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Unstable manifolds of a class of delayed partial differential equations with nondense domain

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Abstract. We present an unstable manifold theory for the abstract delayed semilinear Cauchy problem with nondense domain

$$\frac{du}{dt} = (A + B(t))u(t) + f(t, u_t), \quad t \in \mathbb{R},$$

where $(A, D(A))$ satisfies the Hille–Yosida condition, $(B(t))_{t\in\mathbb{R}}$ is a family of operators in $\mathcal{L}(\overline{D(A)}, X)$ satisfying some measurability and boundedness conditions, and the nonlinear forcing term f satisfies $\|f(t, \phi) - f(t, \psi)\| \le \varphi(t)\|\phi - \psi\|_{\mathcal{C}}$. Here φ belongs to some admissible spaces and $\phi, \psi \in \mathcal{C} := C([-r, 0], X)$.

To reach our goal, we rely mainly on extrapolation theory. First, we develop a new variation of constants formula adapted to our problem. Then, using the characterization of exponential dichotomy, the properties of admissible spaces, the Lyapunov–Perron method as well as useful technical structures we prove the existence of an unstable manifold for our solutions. We also state an exponential attractiveness result concerning the unstable manifold. For illustration, we give an example.

1. Introduction. Unstable manifolds are useful in investigating the dynamics of evolution equations. In this work, we are interested in studying unstable manifolds for the partial differential equation

$$\text{(1.1)} \qquad \begin{cases} \dfrac{du}{dt} = (A + B(t))u(t) + f(t, u_t), & t \ge s, \\ u_s = \varPhi \in \mathcal{C}, \end{cases}$$

where $(A, D(A))$ is a nondensely defined linear operator defined on a Banach space X, $B(t)$, $t \in \mathbb{R}$, is a family of linear operators in $\mathcal{L}(\overline{D(A)}, X)$, $f : \mathbb{R} \times \mathcal{C} \to X$ is a nonlinear operator, $\mathcal{C} := C([-r, 0], X)$, and the history function u_t is defined for $\theta \in [-r, 0]$ by $u_t(\theta) = u(t + \theta)$. Throughout, we

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suppose that A is a *Hille–Yosida operator*, that is,

(H$_1$) there exist $w \in \mathbb{R}$ and $M \geq 1$ such that $(w, \infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda > w. \tag{1.2}$$

Here $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda > w$. Without loss of generality, one assumes that $M = 1$. Otherwise, we can renorm the space X with an equivalent norm for which we obtain (1.2) with $M = 1$.

It should be noted that various types of equations with a nondense domain were studied in the literature. For further information we may refer to [AEO, DS, LiMR2, T1, T2]. These types of equations are often used to describe population dynamics (see [MagR1, MagR2]), reaction-diffusion equations with nonlinear boundary conditions (see [CDMR, HT1]) and some delayed differential equations in Sobolev spaces (see [DLM, LiMR1]).

Regarding the nonautonomous case, several results about the existence and behaviour of solutions have been studied. For instance, we refer the reader to [EJ, GR, Man, R] and references therein. One of the main points of interest related to the asymptotic behaviour of solutions is the (stable, unstable, center-stable) manifold theory which has significant applications in the dynamic of evolution equations such as stability, perturbation, bifurcation etc.

Many results on the existence of stable, unstable and center-stable manifolds were developed for the differential equations

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t, u_t), & t \in [s, \infty), \\ u_s = \Phi, \end{cases} \tag{1.3}$$

where $A(t)$, $t \in \mathbb{R}$, is a family of possibly unbounded linear operators on a Banach space X. Note that if the evolution family $\{U(t,s)\}_{t \geq s}$ arises from the well-posed Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t), & t \geq s, \\ u(s) = \xi_s \in X, \end{cases}$$

then the following system is equivalent to (1.3):

$$\begin{cases} u(t) = U(t,s)u(s) + \displaystyle\int_s^t U(t,\xi) f(\xi, u_\xi)\, d\xi & \text{for } t \geq s, \\ u_s = \Phi \in \mathcal{C}. \end{cases} \tag{1.4}$$

Generally speaking, this system has usually been solved using the fact that f is K-Lipschitz where K is a positive constant and employing thereafter the contraction mapping theorem or the Schauder fixed point theorem to prove the existence of solutions. However, in real-world cases, the function f, which represents the population size or the source of a material in reaction-diffusion phenomena, depends on time (see, for instance, [Mu1, Mu2]). Therefore, one cannot always expect the uniform Lipschitzness of f. By using admissible spaces, the exponential dichotomy of evolution families and the Lyapunov–Perron method (see [HT1, HT2, Hu1, Hu2, Hu3]), the study of such solutions becomes possible even without the Lipschitz condition on f. Moreover, we can construct (stable, unstable, center-stable) manifolds for such solutions, that is, “the manifolds of admissible spaces”.

Motivated by these recent works, we aim to study the existence of unstable manifolds for (1.1), where A just satisfies the Hille–Yosida condition. It is worth noting that since $\overline{D(A)} \neq X$, solutions of our nondensely defined Cauchy problem (1.1) become nonclassical as the range of f is not included in $\overline{D(A)}$. Thus, we cannot deal with equation (1.4) (as in [HT1, HT2]). Moreover, throughout this work, we do not assume that the nonlinear forcing term f is uniformly Lipschitzian.

To achieve our target, we resort to extrapolation theory. We construct a new variation of constants formula adapted to our equation (1.1), under which we can use the Lyapunov–Perron method, the properties of admissible spaces and the characterization of exponential dichotomy, which enables us to construct the unstable manifold for (1.1) under a general condition on the nonlinear term f, without requiring its uniform Lipschitz continuity. Note that our concept of unstable manifolds is reformulated in such a way that we can overcome problems of nonuniqueness of solutions under the generalized φ-Lipschitz condition on the nonlinear forcing term. Furthermore, we will prove that the unstable manifold in question exponentially attracts any mild solution of (1.1). Our principal results appear in Theorems 2.10, 3.5, 3.7 and 4.1.

Note that our conditions implying the existence of unstable manifolds are more general than those in [HT2]. Our results extend those of [GR, HT1, HT2].

This work is organized as follows. In Section 2, we recall some results on extrapolation theory and admissible spaces, and establish a new variation of constants formula for (1.1). In Section 3, we use the variation of constants formula in order to investigate the existence of bounded solutions for (1.1) by virtue of the Lyapunov–Perron method. We therefore construct an unstable manifold for such solutions. Section 4 is devoted to proving that the unstable manifold exponentially attracts any mild solution of (1.1). Finally, in Section 5, we illustrate our theory by an example.

2. Mild solutions, extrapolation spaces, function spaces and admissibility. We first recall some properties of Hille–Yosida operators and extrapolation spaces. For more information, we refer the reader to [GR] and the references therein.

Consider the partial functional differential equation (1.1). It is well known that the part A_0 of A in $X_0 := \overline{D(A)}$ generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 such that $\|T_0(t)\| \leq Me^{wt}$ for all $t \geq 0$. For $\lambda \in \rho(A_0)$, the resolvent $R(\lambda, A_0)$ is the restriction of $R(\lambda, A)$ to X_0. Introducing the norm $\|x\|_{-1} = \|R(\lambda_0, A_0)x\|$ on X_0, for $\lambda_0 \in \rho(A)$, the completion X_{-1} of X_0 with respect to $\|\cdot\|_{-1}$ is called the *extrapolation space* of X_0 with respect to A. Note that a different choice of $\lambda_0 \in \rho(A)$ induces an equivalent norm. The extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ is the unique continuous extension $T_{-1}(t)$ of the operators $T_0(t)$, $t \geq 0$, to X_{-1} and is strongly continuous. Its generator A_{-1} is the unique continuous extension of A_0 to $\mathcal{L}(X_0, X_{-1})$. Furthermore, A_0 and A consist of the parts of A_{-1} respectively in X_0 and X. Finally, for a fixed $\lambda \in \rho(A)$, $R(\lambda, A_{-1})$ is the unique continuous extension of $R(\lambda, A)$ to X_{-1}.

We now give the definition of a mild solution of (1.1). This definition coincides with the one in [GR].

Definition 2.1. Let $\sigma \in \mathbb{R}$, and let $\Phi \in \mathcal{C}$ be such that $\Phi(0) \in \overline{D(A)}$. Then

(2.1)
$$\begin{cases} u(t) = T_0(t-\sigma)\Phi(0) + \int_\sigma^t T_{-1}(t-\tau)(B(\tau)u(\tau) + f(\tau, u_\tau))\,d\tau & \text{for } t \geq \sigma, \\ u_\sigma = \Phi, \end{cases}$$

is a *mild solution* of (1.1).

A function $u = u(\cdot, s, \Phi, f) \in C(\mathbb{R}, X_0)$ which satisfies (2.1) for all $t \geq s$ in $\mathbb{R}$ is called a mild solution of (1.1) on $\mathbb{R}$.

Here, we suppose that

(H_2) $t \mapsto B(t)x$ is strongly measurable for every $x \in X_0$ and there exists $l \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $\|B(\cdot)\| \leq l(\cdot)$.

Now, consider the homogeneous equation

$$\frac{d}{dt}u(t) = (A + B(t))u(t), \qquad t \in \mathbb{R}. \tag{2.2}$$

In order to give another representation of solutions of (2.2), we introduce the following notion.

Definition 2.2. A family $\{U(t,s)\}_{t\geq s}$ of bounded linear operators on a Banach space X is a *strongly continuous, exponential bounded evolution family* if

(i) $U(t,t) = \mathrm{Id}$ and $U(t,r)U(r,s) = U(t,s)$ for all $t \geq r \geq s$,
(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,

(iii) there are constants $K, c \geq 0$ such that $\|U(t,s)x\| \leq Ke^{c(t-s)}\|x\|$ for all $t \geq s$ and $x \in X$.

By [GR], there exists a unique evolution family $\{U_B(t,s)\}_{t\geq s}$ on X_0 that satisfies the variation of constants formula

$$(2.3)\qquad U_B(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-\tau)B(\tau)U_B(\tau,s)x\,d\tau, \qquad t \geq s,\ x \in X_0.$$

Hence, $t \mapsto U_B(t,s)x$ is the unique mild solution on $[s,\infty)$ of the initial value problem

$$\frac{d}{dt}u(t) = (A + B(t))u(t), \qquad t \geq s, \qquad u(s) = x \in X_0.$$

Note that in [GR, Theorem 2.2], the authors established an interesting representation of the mild solution of the modified equation

$$\frac{d}{dt}u(t) = (A + B(t))u(t) + f(t),$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ by means of the family $\{U_B(t,s)\}_{t\geq s}$ of bounded operators. Motivated by this result and proceeding analogically, we will develop a new variation of constants formula for our delayed partial differential equation (1.1) (see Theorem 2.10).

Now, we will introduce the following useful notions and properties of admissible spaces.

Definition 2.3 ([HT2, Hu3]). Let $\mathcal{B}$ denote the Borel algebra and λ the Lebesgue measure on $\mathbb{R}$. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ-nullfunctions) is called a *Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

(1) E is a Banach lattice with respect to a norm $\|\cdot\|_E$, i.e. $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$,

(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$,

(3) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R})$, i.e. for each seminorm p_n of $L_{1,\mathrm{loc}}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n}\|f\|_E$ for all $f \in E$.

Definition 2.4 ([HT2, Hu3]). A Banach function space E is called *admissible* if

(i) there is a constant $M \geq 1$ such that for every compact interval $[a,b] \subset \mathbb{R}$ we have

$$\int_a^b |\varphi(t)|\,dt \le \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E,$$

(ii) for $\varphi \in E$, the function $\Theta_1\varphi$ defined by $\Theta_1\varphi(t) = \int_t^{t+1}\varphi(\tau)d\tau$ belongs to E,

(iii) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined for $\tau \in \mathbb{R}$ by

$$T_\tau^+\varphi(t) = \varphi(t-\tau) \quad \text{for } t \in \mathbb{R},$$
$$T_\tau^-\varphi(t) = \varphi(t+\tau) \quad \text{for } t \in \mathbb{R}.$$

Moreover, there are constants $Q,\ R$ such that $\|T_\tau^+\| \le Q$ and $\|T_\tau^-\| \le R$ for all $\tau \in \mathbb{R}$.

Remark 2.5. If

$$\mathbf{S}(\mathbb{R}) := \Big\{\xi \in L_{1,\mathrm{loc}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_t^{t+1} |\xi(\tau)|\,d\tau < \infty\Big\}$$

is endowed with the norm $\|\xi\|_{\mathbf{S}} := \sup_{t\in\mathbb{R}}\int_t^{t+1}|\xi(\tau)|d\tau$ and E is an admissible Banach function space, it is easy to show that $E \hookrightarrow \mathbf{S}(\mathbb{R})$.

Proposition 2.6 ([HT2, Hu3]). *Let E be an admissible Banach function space.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R})$ be such that $\varphi \ge 0$ and $\Theta_1\varphi \in E$, where Θ_1 is defined as in Definition* 2.4(ii)*. For $\tau > 0$ define*

$$\Theta'_\tau\varphi(t) = \int_{-\infty}^{t} e^{-\tau(t-s)}\varphi(s)\,ds,$$
$$\Theta''_\tau\varphi(t) = \int_{t}^{\infty} e^{-\tau(s-t)}\varphi(s)\,ds.$$

Then $\Theta'_\tau\varphi, \Theta''_\tau\varphi \in E$. In particular, if $\sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\sigma)|d\sigma < \infty$ (this will be satisfied if $\varphi \in E$, see Remark 2.5) *then $\Theta'_\tau\varphi$ and $\Theta''_\tau\varphi$ are bounded. Moreover,*

$$\|\Theta'_\tau\varphi\|_\infty \le \frac{Q}{1-e^{-\tau}}\|\Theta_1\varphi\|_\infty \quad \text{and} \quad \|\Theta''_\tau\varphi\|_\infty \le \frac{R}{1-e^{-\tau}}\|\Theta_1\varphi\|_\infty. \tag{2.4}$$

(b) *E contains all exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed constant $\alpha > 0$.*

(c) *E contains no exponentially growing functions $f(t) = e^{b|t|}$ for $t \in \mathbb{R}$ and any constant $b > 0$.*

Definition 2.7 ([HT2, Hu3]). Let E be an admissible Banach function space and φ be a positive function belonging to E. A function $f : \mathbb{R}\times\mathcal{C} \to X$ is said to be *φ-Lipschitz* if

(i) $\|f(t,0)\| \le \varphi(t)$ for all $t \in \mathbb{R}$,
(ii) $\|f(t,\phi_1) - f(t,\phi_2)\| \le \varphi(t)\|\phi_1 - \phi_2\|_{\mathcal{C}}$ for all $t \in \mathbb{R}$ and all $\phi_1, \phi_2 \in \mathcal{C}$.

REMARK 2.8. If $f(t,\phi)$ is φ-Lipschitz then $\|f(t,\phi)\| \le \varphi(t)(1 + \|\phi\|_{\mathcal{C}})$ for all $\phi \in \mathcal{C}$ and $t \in \mathbb{R}$.

We will assume that

(H_3) $f : \mathbb{R} \times \mathcal{C} \to X$ is φ-Lipschitz, where φ is a positive function belonging to an admissible space E.

The following lemma is useful for the main result of this section.

LEMMA 2.9 ([NS]). *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ and $t \ge s$,*

$$\int_s^t T_{-1}(t-\sigma) f(\sigma)\, d\sigma \in X_0,$$

$$(t,s) \mapsto \int_s^t T_{-1}(t-\sigma) f(\sigma)\, d\sigma \text{ is continuous,}$$

$$\Big\|\int_s^t T_{-1}(t-\sigma) f(\sigma)\, d\sigma\Big\| \le \widetilde{M} \int_s^t e^{\omega(t-\sigma)} \|f(\sigma)\|\, d\sigma \text{ for a constant } \widetilde{M} \ge 1.$$

The following theorem is the main result of this section; it gives a new representation of a mild solution of (1.1) in terms of the evolution family $\{U_B(t,s)\}_{t \ge s}$.

THEOREM 2.10. *Assume that* (H_1)–(H_3) *hold. Let $s \in \mathbb{R}$, and let $\Phi \in \mathcal{C}$ be such that $\Phi(0) \in X_0$. Then equation* (1.1) *has a unique mild solution $u \in C([s,\infty[, X_0)$, given by*

$$\begin{cases} u(t) = U_B(t,s)\Phi(0) + \lim\limits_{\lambda\to\infty} \displaystyle\int_s^t U_B(t,\tau)\lambda R(\lambda, A) f(\tau, u_\tau)\, d\tau & \text{for } t \ge s, \\ u_s = \Phi. \end{cases}$$

Furthermore, for every $t \ge s$, the limit

$$\lim_{\lambda\to\infty} \int_s^t U_B(t,\tau)\lambda R(\lambda, A) f(\tau, u_\tau)\, d\tau \in X_0$$

exists uniformly on compact sets in $\mathbb{R}$.

Proof. For $\lambda > \omega$ and $t \ge s$, set

$$z_\lambda(t,s) = \int_s^t U_B(t,\tau)\lambda R(\lambda, A) f(\tau, u_\tau)\, d\tau.$$

As f is φ-Lipschitz, we have $\|f(\tau,u_\tau)\| \le \varphi(\tau)(1 + \|u_\tau\|_{\mathcal{C}})$. Consequently, $\tau \mapsto f(\tau, u_\tau)$ belongs to $L_{1,\mathrm{loc}}(\mathbb{R}, X)$, because φ is locally integrable. Hence, using (2.3), we get

$$(2.5)\qquad \begin{aligned} z_\lambda(t,s) &= \int_s^t T_0(t-\tau)\lambda R(\lambda,A)f(\tau,u_\tau)\,d\tau \\ &\quad+\int_s^t\Big(\int_\tau^t T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,\tau)\lambda R(\lambda,A)f(\tau,u_\tau)d\sigma\Big)\,d\tau \\ &= \lambda R(\lambda,A_0)\int_s^t T_{-1}(t-\tau)f(\tau,u_\tau)\,d\tau \\ &\quad+\int_s^t\Big(\int_s^\sigma T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,\tau)\lambda R(\lambda,A)f(\tau,u_\tau)\,d\tau\Big)\,d\sigma \\ &= \lambda R(\lambda,A_0)\int_s^t T_{-1}(t-\tau)f(\tau,u_\tau)\,d\tau \\ &\quad+\int_s^t T_{-1}(t-\tau)B(\tau)z_\lambda(\tau,s)\,d\tau,\quad t\ge s. \end{aligned}$$

Set $w(t,s)=\int_s^t T_{-1}(t-\tau)f(\tau,u_\tau)\,d\tau$. Then

$$(2.6)\qquad \begin{aligned} \|z_\mu(t,s)-z_\nu(t,s)\| &\le \|(\mu R(\mu,A_0)-\nu R(\nu,A_0))w(t,s)\| \\ &\quad+\widetilde{M}\int_s^t e^{\omega(t-\tau)}l(\tau)\|z_\mu(\tau,s)-z_\nu(\tau,s)\|\,d\tau. \end{aligned}$$

Lemma 2.9 shows that $w(t,s)$ is a continuous mapping into X_0. Hence

$$\lim_{\mu,\nu\to\infty}\|(\mu R(\mu,A_0)-\nu R(\nu,A_0))w(t,s)\|=0$$

uniformly for $t\ge s$ in compact intervals. From (2.6), we deduce that for $\varepsilon>0$ and $I\subseteq\mathbb{R}$ a compact interval, there is a constant N depending on the length of I such that

$$\|z_\mu(t,s)-z_\nu(t,s)\|\le\varepsilon+N\int_s^t l(\tau)\|z_\mu(\tau,s)-z_\nu(\tau,s)\|\,d\tau$$

for $t\ge s$ in I and $\mu,\nu>\omega$ large enough. Consequently, applying Gronwall's inequality, we obtain

$$\|z_\mu(t,s)-z_\nu(t,s)\|\le\varepsilon e^{N\int_s^t l(\tau)\,d\tau}$$

for $t\ge s$ in I and $\mu,\nu>\omega$ large enough. Thus, $z(t,s)=\lim_{\lambda\to\infty}z_\lambda(t,s)$ exists uniformly for $t\ge s$ in compact intervals. Since A is a Hille–Yosida operator, it follows from the definition of z_λ that

$$\sup\{\|z_\lambda(t,s)\|:\lambda>\omega,\ t\ge s\text{ in }I\}<\infty.$$

Consequently, by applying the Lebesgue dominated convergence theorem to (2.5), it follows that

$$z(t,s)=\int_s^t T_{-1}(t-\tau)B(\tau)z(\tau,s)\,d\tau+\int_s^t T_{-1}(t-\tau)f(\tau,u_\tau)\,d\tau\quad t\ge s. \tag{2.7}$$

Now, consider the function

$$\begin{aligned}u(t)&=U_B(t,s)\Phi(0)+\lim_{\lambda\to\infty}\int_s^t U_B(t,\tau)\lambda R(\lambda,A)f(\tau,u_\tau)\,d\tau\\&=U_B(t,s)\Phi(0)+z(t,s),\quad t\ge s.\end{aligned}$$

From (2.3) and (2.7) it follows that

$$\begin{aligned}u(t)&=U_B(t,s)\Phi(0)+\int_s^t T_{-1}(t-\tau)B(\tau)z(\tau,s)\,d\tau\\&\quad+\int_s^t T_{-1}(t-\tau)f(\tau,u_\tau)\,d\tau\\&=T_0(t-s)\Phi(0)+\int_s^t T_{-1}(t-\tau)(B(\tau)u(\tau)+f(\tau,u_\tau))\,d\tau.\end{aligned}$$

Consequently, u is a mild solution of (1.1).

As for uniqueness, we assume that there exists another mild solution v of (1.1); then

$$u(t)-v(t)=\int_s^t T_{-1}(t-\tau)B(\tau)(u(\tau)-v(\tau))\,d\tau,\quad t\ge s,$$

and an application of Gronwall's inequality yields $u=v$. ■

Theorem 2.10 has the following immediate consequence.

COROLLARY 2.11. *Assume that* (H_1)–(H_3) *hold. Let* $s\in\mathbb{R}$, *and let* $\Phi\in\mathcal{C}$ *be such that* $\Phi(0)\in X_0$. *Then* $u\in C([s,\infty[,X_0)$ *is a mild solution of* (1.1) *if and only if*

$$\begin{cases}u(t)=U_B(t,s)\Phi(0)+\lim_{\lambda\to\infty}\int_s^t U_B(t,\tau)\lambda R(\lambda,A)f(\tau,u_\tau)\,d\tau & \text{for } t\ge s,\\ u_s=\Phi.\end{cases}$$

3. Exponential dichotomy and invariant unstable manifolds. In this section, we prove the existence of unstable manifolds under the condition that the evolution family $\{U_B(t,s)\}_{t\ge s}$ has exponential dichotomy and the nonlinear forcing term f is φ-Lipschitz. We now recall the notion of exponential dichotomy (see for example [DK, GR, He, LaM]).

Definition 3.1. An evolution family $\{U(t,s)\}_{t\geq s}$ on the Banach space X is said to have *exponential dichotomy* on $\mathbb{R}$ if there exist bounded linear projections $P(t)$, $t\in\mathbb{R}$, on X and positive constants L, μ such that

(a) $U(t,s)P(s)=P(t)U(t,s)$, $t\geq s$,
(b) the restriction $U_{|}(t,s):\operatorname{Ker}P(s)\to\operatorname{Ker}P(t)$, $t\geq s$, is an isomorphism,
(c) $\|U(t,s)x\|\leq Le^{-\mu(t-s)}\|x\|$ for $x\in P(s)X$, $t\geq s$,
(d) $\|[U_{|}(t,s)]^{-1}x\|\leq Le^{-\mu(t-s)}\|x\|$ for $x\in\operatorname{Ker}P(t)$, $t\geq s$.

The projections $P(t)$, $t\in\mathbb{R}$, are called the *dichotomy projections* and the constants L, μ the *dichotomy constants*.

By [MiRS, Lemma 4.2], the exponential dichotomy of $\{U(t,s)\}_{t\geq s}$ implies that $K:=\sup_{t\in\mathbb{R}}\|P(t)\|<\infty$ and the map $t\mapsto P(t)$ is strongly continuous. The corresponding Green operator function is defined by

$$\Gamma(t,s)=\begin{cases}P(t)U(t,s), & t>s,\\ -[U_{|}(s,t)]^{-1}(\mathrm{Id}-P(s)), & t<s.\end{cases}\tag{3.1}$$

Then, in view of the exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, we get

$$\|\Gamma(t,s)\|\leq L(1+K)e^{-\mu|t-s|}\quad\text{for all } t\neq s.$$

We will assume the following condition:

(H_4) The evolution family $\{U_B(t,s)\}_{t\geq s}$ has exponential dichotomy with projections $P_B(t)$, $t\in\mathbb{R}$, and constants $L,\mu>0$.

Let

$$\mathcal{C}_A=\{\Phi\in\mathcal{C}:\Phi(0)\in\overline{D(A)}\}$$

denote the phase space of equation (1.1). Then, using the projections $(P_B(t))_{t\in\mathbb{R}}$ on X_0, we define the family of operators $(\mathbb{P}_B(t))_{t\in\mathbb{R}}$ on $\mathcal{C}_A$ by

$$\begin{aligned}&\mathbb{P}_B(t):\mathcal{C}_A\to\mathcal{C}_A,\\ &(\mathbb{P}_B(t)\xi)(\theta)=[U_{B|}(t,t+\theta)]^{-1}(I-P_B(t))\xi(0)\quad\text{for all }\theta\in[-r,0].\end{aligned}\tag{3.2}$$

Since $(\mathbb{P}_B(t))^2=\mathbb{P}_B(t)$, the operators $\mathbb{P}_B(t)$, $t\in\mathbb{R}$, are projections. Furthermore,

$$\operatorname{Im}\mathbb{P}_B(t)=\{\xi\in\mathcal{C}_A:\xi(\theta)=[U_{B|}(t,t+\theta)]^{-1}\mu_1,\ \forall\theta\in[-r,0]\ \text{for some }\mu_1\in\operatorname{Ker}P_B(t)\}.$$

Now, the concept of unstable manifold for the solutions of (1.1) can be defined.

Definition 3.2. A set $\mathcal{U}\subset\mathbb{R}\times\mathcal{C}_A$ is said to be an *unstable manifold* for the solutions of (1.1) if for every $t\in\mathbb{R}$, the phase space $\mathcal{C}_A$ decomposes into a direct sum $\operatorname{Im}\mathbb{P}_B(t)\oplus\operatorname{Ker}\mathbb{P}_B(t)$ such that

$$\sup_{t\in\mathbb{R}}\|\mathbb{P}_B(t)\|<\infty,$$

and there exists a family of Lipschitz continuous mappings

$$\Lambda_t:\operatorname{Im}\mathbb{P}_B(t)\to\operatorname{Ker}\mathbb{P}_B(t),\quad t\in\mathbb{R},$$

with Lipschitz constants independent of t such that

(a) $\mathcal{U}=\{(t,\xi+\Lambda_t(\xi))\in\mathbb{R}\times(\operatorname{Im}\mathbb{P}_B(t)\oplus\operatorname{Ker}\mathbb{P}_B(t)):t\in\mathbb{R},$ $\xi\in\operatorname{Im}\mathbb{P}_B(t)\}$, and we denote

$$\mathcal{U}_t:=\{\xi+\Lambda_t(\xi):(t,\xi+\Lambda_t(\xi))\in\mathcal{U}\},$$

(b) $\mathcal{U}_t$ is homeomorphic to $\operatorname{Im}\mathbb{P}_B(t)$ for all $t\in\mathbb{R}$,

(c) for each $t_0\in\mathbb{R}$ and $\xi\in\mathcal{U}_{t_0}$, there is a unique solution $u(t)$ of (1.1) on $(-\infty,t_0]$ with $u_{t_0}=\xi$ and $\sup_{t\le t_0}\|u_t\|_{\mathcal{C}}<\infty$; furthermore, any two solutions $u_1(t)$ and $u_2(t)$ of (1.1) corresponding to different initial functions $\xi_1,\ \xi_2\in\mathcal{U}_{t_0}$ exponentially attract each other in the sense that there exist positive constants ν and C_ν independent of $t_0\ge 0$ such that

$$\|u_{1t}-u_{2t}\|_{\mathcal{C}}\le C_\nu e^{-\nu(t_0-t)}\|(\mathbb{P}_B(t_0)\xi_1)(0)-(\mathbb{P}_B(t_0)\xi_2)(0)\|\quad\text{for } t\le t_0,$$

(d) $\mathcal{U}$ is invariant under equation (1.1). That is, if $u(t)$, $t\in\mathbb{R}$, is a solution of (1.1) with $u_{t_0}\in\mathcal{U}_{t_0}$ and $\sup_{t\le t_0}\|u_t\|_{\mathcal{C}}<\infty$, then $u_t\in\mathcal{U}_t$ for all $t\in\mathbb{R}$.

Identifying $\operatorname{Im}\mathbb{P}_B(t)\oplus\operatorname{Ker}\mathbb{P}_B(t)$ with $\operatorname{Im}\mathbb{P}_B(t)\times\operatorname{Ker}\mathbb{P}_B(t)$, we get $\mathcal{U}_t=\operatorname{graph}(\Lambda_t)$. In what follows, we will give the form of bounded solutions of equation (1.1).

LEMMA 3.3. *Assume that* (H$_1$)–(H$_4$) *hold. Suppose that $u(t)$ is a solution of equation* (1.1) *in the sense that* $\sup_{t\le t_0}\|u(t)\|<\infty$ *for a fixed* $t_0\in\mathbb{R}$. *Then, for $t\le t_0$, $u(t)$ has the following representation:*

$$\begin{cases} u(t)=[U_{B|}(t_0,t)]^{-1}\mu_1+\lim\limits_{\lambda\to\infty}\displaystyle\int_{-\infty}^{t_0}\Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma, \\ u_{t_0}=\Phi\in\mathcal{C}_A, \end{cases}\tag{3.3}$$

for some $\mu_1\in\operatorname{Ker}P_B(t_0)=(I-P_B(t_0))X_0$, *where* $\Gamma_B(t,\sigma)$ *is the Green operator function defined as in* (3.1).

Proof. Set $z(t)=\lim_{\lambda\to\infty}\int_{-\infty}^{t_0}\Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma$. Owing to (1.2) and Remark 2.8, we get

$$\|z(t)\| \le L(1+K)\int_{-\infty}^{t_0} e^{-\mu|t-\sigma|}\|f(\sigma,u_\sigma)\|\,d\sigma \tag{3.4}$$
$$\le L(1+K)\Big(1+\sup_{t\le t_0}\|u(t)\|\Big)\frac{Q+R}{1-e^{-\mu}}\|\Theta_1\varphi\|_\infty < \infty.$$

Further,

$$\begin{aligned}U_B(t_0,t)z(t) &+ \lim_{\lambda\to\infty}\int_t^{t_0} U_B(t_0,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&= U_B(t_0,t)\Big(\lim_{\lambda\to\infty}\int_{-\infty}^{t_0} \Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\Big)\\
&\quad + \lim_{\lambda\to\infty}\int_t^{t_0} U_B(t_0,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&= \lim_{\lambda\to\infty}\int_{-\infty}^{t_0} U_B(t_0,\sigma)P_B(\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&= \lim_{\lambda\to\infty}\int_{-\infty}^{t_0} \Gamma_B(t_0,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma = z(t_0).\end{aligned}$$

Consequently,

$$z(t_0) = U_B(t_0,t)z(t) + \lim_{\lambda\to\infty}\int_t^{t_0} U_B(t_0,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma.$$

Since $u(t)$ is a solution of equation (1.1), applying Theorem 2.10 we obtain

$$u(t_0) = U_B(t_0,t)u(t) + \lim_{\lambda\to\infty}\int_t^{t_0} U_B(t_0,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma.$$

Thus, $u(t_0)-z(t_0) = U_B(t_0,t)(u(t)-z(t))$. Let $s\le t$. Then

$$\begin{aligned}P_B(t)u(t) &= U_B(t,s)P_B(s)u(s)\\
&\quad + \lim_{\lambda\to\infty}\int_s^{t} U_B(t,\sigma)P_B(\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma.\end{aligned}$$

As

$$\|U_B(t,s)P_B(s)u(s)\| \le LKe^{-\mu(t-s)}\sup_{s\le t_0}\|u(s)\|.$$

Hence, as $s\to-\infty$,

$$P_B(t)u(t) = \lim_{\lambda\to\infty}\int_{-\infty}^{t} U_B(t,\sigma)P_B(\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma = P_B(t)z(t),$$

which means that $u(t)-z(t) \in \operatorname{Ker} P_B(t)$ and so $u(t_0)-z(t_0) = U_B(t_0,t)(u(t) - z(t)) \in \operatorname{Ker} P_B(t_0)$. Setting $\mu_1 = u(t_0) - z(t_0)$ gives

$$u(t) = [U_{B|}(t_0,t)]^{-1}\mu_1 + \lim_{\lambda\to\infty} \int_{-\infty}^{t_0} \Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma. \blacksquare$$

REMARK 3.4. It should be noted that the converse of Lemma 3.3 is also true by a direct computation. Then all solutions of equation (3.3) satisfy equation (1.1). Equation (3.3) is called the *Lyapunov–Perron equation*.

Now, we will give our result about the existence-uniqueness and the exponential stability of solutions of equation (1.1).

THEOREM 3.5. *Assume that* (H_1)–(H_4) *hold. Set*

$$H := \frac{L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-\mu}}. \tag{3.5}$$

If $H < 1$, then for each $\xi \in \operatorname{Im}\mathbb{P}_B(t_0)$ there exists a unique solution of equation (1.1) *on $(-\infty,t_0]$ satisfying $\mathbb{P}_B(t_0)u_{t_0} = \xi$ and $\sup_{t\le t_0}\|u_t\|_{\mathcal{C}} < \infty$. Moreover, for any two solutions $u(t)$, $v(t)$ corresponding to different initial functions $\xi_1,\ \xi_2 \in \operatorname{Im}\mathbb{P}_B(t_0)$,*

$$\|u(t)-v(t)\|_{\mathcal{C}} \le C_\nu e^{-\nu(t_0-t)}\|\xi_1(0)-\xi_2(0)\| \quad \textit{for all } t \le t_0,$$

where ν is a positive constant satisfying

$$0 < \nu < \mu + \ln(1 - L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty),$$

and

$$C_\nu := \frac{Le^{\mu r}}{1 - \frac{L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-(\mu-\nu)}}}.$$

REMARK 3.6. The existence-uniqueness result of Theorem 3.5 can also be obtained if $H < e^{\mu r}$.

Proof. Consider the Banach space $BC((-\infty,t_0],X_0)$ of bounded, continuous and X_0-valued functions defined on $(-\infty,t_0]$, endowed with the uniform norm topology. Set $\mu_1 := \xi(0)$ and consider the nonlinear operator given by

$$(Fu)(t) = [U_{B|}(t_0,t)]^{-1}\mu_1 + \lim_{\lambda\to\infty} \int_{-\infty}^{t_0} \Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \text{ for } t \le t_0.$$

Since $\mu_1 \in \operatorname{Ker} P_B(t_0)$, in view of (3.4) the operator F maps $BC((-\infty,t_0],X_0)$ into itself. On the other hand, using (1.2) we obtain, for $t \le t_0$,

$$\begin{aligned}\|(Fu)(t)-(Fv)(t)\| &\le L(1+K)\int_{-\infty}^{t_0} e^{-\mu|t-\sigma|}\varphi(\sigma)\|u_\sigma - v_\sigma\|_{\mathcal{C}}\,d\sigma \\ &\le He^{-\mu r}\sup_{t\le t_0}\|u(t)-v(t)\|.\end{aligned}$$

Accordingly,

$$\sup_{t\le t_0}\|(Fu)(t)-(Fv)(t)\| \le He^{-\mu r}\sup_{t\le t_0}\|u(t)-v(t)\|.$$

Since $H<1$, F is a strict contraction. By the Banach fixed point theorem, there exists a unique $u(\cdot)\in BC((-\infty,t_0],X)$ such that $Fu=u$. Therefore, u is the unique mild solution of (3.3) with the initial condition

$$u_{t_0}(\theta)=[U_{B|}(t_0,t_0+\theta)]^{-1}\mu_1+\lim_{\lambda\to\infty}\int_{-\infty}^{t_0}\Gamma_B(t_0+\theta,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma$$

for all $\theta\in[-r,0]$ and $(I-P_B(t_0))u(t_0)=\mu_1=\xi(0)$. Thus, $\mathbb{P}_B(t_0)u_{t_0}=\xi$ by the definition of $\mathbb{P}_B(t_0)$.

Now, assume that $u(t)$ and $v(t)$ are two solutions of (3.3) corresponding to initial functions $\xi_1,\xi_2\in\operatorname{Im}\mathbb{P}_B(t_0)$. Set $\mu_1:=\xi_1(0)$ and $\mu_2:=\xi_2(0)$. Then for $t\le t_0$,

$$\|u(t)-v(t)\|\le Le^{-\mu(t_0-t)}\|\mu_1-\mu_2\|+L(1+K)\int_{-\infty}^{t_0}e^{-\mu|t-\sigma|}\varphi(\sigma)\|u_\sigma-v_\sigma\|_{\mathcal{C}}\,d\sigma.$$

This yields, for $t\le t_0$ and $\theta\in[-r,0]$,

$$\|u_t-v_t\|_{\mathcal{C}}\le Le^{-\mu(t_0-t)}\|\mu_1-\mu_2\|+L(1+K)e^{\mu r}\int_{-\infty}^{t_0}e^{-\mu|t-\sigma|}\varphi(\sigma)\|u_\sigma-v_\sigma\|_{\mathcal{C}}\,d\sigma.$$

Set $g(t)=\|u_t-v_t\|_{\mathcal{C}}$. Then $\sup_{t\le t_0}g(t)<\infty$ and

$$g(t)\le Le^{-\mu(t_0-t)}\|\mu_1-\mu_2\| + L(1+K)e^{\mu r}\int_{-\infty}^{t_0}e^{-\mu|t-\sigma|}\varphi(\sigma)g(\sigma)\,d\sigma,\qquad t\le t_0. \tag{3.6}$$

Now, we apply the cone inequality theorem (see [Hu3, Theorem 2.8]) to $L_\infty(-\infty,t_0]$, the Banach space of real valued functions essentially bounded on $(-\infty,t_0]$, equipped with the uniform norm topology. Here, the cone C is the set of all positive functions. Consider the linear operator D defined for $\kappa\in L_\infty(-\infty,t_0]$ by

$$(D\kappa)(t)=L(1+K)e^{\mu r}\int_{-\infty}^{t_0}e^{-\mu|t-\sigma|}\varphi(\sigma)\kappa(\sigma)\,d\sigma,\quad t\le t_0.$$

Using Proposition 2.6, we obtain

$$\sup_{t\le t_0}(D\kappa)(t)=\sup_{t\le t_0}L(1+K)e^{\mu r}\int_{-\infty}^{t_0}e^{-\mu|t-\sigma|}\varphi(\sigma)\kappa(\sigma)\,d\sigma\le H\|\kappa\|_\infty.$$

Thus, D is a bounded linear operator satisfying $\|D\|<1$. Clearly, the cone C is invariant under the operator D. Also, we can rewrite inequality (3.6) as

$$g \le Dg + w \quad \text{where} \quad w(t) = Le^{-\mu(t_0-t)}\|\mu_1 - \mu_2\|.$$

It follows from the cone inequality theorem that $g \le h$, where h is the solution in $L_\infty(-\infty, t_0]$ of the equation $h = Dh + w$. That is,

$$h(t) = Le^{-\mu(t_0-t)}\|\mu_1 - \mu_2\| + L(1+K)e^{\mu r}\int_{-\infty}^{t_0} e^{-\mu|t-\sigma|}\varphi(\sigma)h(\sigma)\,d\sigma.$$

In order to estimate h, set $z(t) = e^{\nu(t_0-t)}h(t)$ for $t \le t_0$. Then

$$z(t) = Le^{-(\mu-\nu)(t_0-t)}\|\mu_1 - \mu_2\| + L(1+K)e^{\mu r}\int_{-\infty}^{t_0} e^{-\mu|t-\sigma|+\nu(\sigma-t)}\varphi(\sigma)z(\sigma)\,d\sigma. \tag{3.7}$$

Consider the linear operator W defined on $L_\infty(-\infty, t_0]$ by

$$(W\xi)(t) = L(1+K)e^{\mu r}\int_{-\infty}^{t_0} e^{-\mu|t-\sigma|+\nu(\sigma-t)}\varphi(\sigma)\xi(\sigma)\,d\sigma \quad \text{for all } t \le t_0.$$

It is clear that W is a bounded linear operator and

$$\|W\| \le \frac{L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1 - e^{-(\mu-\nu)}}.$$

One can rewrite (3.7) as

$$z = Wz + \widetilde{w} \quad \text{where} \quad \widetilde{w}(t) = Le^{-(\mu-\nu)(t_0-t)}\|\mu_1 - \mu_2\|. \tag{3.8}$$

Note that $\|W\| < 1$ if

$$0 < \nu < \mu + \ln(1 - L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty). \tag{3.9}$$

Equation (3.8) as well as condition (3.9) lead to the existence of a unique $z \in L_\infty(-\infty, t_0]$ such that $z = (I-W)^{-1}\widetilde{w}$. Then

$$\begin{aligned}\|z\| &= \|(I-W)^{-1}\widetilde{w}\|_\infty \le \frac{L}{1 - \|W\|}\|\mu_1 - \mu_2\| \\ &\le \frac{L}{1 - \frac{L(1+K)e^{\mu r}(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-(\mu-\nu)}}}\|\mu_1 - \mu_2\| =: C_\nu\|\mu_1 - \mu_2\|.\end{aligned}$$

It follows that $z(t) \le C_\nu\|\mu_1 - \mu_2\|$ for $t \le t_0$. Thus,

$$\begin{aligned}g(t) &= \|u_t - v_t\|_{\mathcal{C}} \\ &\le h(t) = e^{-\nu(t_0-t)}z(t) \le C_\nu e^{-\nu(t_0-t)}\|\mu_1 - \mu_2\| \quad \text{for all } t \le t_0. \ \blacksquare\end{aligned}$$

Now, we will give our result about the existence of an unstable manifold for solutions of (1.1).

THEOREM 3.7. *Assume that* (H_1)–(H_4) *hold. Let*

$$H < \min\left\{1, \frac{e^{\mu r}}{1+L}\right\}, \tag{3.10}$$

where H *is defined as in* (3.5). *Then there exists an unstable manifold* $\mathcal{U}$ *for the solutions of equation* (1.1).

Proof. As $\mathbb{P}_B(t)$ is a projector, the phase space $\mathcal{C}_A$ splits into the direct sum $\operatorname{Im}\mathbb{P}_B(t) \oplus \operatorname{Ker}\mathbb{P}_B(t)$ where the projections $\mathbb{P}_B(t)$, $t \in \mathbb{R}$, are defined as in (3.2). It is easy to show that $\sup_{t\in\mathbb{R}} \|\mathbb{P}_B(t)\| < \infty$. In order to construct an unstable manifold $\mathcal{U} = \{(t, \mathcal{U}_t) : t \in \mathbb{R}\}$ for the solutions of (1.1), we determine for $t \in \mathbb{R}$ the surface

$$\mathcal{U}_t := \{\xi + \Lambda_t(\xi) : \xi \in \operatorname{Im}\mathbb{P}_B(t)\} \subset \mathcal{C}_A,$$

where the operator Λ_t is defined for each $t \in \mathbb{R}$ by

$$\Lambda_t(\xi)(\theta) = \lim_{\lambda\to\infty} \int_{-\infty}^{t} \Gamma_B(t+\theta, \sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma)\, d\sigma \quad \text{for all } \theta \in [-r, 0].$$

Note that $u(\cdot)$ is the solution of (1.1) on $(-\infty, t]$ such that $\mathbb{P}_B(t)u_t = \xi$. The existence and the uniqueness of $u(\cdot)$ is ensured by Theorem 3.5. The definition of the Green operator Γ_B yields $\Lambda_t(\xi) \in \operatorname{Ker}\mathbb{P}_B(t)$. Now, we check the conditions of Definition 3.2.

Let us first prove the uniform Lipschitz continuity of Λ_t independently of t. In fact, for $\xi_1, \xi_2 \in \operatorname{Im}\mathbb{P}_B(t)$ we have

$$\begin{aligned} &\|\Lambda_t(\xi_1)(\theta) - \Lambda_t(\xi_2)(\theta)\| \\ &\quad\le L(1+K) \int_{-\infty}^{t} e^{-\mu|t+\theta-\sigma|}\varphi(\sigma)\|u_\sigma - v_\sigma\|\, d\sigma \\ &\quad\le L(1+K)\sup_{\sigma\le t}\|u_\sigma - v_\sigma\|_{\mathcal{C}} \Big(\int_{-\infty}^{t+\theta} e^{-\mu(t+\theta-\sigma)}\varphi(\sigma)\, d\sigma \\ &\qquad\qquad + \int_{t+\theta}^{t} e^{-\mu(\sigma-t-\theta)}\varphi(\sigma)\, d\sigma\Big) \\ &\quad\le \frac{L(1+K)(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-\mu}} \sup_{\sigma\le t}\|u_\sigma - v_\sigma\|_{\mathcal{C}}. \end{aligned} \tag{3.11}$$

The Lyapunov–Perron equation (3.3) for $u(\cdot)$ and $v(\cdot)$ implies that for $\tau \le t$ and $\theta \in [-r, 0]$,

$$\|u(\tau+\theta) - v(\tau+\theta)\| \le \|[U_{B|}(t, \tau+\theta)]^{-1}(\mu_1 - \mu_2)\|$$

$$+\Big\|\lim_{\lambda\to\infty}\int_{-\infty}^{t}\Gamma_B(\tau+\theta,\sigma)\lambda R(\lambda,A)(f(\sigma,u_\sigma)-f(\sigma,v_\sigma))\,d\sigma\Big\|$$

$$\le L\|\mu_1-\mu_2\|+L(1+K)\sup_{\sigma\le t}\|u_\sigma-v_\sigma\|_{\mathcal{C}}\int_{-\infty}^{t}e^{-\mu|\tau+\theta-\sigma|}\varphi(\sigma)\,d\sigma$$

$$\le L\|\mu_1-\mu_2\|+L(1+K)\frac{Q+R}{1-e^{-\mu}}\|\Theta_1\varphi\|_\infty\sup_{\sigma\le t}\|u_\sigma-v_\sigma\|_{\mathcal{C}}.$$

This gives

$$\sup_{\sigma\le t}\|u_\sigma-v_\sigma\|_{\mathcal{C}}\le L\|\xi_1-\xi_2\|_{\mathcal{C}}+\frac{L(1+K)(Q+R)\|\Theta_1\varphi\|_\infty}{1-e^{-\mu}}\sup_{\sigma\le t}\|u_\sigma-v_\sigma\|_{\mathcal{C}}.$$

Then

$$\sup_{\tau\le t}\|u_\tau-v_\tau\|_{\mathcal{C}}\le\frac{L}{1-He^{-\mu r}}\|\xi_1-\xi_2\|_{\mathcal{C}}.$$

Substituting this inequality into (3.11) gives

$$\begin{aligned}\|\Lambda_{t_0}(\xi_1)-\Lambda_{t_0}(\xi_2)\|_{\mathcal{C}}&=\sup_{\theta\in[-r,0]}\|\Lambda_{t_0}(\xi_1)(\theta)-\Lambda_{t_0}(\xi_2)(\theta)\|\\&\le\frac{LHe^{-\mu r}}{1-He^{-\mu r}}\|\xi_1-\xi_2\|_{\mathcal{C}}.\end{aligned}$$

This implies the uniform Lipschitz continuity of Λ_{t_0} independently of t_0.

To prove that $\mathcal{U}_t$ is homeomorphic to $\operatorname{Im}\mathbb{P}_B(t)$ for each $t\in\mathbb{R}$, consider the operator $T:\operatorname{Im}\mathbb{P}_B(t)\to\mathcal{U}_t$ defined by $T\xi:=\xi+\Lambda_t(\xi)$ for all $\xi\in\operatorname{Im}\mathbb{P}_B(t)$. Owing to the implicit function theorem for Lipschitz continuous mappings (see [MiW, Lemma 2.7]), we find that T is a homeomorphism under the condition $\frac{LHe^{-\mu r}}{1-He^{-\mu r}}<1$. Hence, condition (b) of Definition 3.2 is satisfied. Theorem 3.5 gives condition (c).

It remains to prove (d). Let $u(\cdot)$ be a solution of (1.1) such that $u_{t_0}\in\mathcal{U}_{t_0}$ and $\sup_{t\le t_0}\|u(t)\|<\infty$. We aim to show that $u_t\in\mathcal{U}_t$ for all $t\in\mathbb{R}$.

First, let $t\ge t_0$. Then

$$\begin{aligned}u_{t_0}(\theta)&=[U_{B|}(t_0,t_0+\theta)]^{-1}\mu_1\\&\quad+\lim_{\lambda\to\infty}\int_{-\infty}^{t_0}\Gamma_B(t_0+\theta,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\qquad\text{for }\theta\in[-r,0],\end{aligned}$$

where $\mu_1\in\operatorname{Ker}P_B(t_0)$. Let $u_1(\cdot)$ be a solution of (1.1) with the initial condition $u_{1t_0}=u_{t_0}$. Set

$$v(\tau)=\begin{cases}u(\tau)&\text{for }\tau\le t_0,\\u_1(\tau)&\text{for }\tau\in[t_0,t].\end{cases}$$

It is clear that $v(t)$ is continuous and bounded on $(-\infty,t]$ and $u_t=v_t$. Let us show that $v_t\in\mathcal{U}_t$. Indeed, for $\tau\in[t_0,t]$, according to Theorem 2.10 we have

$$v(\tau) = u_1(\tau) = U_B(\tau, t_0)v(t_0) + \lim_{\lambda\to\infty} \int_{t_0}^{\tau} U_B(\tau, \sigma)\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$= U_B(\tau, t_0)\Big(\mu_1 + \lim_{\lambda\to\infty} \int_{-\infty}^{t_0} \Gamma_B(t_0, \sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma)\, d\sigma\Big)$$

$$+ \lim_{\lambda\to\infty} \int_{t_0}^{\tau} U_B(\tau, \sigma)\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$= U_B(\tau, t_0)\mu_1 + \lim_{\lambda\to\infty} \int_{-\infty}^{t_0} U_B(\tau, \sigma)P_B(\sigma)\lambda R(\lambda, A) f(\sigma, u_\sigma)\, d\sigma$$

$$+ \lim_{\lambda\to\infty} \int_{t_0}^{\tau} U_B(\tau, \sigma)P_B(\sigma)\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$+ \lim_{\lambda\to\infty} \int_{t_0}^{\tau} U_B(\tau, \sigma)(I - P_B(\sigma))\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$\le U_B(\tau, t_0)\mu_1 + \lim_{\lambda\to\infty} \int_{t_0}^{\tau} U_B(\tau, \sigma)(I - P_B(\sigma))\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$+ \lim_{\lambda\to\infty} \int_{-\infty}^{\tau} \Gamma_B(\tau, \sigma)\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma.$$

Set

$$\mu_2 := U_B(t, t_0)\mu_1 + \lim_{\lambda\to\infty} \int_{t_0}^{t} U_B(t, \sigma)(I - P_B(\sigma))\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma \in \operatorname{Ker} P_B(t).$$

Then

$$[U_{B|}(t, \tau)]^{-1}\mu_2 = U_B(\tau, t_0)\mu_1 + \lim_{\lambda\to\infty} \int_{t_0}^{\tau} U_B(\tau, \sigma)(I - P_B(\sigma))\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$+ \lim_{\lambda\to\infty} \int_{\tau}^{t} [U_{B|}(\sigma, \tau)]^{-1}(I - P_B(\sigma))\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma.$$

Hence,

$$v(\tau) = [U_{B|}(t, \tau)]^{-1}\mu_2 - \lim_{\lambda\to\infty} \int_{\tau}^{t} [U_{B|}(\sigma, \tau)]^{-1}(I - P_B(\sigma))\lambda R(\lambda, A) f(\sigma, v_\sigma)\, d\sigma$$

$$+\lim_{\lambda\to\infty}\int_{-\infty}^{\tau} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,v_\sigma)\,d\sigma$$

$$= [U_{B|}(t,\tau)]^{-1}\mu_2 + \lim_{\lambda\to\infty}\int_{-\infty}^{t} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,v_\sigma)\,d\sigma.$$

For $\tau \le t_0$, we have

$$\begin{aligned}
v(\tau) = u(\tau) &= [U_{B|}(t_0,\tau)]^{-1}\mu_1 + \lim_{\lambda\to\infty}\int_{-\infty}^{t_0} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \\
&= [U_{B|}(t_0,\tau)]^{-1}\mu_1 \\
&\quad - \lim_{\lambda\to\infty}\int_{\tau}^{t_0} [U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \\
&\quad + \lim_{\lambda\to\infty}\int_{-\infty}^{\tau} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \\
&= [U_{B|}(t,\tau)]^{-1}\mu_2 \\
&\quad - \lim_{\lambda\to\infty}\int_{\tau}^{t} [U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,v_\sigma)\,d\sigma \\
&\quad + \lim_{\lambda\to\infty}\int_{-\infty}^{\tau} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \\
&= [U_{B|}(t,\tau)]^{-1}\mu_2 + \lim_{\lambda\to\infty}\int_{-\infty}^{t} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,v_\sigma)\,d\sigma.
\end{aligned}$$

Therefore, for all $\tau \le t$, there exists $\mu_2 \in \operatorname{Ker} P_B(t)$ such that

$$v(\tau) = [U_{B|}(t,\tau)]^{-1}\mu_2 + \lim_{\lambda\to\infty}\int_{-\infty}^{t} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,v_\sigma)\,d\sigma.$$

Consequently, $v_t \in \mathcal{U}_t$ and hence $u_t = v_t \in \mathcal{U}_t$ for all $t > t_0$.

Now, for $t < t_0$, we will also prove that $u_t \in \mathcal{U}_t$. Indeed, for $\tau \le t \le t_0$, we have

$$\begin{aligned}
u(\tau) &= [U_{B|}(t_0,\tau)]^{-1}\mu_1 + \lim_{\lambda\to\infty}\int_{-\infty}^{t_0} \varGamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \\
&= [U_{B|}(t_0,\tau)]^{-1}\mu_1 \\
&\quad - \lim_{\lambda\to\infty}\int_{\tau}^{t_0} [U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \\
&\quad + \lim_{\lambda\to\infty}\int_{-\infty}^{\tau} U_B(\tau,\sigma)P_B(\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma.
\end{aligned}$$

Set

$$\mu_2 := [U_{B|}(t_0,t)]^{-1}\mu_1 - \lim_{\lambda\to\infty}\int_t^{t_0}[U_{B|}(\sigma,t)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma \in \operatorname{Ker} P_B(t).$$

Then

$$\begin{aligned}
&[U_{B|}(t,\tau)]^{-1}\mu_2\\
&= [U_{B|}(t_0,\tau)]^{-1}\mu_1 - \lim_{\lambda\to\infty}\int_t^{t_0}[U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&= [U_{B|}(t_0,\tau)]^{-1}\mu_1 - \lim_{\lambda\to\infty}\int_\tau^{t_0}[U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&\quad + \lim_{\lambda\to\infty}\int_\tau^{t}[U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
u(\tau) &= [U_{B|}(t,\tau)]^{-1}\mu_2\\
&\quad - \lim_{\lambda\to\infty}\int_\tau^{t}[U_{B|}(\sigma,\tau)]^{-1}(I-P_B(\sigma))\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&\quad + \lim_{\lambda\to\infty}\int_{-\infty}^{\tau} U_B(\tau,\sigma)P_B(\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma\\
&= [U_{B|}(t,\tau)]^{-1}\mu_2 + \lim_{\lambda\to\infty}\int_{-\infty}^{t} \Gamma_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma,u_\sigma)\,d\sigma
\end{aligned}$$

for all $\tau \le t$. Hence $u_t \in \mathcal{U}_t$. ∎

REMARK 3.8. If $H < \frac{e^{\mu r}}{1+L}$, all the conditions defining an unstable manifold for (1.1) are satisfied expect for the exponential attraction of solutions with different initial conditions, that is, condition (c). That is why we have assumed that $H < \min\{1, \frac{e^{\mu r}}{1+L}\}$.

4. Exponential attraction. With the established theory of unstable manifolds for the differential equation (1.1), we propose to show that the unstable manifold $\mathcal{U} = \{\mathcal{U}_t\}_{t\in\mathbb{R}}$ exponentially attracts all solutions of (1.1).

THEOREM 4.1. *Assume that* (H_1)–(H_4) *and* (3.10) *hold. Let*

$$\tilde{c} := L(1+K)\left(\frac{LH}{1-He^{-\mu r}}L + e^{\mu r}c\right) < 1,$$

where c is a constant defined below. Then the unstable manifold $\mathcal{U} = \{\mathcal{U}_t\}_{t\in\mathbb{R}}$ *exponentially attracts all mild solutions of* (1.1). *This means that for any mild solution* $u(\cdot)$ *of* (1.1) *with initial function* u_τ*, there exists a solution* $\tilde{u}(\cdot)$ *lying in* $\mathcal{U}$ (*that is,* $\tilde{u}_t \in \mathcal{U}_t$ *for all* $t \in \mathbb{R}$) *and constants* $N, \delta > 0$ *such that*

$$\|u_t - \tilde{u}_t\|_{\mathcal{C}} \le N e^{-\delta(t-\tau)} \|u_\tau - \tilde{u}_\tau\|_{\mathcal{C}} \quad \textit{for all } t \ge \tau.$$

Proof. Let $\tau \in \mathbb{R}$ and introduce the space

$$C_{\tau,\mu} := \Big\{ v \in C([\tau - r, \infty), X) : \sup_{t \ge \tau - r} e^{\mu(t-\tau)} \|v(t)\| < \infty \Big\},$$

equipped with the norm $|v|_\mu := \sup_{t\ge\tau-r} e^{\mu(t-\tau)}\|v(t)\|$. It is easy to see that $C_{\tau,\mu}$ is a Banach space. To achieve our goal, we will find $\tilde{u}(\cdot)$ in the form $\tilde{u}(\cdot) = u(\cdot) + v(\cdot)$ such that $v \in C_{\tau,\mu}$.

It is clear that $\tilde{u}(\cdot)$ is a solution of (1.1) if and only if $v(\cdot)$ is a solution of the equation

$$\begin{aligned} v(t) &= U_B(t,\tau)v(\tau) \\ &\quad + \lim_{\lambda\to\infty} \int_\tau^t U_B(t,\sigma)\lambda R(\lambda, A)(f(\sigma, u_\sigma + v_\sigma) - f(\sigma, u_\sigma))\, d\sigma. \end{aligned} \tag{4.1}$$

Set

$$\tilde{f}(t, v_t) = f(t, u_t + v_t) - f(t, u_t).$$

Then $\tilde{f} : \mathbb{R} \times \mathcal{C} \to X$ is also φ-Lipschitz. Further, one can remark that $\tilde{f}(t, 0) = 0$. Equation (4.1) can then be rewritten as

$$v(t) = U_B(t,\tau)v(\tau) + \lim_{\lambda\to\infty} \int_\tau^t U_B(t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma. \tag{4.2}$$

Using the same method as in the proof of Lemma 3.3 and Remark 3.4, we prove that the solution of (4.2) is bounded on $[\tau - r, \infty)$ if and only if it satisfies

$$v(t) = U_B(t,\tau)\mu_0 + \lim_{\lambda\to\infty} \int_\tau^\infty \Gamma_B(t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma \tag{4.3}$$

for some $\mu_0 \in \operatorname{Im} P_B(\tau)$ and $t \ge \tau$. Furthermore, set

$$(Cv)(t) := \begin{cases} U_B(t,\tau)\mu_0 + \lim\limits_{\lambda\to\infty} \displaystyle\int_\tau^\infty \Gamma_B(t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma \quad \text{for } t \ge \tau, \\[2ex] U_B(2\tau - t,\tau)\mu_0 + \lim\limits_{\lambda\to\infty} \displaystyle\int_\tau^\infty \Gamma_B(2\tau - t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma \\ \hfill \text{for } \tau - r \le t \le \tau. \end{cases}$$

Using the fixed point theorem, one can show that C has a fixed point v in the Banach space $BC([\tau - r, \infty), X)$ such that

$$v(t) = U_B(2\tau - t, \tau)\mu_0 + \lim_{\lambda\to\infty} \int_\tau^\infty \Gamma_B(2\tau - t, \sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma \quad \text{for } \tau - r \le t \le \tau. \tag{4.4}$$

Now, we choose $\mu_0 \in \operatorname{Im} P_B(\tau)$ such that $\tilde{u}_\tau = u_\tau + v_\tau \in \mathcal{U}_\tau$. This means that

$$(I - \mathbb{P}_B(\tau))(u_\tau + v_\tau)(\theta) = \Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau + v_\tau))(\theta). \tag{4.5}$$

Thus,

$$\begin{aligned} \mu_0 &= (v_\tau - \mathbb{P}_B(\tau)v_\tau)(0) \\ &= -\mathbb{P}_B(\tau)u(\tau) + \Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau + v_\tau))(0). \end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.3) and (4.4), it follows that

(4.7)
$$v(t) = \begin{cases} U_B(t,\tau)\big(-\mathbb{P}_B(\tau)u(\tau) + \Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau + v_\tau))(0)\big) \\ \quad + \lim_{\lambda\to\infty} \int_\tau^\infty \Gamma_B(t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma & \text{for } t \ge \tau, \\ U_B(2\tau - t,\tau)\big(-\mathbb{P}_B(\tau)u(\tau) + \Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau + v_\tau))(0)\big) \\ \quad + \lim_{\lambda\to\infty} \int_\tau^\infty \Gamma_B(2\tau - t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma & \text{for } \tau - r \le t \le \tau. \end{cases}$$

Consequently, $\tilde{u}(\cdot)$ satisfies (1.1) and belongs to $\mathcal{U}_\tau$ if and only if $v(\cdot)$ satisfies (4.7). Accordingly, we look for solutions to (4.7) in the Banach space $C_{\tau,\mu}$. Define a mapping $\mathcal{F}$ as follows:

$$(\mathcal{F}v)(t) = \begin{cases} U_B(t,\tau)\big(-\mathbb{P}_B(\tau)u(\tau) + \Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau + v_\tau))(0)\big) \\ \quad + \lim_{\lambda\to\infty} \int_\tau^\infty \Gamma_B(t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma \quad \text{for } t \ge \tau, \\ U_B(2\tau - t,\tau)\big(-\mathbb{P}_B(\tau)u(\tau) + \Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau + v_\tau))(0)\big) \\ \quad + \lim_{\lambda\to\infty} \int_\tau^\infty \Gamma_B(2\tau - t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\, d\sigma \\ \qquad\qquad \text{for } \tau - r \le t \le \tau. \end{cases}$$

Then, for $\tau - r \le t \le \tau$, we have

$$\begin{aligned} e^{\mu(t-\tau)}\|(\mathcal{F}v)(t)\| &\le e^{\mu(t-\tau)}\|U_B(2\tau - t,\tau)\mu_0\| \\ &\quad + e^{\mu(t-\tau)} \lim_{\lambda\to\infty} \int_\tau^\infty \|\Gamma_B(2\tau - t,\sigma)\lambda R(\lambda, A)\tilde{f}(\sigma, v_\sigma)\|\, d\sigma, \end{aligned}$$

$$\leq Le^{2\mu(t-\tau)}\|\mu_0\| + L(1+K)e^{\mu r}|v|_\mu \times \Big(\int_\tau^{2\tau-t} e^{-2\mu(\tau-t)}\varphi(\sigma)\,d\sigma + \int_{2\tau-t}^{\infty} e^{-2\mu(\sigma-\tau)}\varphi(\sigma)\,d\sigma\Big).$$

As φ is locally integrable, from (2.4) it follows that there exists $c > 0$ such that, for all $\tau - r \leq t \leq \tau$,

$$e^{\mu(t-\tau)}\|(\mathcal{F}v)(t)\| \leq L\|\mu_0\| + cL(1+K)e^{\mu r}|v|_\mu.$$

Similarly, for $t \geq \tau$, we have

$$\begin{aligned} e^{\mu(t-\tau)}\|(\mathcal{F}v)(t)\| &\leq e^{\mu(t-\tau)}\|U_B(t,\tau)\mu_0\| \\ &\quad + e^{\mu(t-\tau)}\lim_{\lambda\to\infty}\int_\tau^\infty \|\Gamma_B(t,\sigma)\lambda R(\lambda,A)\tilde{f}(\sigma,v_\sigma)\|\,d\sigma \\ &\leq L\|\mu_0\| \\ &\quad + L(1+K)e^{\mu r}|v|_\mu\Big(\int_\tau^t \varphi(\sigma)\,d\sigma + \int_t^\infty e^{-2\mu(\sigma-t)}\varphi(\sigma)\,d\sigma\Big) \\ &\leq L\|\mu_0\| + cL(1+K)e^{\mu r}|v|_\mu. \end{aligned}$$

This yields

$$|\mathcal{F}v|_\mu \leq L\|\mu_0\| + cL(1+K)e^{\mu r}|v|_\mu.$$

Using the Lipschitz condition on Λ_τ, it follows that

$$\begin{aligned} \|\mu_0\| &\leq \|\Lambda_\tau(\mathbb{P}_B(\tau)u_\tau)(0) - P_B(\tau)u(\tau)\| \\ &\quad + \|\Lambda_\tau(\mathbb{P}_B(\tau)(u_\tau+v_\tau))(0) - \Lambda_\tau(\mathbb{P}_B(\tau)u_\tau)(0)\| \\ &\leq \|\Lambda_\tau(\mathbb{P}_B(\tau)u_\tau) - (I-\mathbb{P}_B(\tau))u_\tau\|_{\mathcal{C}} + \frac{LHe^{-\mu r}}{1-He^{-\mu r}}L(1+K)\|v_\tau\|_{\mathcal{C}} \\ &\leq \|\Lambda_\tau(\mathbb{P}_B(\tau)u_\tau) - (I-\mathbb{P}_B(\tau))u_\tau\|_{\mathcal{C}} + \frac{LH}{1-He^{-\mu r}}L(1+K)|v|_\mu. \end{aligned}$$

Consequently,

$$\begin{aligned} |\mathcal{F}v|_\mu &\leq L\|\Lambda_\tau(\mathbb{P}_B(\tau)u_\tau) - (I-\mathbb{P}_B(\tau))u_\tau\|_{\mathcal{C}} \\ &\quad + \frac{LH}{1-He^{-\mu r}}L^2(1+K)|v|_\mu + L(1+K)e^{\mu r}c|v|_\mu \\ &\leq L\|\Lambda_\tau(\mathbb{P}_B(\tau)u_\tau) - (I-\mathbb{P}_B(\tau))u_\tau\|_{\mathcal{C}} + \tilde{c}|v|_\mu. \end{aligned} \tag{4.8}$$

This means that $\mathcal{F}v$ belongs to $C_{\tau,\mu}$.

Now, let us prove that $\mathcal{F}$ is a contraction. Indeed, let $v, w \in C_{\tau,\mu}$. Then, for $t \in [\tau - r, \tau]$,

$$
\begin{aligned}
&e^{\mu(t-\tau)}\|(\mathcal{F}v)(t)-(\mathcal{F}w)(t)\| \\
&\qquad\le Le^{\mu(t-\tau)}e^{-\mu(\tau-t)}\|\psi_0-\xi_0\| \\
&\qquad\quad + L(1+K)e^{\mu(t-\tau)}\int_\tau^\infty e^{-\mu|2\tau-t-\sigma|}\varphi(\sigma)\|v_\sigma-w_\sigma\|_{\mathcal{C}}\,d\sigma \\
&\qquad\le L\|\psi_0-\xi_0\| + L(1+K)e^{\mu r}c|v-w|_\mu,
\end{aligned}
$$

and for $t \ge \tau$,

$$
\begin{aligned}
&e^{\mu(t-\tau)}\|(\mathcal{F}v)(t)-(\mathcal{F}w)(t)\| \\
&\qquad\le L\|\psi_0-\xi_0\| + L(1+K)e^{\mu(t-\tau)}\int_\tau^\infty e^{-\mu|t-\sigma|}\varphi(\sigma)\|v_\sigma-w_\sigma\|_{\mathcal{C}}\,d\sigma \\
&\qquad\le L\|\psi_0-\xi_0\| + L(1+K)e^{\mu r}c|v-w|_\mu.
\end{aligned}
$$

Therefore,

$$
|\mathcal{F}_v-\mathcal{F}_w|_\mu \le L\|\psi_0-\xi_0\| + cL(1+K)e^{\mu r}|v-w|_\mu.
$$

On the other hand,

$$
\begin{aligned}
\|\psi_0-\xi_0\| &= \|\varLambda_\tau(\mathbb{P}_B(\tau)(u_\tau+v_\tau))(0)-\varLambda_\tau(\mathbb{P}_B(\tau)(u_\tau+w_\tau))(0)\| \\
&\le \frac{LHe^{-\mu r}}{1-He^{-\mu r}}\|\mathbb{P}_B(\tau)(v_\tau-w_\tau)\|_{\mathcal{C}} \\
&\le \frac{LHe^{-\mu r}}{1-He^{-\mu r}}L(1+K)e^{\mu r}|v-w|_\mu.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|\mathcal{F}v-\mathcal{F}w|_\mu &\le L(1+K)\Big(\frac{LH}{1-He^{-\mu r}}L+e^{\mu r}c\Big)|v-w|_\mu \\
&\le \tilde{c}|v-w|_\mu.
\end{aligned}
$$

Since $\tilde{c}<1$, this proves that $\mathcal{F}$ is a contraction on $C_{\tau,\mu}$. Therefore, $\mathcal{F}$ has a unique fixed point v belonging to $C_{\tau,\mu}$. By (4.8), this yields

$$
|v|_\mu \le \frac{L}{1-\tilde{c}}\|\varLambda_\tau(\mathbb{P}_B(\tau)u_\tau)-(I-\mathbb{P}_B(\tau))u_\tau\|_{\mathcal{C}}.
$$

This means that by virtue of (4.5),

$$
\begin{aligned}
\|\tilde{u}_t-u_t\|_{\mathcal{C}} = \|v_t\|_{\mathcal{C}} &\le e^{\mu r}e^{-\mu(t-\tau)}|v|_\mu \\
&\le e^{\mu r}\frac{L}{1-\tilde{c}}e^{-\mu(t-\tau)}\big(\|\varLambda_\tau(\mathbb{P}_B(\tau)u_\tau)-\varLambda_\tau(\mathbb{P}_B(\tau)\tilde{u}_\tau)\|_{\mathcal{C}} \\
&\qquad\qquad + \|(I-\mathbb{P}_B(\tau))(u_\tau-\tilde{u}_\tau)\|_{\mathcal{C}}\big)
\end{aligned}
$$

$$\leq e^{\mu r}\frac{L}{1-\tilde{c}}e^{-\mu(t-\tau)}\bigg(\frac{LHe^{-\mu r}}{1-He^{-\mu r}}L(1+K)\|u_\tau-\tilde{u}_\tau\|_{\mathcal{C}} + (L(1+K)+1)\|u_\tau-\tilde{u}_\tau\|_{\mathcal{C}}\bigg)$$

$$\leq e^{\mu r}\frac{L}{1-\tilde{c}}e^{-\mu(t-\tau)}\bigg(\frac{LHe^{-\mu r}}{1-He^{-\mu r}}L(1+K)+L(1+K)+1\bigg) \times \|u_\tau-\tilde{u}_\tau\|_{\mathcal{C}}$$

for all $t \geq \tau$. This completes the proof. ■

5. Application. Now, we propose to discuss the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t}x(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}x(t,\xi) - \alpha x(t,\xi) \\ \quad +\beta e^{-\varepsilon|t|}\displaystyle\int_{-r}^{0}\ln(1+|x(t+\theta,\xi)|)\,d\theta & \text{for } t \geq s,\ \xi\in[0,2\pi], \\ x(t,0)=x(t,2\pi)=0 & \text{for } t\in\mathbb{R}, \\ x_s(\theta,\xi)=\Phi(\theta,\xi) & \text{for } -r\leq\theta\leq 0,\ 0\leq\xi\leq 2\pi, \end{cases} \tag{5.1}$$

where $\alpha,\varepsilon>0$, and $\beta\neq 0$.

Consider the Banach space $X := C([0,2\pi],\mathbb{R})$ of all continuous functions from $[0,2\pi]$ into $\mathbb{R}$. Let the operator $C : X \supset D(C) \to X$ be defined by

$$D(C) = \{\phi\in C^2([0,2\pi],\mathbb{R}) : \phi(0)=\phi(2\pi)=0\}$$
$$(C\phi)(x) = \frac{\partial^2}{\partial x^2}\phi(x) - \alpha\phi(x), \quad x\in[0,2\pi].$$

Then $(C, D(C))$ is a Hille–Yosida operator. The spectrum of the part of C in $X_0 = \overline{D(C)} = \{\psi\in C([0,2\pi],\mathbb{R}) : \psi(0)=\psi(2\pi)=0\}$ is the set $\{-n^2-\alpha : n=1,2,\ldots\}$. It can be seen that C generates an analytic semigroup $(e^{tC})_{t\geq 0}$ on X_0. According to the spectral mapping theorem (see for instance [EN]), $(e^{tC})_{t\geq 0}$ is hyperbolic. Therefore, the evolution family $(U_B(t,s))_{t\geq s}$ corresponding to C (that is, $U_B(t,s) := e^{(t-s)C}$) has exponential dichotomy with dichotomy constants L, μ. Consequently, system (5.1) can be rewritten as the following abstract Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}x(t,\cdot) = (A+B(t))x(t,\cdot) + f(t,x_t(\theta,\cdot)) & \text{for } t\geq s, \\ x_s(\theta,\cdot) = \Phi(\theta,\cdot)\in\mathcal{C}_A & \text{for } -r\leq\theta\leq 0, \end{cases}$$

where $A+B(t) := C$ and $f:\mathbb{R}\times\mathcal{C}\to X$ is defined by

$$f(t,\Phi)(x) = \beta e^{-\varepsilon|t|}\int_{-r}^{0}\ln(1+|\Phi(s)(x)|)\,ds, \quad x\in[0,2\pi].$$

Let us show that f is φ-Lipschitz with φ belonging to an admissible space E. Indeed, condition (i) of φ-Lipschitzness is evident. Now, let us prove (ii). Since $\ln(1+\sigma) \le \sigma$ for $\sigma \ge 0$, we have

$$\begin{aligned}|f(t,\phi)(x) - f(t,\psi)(x)| &\le |\beta| e^{-\varepsilon|t|} \int_{-r}^{0} \left| \ln\left(\frac{1+|\phi(s)(x)|}{1+|\psi(s)(x)|} \right) \right| ds \\ &= |\beta| e^{-\varepsilon|t|} \int_{-r}^{0} \left| \ln\left(1 + \frac{|\phi(s)(x)| - |\psi(s)(x)|}{1+|\psi(s)(x)|} \right) \right| ds \\ &\le |\beta| r e^{-\varepsilon|t|} \sup_{s\in[-r,0]} \sup_{x\in[0,2\pi]} |\phi(s)(x) - \psi(s)(x)|.\end{aligned}$$

Then

$$\|f(t,\phi) - f(t,\psi)\| \le |\beta| r e^{-\varepsilon|t|} \|\phi-\psi\|_{\mathcal{C}}.$$

This shows that f is φ-Lipschitz with $\varphi(t) = |\beta| r e^{-\varepsilon|t|} \in L^p(\mathbb{R})$, an admissible space (for any $p \ge 1$). Note that the constants Q and R in Definition 2.4 are defined by $Q = R = 1$ and $\|\Theta_1 \varphi\|_\infty \le 2|\beta| r/\varepsilon$. Theorem 3.7 shows that if

$$\frac{|\beta| r}{\varepsilon} \le \min\left\{ \frac{1-e^{-\mu}}{4L(1+L)(1+K)}, \frac{1-e^{-\mu}}{4L(1+K)e^{\mu r}} \right\}$$

then there exists an unstable manifold $\mathcal{U}$ for the mild solutions of (5.1).

Note that $\varphi \in L^1(\mathbb{R})$. Using Theorem 4.1, one can take $c := \|\varphi\|_{L^1}$. Then the unstable manifold $\mathcal{U}$ exponentially attracts any mild solution of (1.1) if

$$\tilde{c} = L(1+K)\left(\frac{LH}{1-He^{-\mu r}} L + e^{\mu r} \|\varphi\|_{L_1} \right) < 1,$$

i.e.

$$L(1+K)\left(\frac{4L^3 |\beta| r e^{\mu r}(1+K)}{\varepsilon(1-e^{-\mu}) - 4L|\beta| r(1+K)} + \frac{2|\beta| r e^{\mu r}}{\varepsilon} \right) < 1.$$

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