) is also satisfied. Thus $\{\mathcal{F}_i, r_i\}$ form a cell structure.

Define $\varphi : \mathcal{F}^* \to X$ by assigning $\varphi(\pi(x)) = p$ to each $x \in \mathcal{F}_\infty$, where p is a point in $\bigcap_{i \geq 1} x(i)$. We need to prove φ is a homeomorphism.

The point p is unique since if $q \in X$ with $q \neq p$ then as above there exists j such that $\mathrm{st}^2_{r_j}(x(j)) \subset X \setminus \{q\}$. So $x \not\sim y$ for any $y \in \mathcal{F}_\infty$ such that $\{q\} = \bigcap y(i)$ and the function φ is well-defined and injective.

Let $p \in X$. Define $\mathcal{A}_i = \{F \in \mathcal{F}_i \mid p \in F\}$ for each i. Let U be an open subset of X containing p such that U intersects only finitely many $F \in \mathcal{F}_i$, and let $j \geq i$ be such that $\bigcup \mathcal{A}_j \subset U$. Then $g^j_i(\mathcal{A}_j)$ is a finite (non-empty) set. This means that the assumptions of Lemma 3.2 are satisfied. Consequently, there exists $x \in \mathcal{A}_\infty$. Since $p \in \bigcap_i x(i)$, we have $p = \varphi(\pi(x))$. Hence, φ is a surjection.

Let $U \subset X$ be an open set and let $p = \varphi(\pi(x)) \in U$. Then there exists i such that each set $F \in \mathcal{F}_i$ containing p is contained in U. Now $x \in \mathcal{F}_\infty$ and $\varphi(\pi(x)) \in \bigcap_j x(j)$ (where $\pi : \mathcal{F}_\infty \to \mathcal{F}^*$ is the quotient map). Hence, $\varphi(\pi(\langle F \rangle)) \subset F$ for $F \in \mathcal{F}_i$. Let $F_x = \{F \in \mathcal{F}_i \mid p \in F\}$. Then $\varphi \circ \pi(\langle F_x \rangle) \subset U$. So $\varphi \circ \pi$ is continuous. Since π is a quotient map, φ is continuous by [6, 2.4.2].

Let $G \subset \mathcal{F}^*$ be a closed set. It remains to prove that $\varphi(G)$ is closed. Let $p \in X \setminus \varphi(G)$. Suppose that for each i there exists $F \in \mathcal{F}_i$ containing p and having non-empty intersection with $\varphi(G)$. Let $\mathcal{A}_i = \{F \in \mathcal{F}_i \mid p \in F, F \cap \varphi(G) \neq \emptyset\}$. The sets $\mathcal{A}_j$ satisfy Lemma 3.2. Hence there exists $x \in \mathcal{A}_\infty$. Then $p \in \bigcap_i x(i)$ and $x(i) \cap \varphi(G) \neq \emptyset$ for all i. Let $\mathcal{B}_i = \{y(i) \mid y \in \pi^{-1}(G), y(i) \cap x(i) \neq \emptyset\}$. By (iv), the $\mathcal{B}_j$ are non-empty sets satisfying Lemma 3.2. Thus, there exists $z \in \mathcal{B}_\infty$. For each i the intersection $z(i) \cap x(i)$ is non-empty and there is $y \in \pi^{-1}(G)$ such that $z(i) = y(i)$. Since G is closed and each basic neighbourhood of z meets $\pi^{-1}(G)$, we have $z \in \pi^{-1}(G)$. So $z \sim x$ and $x \in \pi^{-1}(G)$. Hence $\pi(x) \in G$ and $p = \varphi(\pi(x)) \in \varphi(G)$. This is a contradiction. Hence, there exists i such that no $F \in \mathcal{F}_i$ having non-empty intersection with $\varphi(G)$ contains p. Then $\bigcup\{F \in \mathcal{F}_i \mid F \cap \varphi(G) \neq \emptyset\}$ is a closed set containing the set $\varphi(G)$ and not containing p. This means that $\varphi(G)$ is closed. Hence, φ is a closed map, which finishes the proof that φ is a homeomorphism. By Theorem 3.6, X is a completely metrizable space. ∎

COROLLARY 4.2. *Let X be a complete metric space and let $\{\mathcal{F}_i\}$ be a sequence of closed locally finite covers of X such that $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ and the supremum of diameters of elements of $\mathcal{F}_i$ goes to 0 as $i \to \infty$. Then one can form as above a complete cell structure which determines the space X.*

Proof. The hypotheses of Theorem 4.1 are satisfied, so we have a cell structure. It is complete because every Cauchy sequence of cells in $\bigcup \mathcal{F}_i$ determines a Cauchy sequence in the complete space X. ∎

THEOREM 4.3. *Every completely metrizable space is obtainable from a complete cell structure.*

Proof. Let d be a complete metric for X. Since (X, d) is a paracompact space, for each $\epsilon > 0$ there exists an open locally finite cover of X of mesh less than ϵ. By taking closures we get a closed locally finite cover of mesh less than ϵ. We inductively define closed locally finite covers $\mathcal{F}_i$ of X of mesh less than $1/n$ and such that $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ for each i. Suppose $\mathcal{F}_1, \ldots, \mathcal{F}_i$ are already defined. Then there exists a locally finite closed cover $\mathcal{F}$ consisting of sets of diameter less then $1/(n+1)$. We define $\mathcal{F}_{i+1}$ as the set of non-empty intersections of elements of $\mathcal{F}_i$ with elements of $\mathcal{F}$. The theorem now follows by Corollary 4.2. ∎

REMARK 4.4. All results in this section can be proved using open covers instead of closed covers if we insist that the $(i+1)$st cover closure refines the ith cover.

5. Examples

EXAMPLE 5.1 (The real line). For $i \geq 0$ let R_i denote the set of finite decimal fractions having i digits after the decimal point. In particular, R_0 denotes the set of integers. Two elements of R_i are said to be *adjacent* if their difference does not exceed 10^{-i}. The map $r_i^{i+1} : R_{i+1} \to R_i$ is defined by forgetting the last digit of $x \in R_{i+1}$. If $x, y \in R_\infty$ with $x \neq y$ but $x \sim y$ then $x(i)$ is adjacent to $y(i)$ for each i. Let j be the first coordinate such that $x(j) \neq y(j)$. Suppose first $j > 0$. We suppose without loss of generality that $x(j) < y(j)$. Then $y(j) = x(j) + 10^{-j}$ and for each $i > j$ all the digits in $x(i)$ to the right of the jth place are 9's and all the digits in $y(i)$ to the right of the jth place are 0's. The case $j = 0$ is similar. The topological space R^* determined by the cell structure

$$(*_1) \qquad R_0 \overset{r_0^1}{\longleftarrow} R_1 \overset{r_1^2}{\longleftarrow} R_2 \leftarrow \cdots$$

is homeomorphic to the real line $\mathbb{R}$.

If $G_1, \ldots, G_k$ are graphs then the Cartesian product $G_1 \times \cdots \times G_k$ is a graph with the relation obtained by declaring x to be related to y in

$G_1 \times \cdots \times G_k$ if and only if for each $i \in \{1, \ldots, k\}$ the ith coordinate of x is related to the ith coordinate of y.

Example 5.2 (Euclidean n-space). Let $(R_i)^n$ (resp. $(r_i^{i+1})^n$) be the Cartesian product of n copies of the graph R_i (resp. of the map r_i^{i+1}) of Example 5.1. The topological space $(R^n)^* = (R_\infty)^n/\sim$ determined by the cell structure

$$(*_2) \qquad (R_0)^n \xleftarrow{(r_0^1)^n} (R_1)^n \xleftarrow{(r_1^2)^n} (R_2)^n \leftarrow \cdots$$

is homeomorphic to the Euclidean space $\mathbb{R}^n$.

Example 5.3 (The Hilbert space). With the notation of Example 5.2 consider the cell structure

$$(*_3) \qquad (R^0)_1 \leftarrow (R^1)_2 \leftarrow (R^2)_3 \leftarrow \cdots$$

where the bonding map from $(R^i)_{i+1}$ to $(R^{i-1})_i$ is defined by forgetting the last coordinate of points in $(R_i)^{i+1}$ and then applying $(r_{i-1}^i)^i$. By abuse of notation let $(R^\infty)_\infty$ denote the set of threads of this cell structure. The topological space determined by this cell structure is the Hilbert space R^∞.

Example 5.4 (Polish spaces). Every Polish space embeds in the Hilbert space R^∞ as a closed subset. Hence, restricting the maps in Example 5.3 to the appropriate subset of $(R^\infty)_\infty$ we get a cell structure which determines the space.

6. Some classes of spaces. By placing restrictions on the graphs in the cell structure $(*)$ we are able to determine specific classes of spaces. Namely:

1. If r_i is the diagonal equivalence relation (defined by setting $(x, y) \in r_i$ if and only if $x = y$) on the graph G_i for each i then the class of spaces determined by such cell structures is the class of 0-dimensional, completely metrizable spaces. (By [5, 4.1.3 and 4.1.25], the product of countably many 0-dimensional metric spaces is 0-dimensional.)
2. If the graphs G_i are finite, the corresponding class of spaces is the class of metric compacta.
3. If the graphs G_i are finite and connected, the corresponding class of spaces is the class of metric continua.
4. If the graphs G_i are finite trees, the corresponding class of spaces is the class of metric treelike continua.
5. If the graphs G_i are finite and linear, the corresponding class of spaces is the class of chainable metric continua.
6. If the graphs G_i are of dimension $\le n$ (i.e. each set of mutually adjacent cells in G_i has cardinality $\le n + 1$), the corresponding class of spaces is the class of at most n-dimensional completely metrizable spaces.

We only prove (6). If the graphs G_i are of dimension $\le n$, the quotient map $\pi : G_\infty \to G^* = G_\infty/\!\sim$ is of multiplicity $\le n+1$. By the Morita theorem [5, 4.3.3], the dimension of G^* does not exceed the dimension of G_∞ plus n since π is closed and G_∞ is 0-dimensional and metrizable.

If X is a metrizable space of dimension not exceeding n then there is a sequence of open covers $\{\mathcal{U}_i\}_{i=1}^\infty$ of X such that $\mathcal{U}_i$ has mesh $\le 1/i$, $\mathcal{U}_{i+1}$ star closure refines $\mathcal{U}_i$ and such that if $\mathcal{V} \subset \mathcal{U}_i$ is such that each pair of elements of $\mathcal{V}$ meet then the cardinality of $\mathcal{V}$ does not exceed $n+1$. This condition is a consequence of the Ostrand theorem [5, p. 237]. As graphs, the $\mathcal{U}_i$ have dimension $\le n$. Define, as in Theorem 4.1, $g_i^{i+1} : \mathcal{U}_{i+1} \to \mathcal{U}_i$ such that for each $V \in \mathcal{U}_{i+1}$ we have $\operatorname{cl}(\bigcup \operatorname{st}(V, \mathcal{U}_{i+1})) \subset g_i^{i+1}(V)$.

7. Cell maps and maps of completely metrizable spaces. In this section we shall define cell maps between cell structures, and show that a cell map between two complete cell structures induces a continuous function between the spaces determined by the cell structures. Conversely, each mapping between two completely metrizable spaces is obtainable in this way.

Let

$$(*)\qquad G_0 \overset{g_0^1}{\longleftarrow} G_1 \overset{g_1^2}{\longleftarrow} G_2 \leftarrow \cdots,$$

$$(*')\qquad H_0 \overset{h_0^1}{\longleftarrow} H_1 \overset{h_1^2}{\longleftarrow} H_2 \leftarrow \cdots$$

be cell structures where the symmetric and reflexive relations on G_i and H_i are r_i and r'_i, respectively. Let π and π' be the quotient maps of G_∞ onto G^* and of H_∞ onto H^*, respectively. We also suppose that G_0 and H_0 each have exactly one element.

A function $f : \bigcup G_i \to \bigcup H_i$ is called a *cell map* of $(*)$ to $(*')$ if f takes close cells to close cells and f preserves Cauchy sequences. The next proposition follows from the definition.

PROPOSITION 7.1. *The composition of cell maps is a cell map.*

LEMMA 7.2. *Let $f : \bigcup G_i \to \bigcup H_i$ be a cell map of the cell structure $(*)$ to the complete cell structure $(*')$. Then f induces a function $\hat{f} : G^* \to H^*$ defined as follows: for a thread x in G_∞, $\hat{f}(\pi(x)) = \pi'(y)$ where y is a thread in H_∞ such that $f(x)$ converges to y.*

Proof. Let $x \in G_\infty$. Since x is a Cauchy sequence, $f(x)$ is Cauchy by the definition of a cell map. Let $y \in H_\infty$, so $f(x)$ converges to y. Let $v \in G_\infty$ be such that $x \sim v$ and let $w \in H_\infty$ be such that $f(v)$ converges to w. We show $w \sim y$. Since f is a cell map, $f(x(i))$ and $f(v(j))$ are close for all i and j. Hence, $w(i) \in \operatorname{st}^3_{r_i}(y(i))$ for each i. By (vi) applied twice, it follows that $y(i) \sim w(i)$ for each i. Thus $y \sim w$, and so $\hat{f}$ is well-defined. ∎

LEMMA 7.3 (Error estimation). *Let $f : \bigcup G_i \to \bigcup H_i$ be a cell map of the cell structure $(*)$ to the complete cell structure $(*')$. If $a \in G_i$ then $\hat{f}(\pi(\langle a\rangle)) \subset \pi'(\langle \mathrm{st}_{r'_j}(f(a))\rangle)$ where $j = \deg(f(a))$.*

Proof. Let $x \in \langle a\rangle$ and let $y \in H_\infty$ so that $f(x)$ converges to y. Then $f(a) = f(x(i)) \in \mathrm{st}_{r'_j}(y(j))$. So $\hat{f}(\pi(x)) \in \pi'(\langle \mathrm{st}_{r'_j}(f(x(i)))\rangle)$. ■

THEOREM 7.4. *Let $f : \bigcup G_i \to \bigcup H_i$ be a cell map of the cell structure $(*)$ to the complete cell structure $(*')$. Then the induced map $\hat{f} : G^* \to H^*$ is continuous.*

Proof. Let $x \in G_\infty$ and let U be a neighborhood of $\hat{f}(\pi(x)) \in H^*$. For each i let

$$A_i = \{a \in \mathrm{st}_{r_i}(x(i)) \mid \hat{f}(\pi(\langle a\rangle)) \not\subseteq U\}.$$

Suppose that $A_i \neq \emptyset$ for all i. By Lemma 3.1, there exists $y \in A_\infty$. Then $y \sim x$, $\hat{f}(\pi(y)) = \hat{f}(\pi(x)) \in U$ and $\hat{f}(\pi(\langle y(i)\rangle)) \not\subseteq U$ for all i. By completeness of $(*')$, there exists $z \in H_\infty$ such that $f(y)$ converges to z. Then $\pi'(z) \in U$ since $\pi'(z) = \hat{f}(\pi(y))$. By Lemma 7.3, we have $\hat{f}(\pi(\langle y(i)\rangle)) \subset \pi'(\langle \mathrm{st}_{r'_{j(i)}}(f(y(i)))\rangle)$ where $j(i) = \deg(f(y(i)))$ for all i. But $\mathrm{st}_{r'_{j(i)}}(f(y(i))) \subset \mathrm{st}^2_{r'_{j(i)}}(z(j(i)))$. Consequently, we have $\hat{f}(\pi(\langle y(i)\rangle)) \subset \pi'(\langle \mathrm{st}^2_{r'_{j(i)}}(z(j(i)))\rangle)$. But $\pi'(\langle \mathrm{st}^2_{r'_{j(i)}}(z(j(i)))\rangle) \subset U$ for sufficiently large i by (iv). Hence, $\hat{f}(\pi(\langle y(i)\rangle)) \subset U$ for that i, a contradiction. Hence, $A_m = \emptyset$ for some m. Consequently, the set $V = G^* \setminus \pi(\langle G_m \setminus \mathrm{st}_{r_m}(x(m))\rangle)$ is a neighbourhood of $\pi(x)$ contained in $\hat{f}^{-1}(U)$. ■

THEOREM 7.5. *Let $(*)$ be a cell structure and $(*')$ be a complete cell structure as at the beginning of this section. Let $F : G^* \to H^*$ be a continuous map and let H_0 consist of one element. Then there exists a cell map $f : \bigcup G_i \to \bigcup H_i$ of $(*)$ to $(*')$ such that $F = \hat{f}$.*

Proof. For $a \in G_i$ let $f(a) \in h_l(\pi'^{-1}(F(\pi(\langle a\rangle))))$ where l is the largest number such that $s = h_l(\pi'^{-1}(F(\pi(\langle \mathrm{st}_{r_i}(a)\rangle))))$ is a simplex (i.e. any two of its elements are adjacent) or $l = i$ otherwise.

Let a and b be close, $a, b \in \bigcup G_i$. Let $i = \deg(a)$ and $j = \deg(b)$. We may assume that $i \leq j$. Then a and $g_i^j(b)$ are adjacent and $\langle b\rangle \subset \langle \mathrm{st}_{r_i}(a)\rangle$. Let $m = \deg(f(a))$, $k = \deg(f(b))$, and let $n = \min(k, m)$. Then

$$h_n(\pi'^{-1}(F(\pi(\langle b\rangle)))) \subset h_n(\pi'^{-1}(F(\pi(\langle \mathrm{st}_{r_i}(a)\rangle)))).$$

Since

$$h_n^k(f(b)) \in h_n(\pi'^{-1}(F(\pi(\langle b\rangle)))), \qquad h_n^m(f(a)) \in h_n(\pi'^{-1}(F(\pi(\langle \mathrm{st}_{r_i}(a)\rangle)))),$$

where the last set is a simplex, $f(a)$ and $f(b)$ are adjacent. Hence $f(a)$ and $f(b)$ are close. This means the map f preserves closeness.

Let x be a Cauchy sequence in $\bigcup G_i$, $y \in G_\infty$ be a thread to which x converges, $p = \pi(y)$ and $q = F(p)$. Let $j \in \mathbb{N}$. The sets $\pi'(\langle b \rangle)$, for $b \in H_j$, form a closed locally finite cover of H^*. Let $U \subset H^*$ be an open set containing q such that $U \cap \pi'(\langle b \rangle) \neq \emptyset$ only for finitely many $b \in H_j$. Let $V = H^* \setminus \bigcup\{\pi'(\langle b \rangle) \mid q \notin \pi'(\langle b \rangle),\ b \in H_j\}$. Then V is an open set containing q such that $h_j(\pi'^{-1}(V))$ is a simplex. Now $W = \pi^{-1}(F^{-1}(V))$ is an open set in G_∞ containing the compact set $[y]$. Then there exists k' such that $\langle \mathrm{st}^2_{r_{k'}}(y(k')) \rangle \subset W$. Consequently, $\langle \mathrm{st}_{r_m}(x(n)) \rangle \subset W$ for $m = \deg(x(n)) \geq k'$. Hence, we have $F(\pi(\langle \mathrm{st}_{r_m}(x(n)) \rangle)) \subset V$ and the set $h_j(\pi'^{-1}(F(\pi(\langle \mathrm{st}_{r_m}(x(n)) \rangle))))$ is a simplex. This implies that $f(x)$ is a Cauchy sequence.

It is not difficult to see that $F = \hat{f}$. ■

8. Real functions. The case of real functions is of special interest. Represent the real numbers by the cell structure of Example 5.1:

$$\{R\} \leftarrow R_0 \leftarrow R_1 \leftarrow \cdots$$

where R_i denotes the set of finite decimal fractions having i digits after the decimal point.

A cell map $f : \{R\} \cup \bigcup_{i=0}^{\infty} R_i \to \{R\} \cup \bigcup_{i=0}^{\infty} R_i$ assigns to each finite decimal fraction another finite decimal fraction. It may be regarded as a kind of computation of the continuous function $F : R \to R$ induced by f. (Here we may ignore terms of the form $f(x) = *$ since f takes Cauchy sequences to Cauchy sequences.) If $a \in R$ and $x \in R_i$ are such that $|a - x| \leq 10^{-i}$ and $f(x) \in R_j$ then $|f(x) - F(a)| \leq 2 \cdot 10^{-j}$. It follows from the definition of cell maps that by choosing $x \in R_i$ with i sufficiently large we make $f(x)$ approximate $F(a)$ arbitrarily closely.

Analogous comments apply to real functions of several variables.

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