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GRAPH AND POINTWISE UPPER KURATOWSKI LIMITS

BY

MILAN MATEJDES (Trnava)

Abstract. We deal with a certain relationship between the graph upper Kuratowski limit $\mathrm{Ls}\,\mathrm{Gr}\,F_n$ of a sequence of graphs of lower quasi continuous multifunctions and the pointwise upper Kuratowski limit $\mathrm{Ls}\,F_n(x)$ of a sequence of their values. Namely, we will study under which conditions there is a residual set on which both Kuratowski limits are identical.

1. Introduction. A motivation of this work comes from the paper [2] (see also [1]) which solves a few problems concerning the graph and pointwise convergence of sequences of quasi-continuous functions. More precisely, a connection between the upper Kuratowski limit $\mathrm{Ls}\,\mathrm{Gr}\,f_n$ of a sequence $\{\mathrm{Gr}\,f_n : n \in \omega\}$ of the graphs of quasi-continuous functions and the upper Kuratowski limit $\mathrm{Ls}\,f_n$ of a sequence $\{f_n(x) : n \in \omega\}$ of their values was studied and one of the main theorems of [2] guarantees the existence of a dense G_δ-set A such that for any $x \in A$, if $(x, f(x))$ is a cluster point of a sequence $\{\mathrm{Gr}\,f_n : n \in \omega\}$, then $f(x)$ is a cluster point of $\{f_n(x) : n \in \omega\}$, provided f is a cliquish selection of the multifunction $F : x \mapsto [\mathrm{Ls}\,\mathrm{Gr}\,f_n]_x$, where $[\cdot]_x$ denotes the x-section of a subset of $X \times Y$.

In [9], the results of [2] were extended to sequences of lower quasi-continuous multifunctions. Also in [9] the existence of a selection of $F : x \mapsto [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ which is quasi-continuous on an open dense set was proved. In [2, Theorem 1] such a selection was implicitly supposed to exist. The results of [2] and [9] stimulate a further question: under which conditions the multifunction $F : x \mapsto [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ is identical with the pointwise upper Kuratowski limit $\mathrm{Ls}\,F_n$ of the sequence $\{F_n(x) : n \in \omega\}$ of lower quasi-continuous multifunctions on a dense G_δ-set. More precisely, is there a dense G_δ-subset A of X such that $[\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = [\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x$ for any $x \in A$?

2. Basic definitions and survey of some results. In what follows, X is a nonempty topological space. We denote by $\bar{A}$ and A° the closure and

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the interior of A in X, respectively. A set A is *quasi-open* if for any open set G intersecting A, there is a nonempty open set $H \subset A \cap G$, equivalently $A \subseteq \overline{A^\circ}$. If $A \subseteq X \times Y$, then $[A]_x = \{y \in Y : (x,y) \in A\}$ denotes the x-section of A; and ω denotes the natural numbers. A topological space X is called *σ-compact* if $X = \bigcup_{n\in\omega} C_n$, where C_n is compact for any $n \in \omega$.

Definition 2.1. Let $\{A_n : n \in \omega\}$ be a sequence of nonempty subsets of a topological space. The *upper Kuratowski limit* of $\{A_n : n \in \omega\}$, denoted by $\operatorname{Ls} A_n$, is defined as the set of all points x such that each neighborhood of x intersects A_n for infinitely many n. A point $x \in \operatorname{Ls} A_n$ is called a *cluster point* of $\{A_n : n \in \omega\}$. It is clear that $\operatorname{Ls} A_n = \bigcap_{n\in\omega} \overline{\bigcup_{k\geq n} A_k}$ and $\operatorname{Ls} A_n$ is a closed set.

If X, Y are topological spaces, $\emptyset \neq A \subset X$, and $F : A \to Y$ is a nonempty-valued multifunction from A to Y, then we write $F^-(V) = \{x \in A : F(x) \cap V \neq \emptyset\}$ and $F^+(V) = \{x \in A : F(x) \subset V\}$ for $V \subset Y$, and $\operatorname{Gr} F = \{(x,y) : y \in F(x)\}$ is the *graph* of F. If for a function $f : X \to Y$, $f(x) \in F(x)$ for any $x \in A \subset X$, then we say f is a *selection* of F *on* A. If $A = X$, we say f is a *selection* of F.

Definition 2.2. If $\{F_n : n \in \omega\}$ is a sequence of multifunctions from A to Y, then $\operatorname{Ls}\operatorname{Gr} F_n$ is called the *graph upper Kuratowski limit* of $\{F_n : n \in \omega\}$, and we denote by $\operatorname{Ls} F_n$ the *pointwise upper Kuratowski limit* of $\{F_n : n \in \omega\}$, defined as the multifunction from A to Y which maps each $x \in A$ to $\operatorname{Ls} F_n(x)$, provided it is nonempty for any $x \in A$. It is clear $\operatorname{Ls} F_n(x) \subset [\operatorname{Ls}\operatorname{Gr} F_n]_x$ for any $x \in X$.

Definition 2.3 ([10]). A multifunction $F : X \to Y$ is *lower quasi-continuous at a point* x if for any open set G intersecting $F(x)$ and any open set H containing x there is a nonempty open set $U \subset H$ such that $G \cap F(u) \neq \emptyset$ for any $u \in U$. Moreover, F is *lower quasi-continuous* if it is so at any point. For a function $f : X \to Y$, we only consider quasi-continuity at x.

Now we recall two main results of [2] and [9].

Theorem 2.4 (for a sequence of quasi-continuous functions see [2]). *Let X be a Baire space, and Y be a developable space. If $f : X \to Y$ is a function which is quasi-continuous at any point of a dense set $B_1 \subset X$, $\{F_n : n \in \omega\}$ is a sequence of lower quasi-continuous multifunctions from X to Y and $f(x) \in [\operatorname{Ls}\operatorname{Gr} F_n]_x$ for any x from a dense set B_2, then there is a dense G_δ-set A such that $f(x) \in \operatorname{Ls} F_n(x)$ for any $x \in A$, i.e., $f(x)$ is a cluster point of $\{F_n(x) : n \in \omega\}$.*

The function f from the theorem above (in [2], f is supposed to be cliquish) seems to be redundant, since its existence is guaranteed by the next theorem from [9] (also in [2, Theorem 4] for a sequence of equi-quasi-continuous functions there are no assumptions on f).

THEOREM 2.5. *Let X be a Baire space, Y be a T_1-regular σ-compact space and suppose the x-section of $\mathrm{Ls}\,\mathrm{Gr}\,F_n$ is nonempty for any $x \in X$. Then there is a function $f : X \to Y$ such that $\mathrm{Gr}\, f \subset \mathrm{Ls}\,\mathrm{Gr}\,F_n$ (i.e., f is a selection of $F : x \mapsto [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$) and f is quasi-continuous on an open dense set in X.*

3. Main results. As mentioned above, the main question is whether there is a dense G_δ-set A such that $[\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = [\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x$ for any $x \in A$ and a crucial point is to find not only a selection of $F : x \mapsto [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ which is quasi-continuous on an open dense set, but also the existence of a suitable dense quasi-continuous representation of $F : x \mapsto [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ is significant (see Theorem 3.2 below).

THEOREM 3.1. *Let X be Baire and Y be metric σ-compact. If $F : X \to Y$ has closed graph, then for any nonempty compact subset Z of Y such that $A := F^-(Z)$ is nonempty, there is a selection $f_Z : X \to Y$ of F such that $f_Z(x) \in Z$ for $x \in A$ and f_Z is quasi-continuous at any point of X except for a closed nowhere dense set.*

Proof. Since F has closed graph, for any open set G with compact complement $F^+(G)$ is open. So F is u-upper Baire continuous (see [8]). By [8, Corollary 1], F has a selection $f_0 : X \to Y$ which is quasi-continuous except for a nowhere dense set N_0. Since F has closed graph and Z is compact, the set $A = F^-(Z)$ is closed.

If $A^\circ = \emptyset$ (i.e., A is nowhere dense), then we define $f_Z(x) = f_0(x)$ if $x \in X \setminus A$, and $f_Z(x)$ is an arbitrary point from $F(x) \cap Z$ if $x \in A$. Then f_Z is quasi-continuous at any point from $X \setminus (A \cup \overline{N}_0)$. Since $A \cup \overline{N}_0$ is closed nowhere dense, f_Z fulfils all conditions of our theorem.

Suppose $A^\circ \neq \emptyset$. Define $F_Z : A^\circ \to Y$ by $F_Z(x) = F(x) \cap Z$. Then F_Z has closed graph in $A^\circ \times Y$, the subspace A° is Baire and by [8, Corollary 1], there is a selection $f_1 : A^\circ \to Y$ of F_Z which is quasi-continuous everywhere except on a nowhere dense subset N_1 in the subspace A°, so N_1 is also nowhere dense in X. Define $f_Z : X \to Y$ by $f_Z(x) = f_0(x)$ if $x \notin A$, $f_Z(x) = f_1(x)$ if $x \in A^\circ$ and $f_Z(x) \in F(x) \cap Z$ if $x \in A \setminus A^\circ$. Then f_Z is a selection of F which is quasi-continuous at any point $x \notin \overline{N}_0 \cup \overline{N}_1 \cup (A \setminus A^\circ)$ which is a closed nowhere dense set, and if $x \in A$, then $f_Z(x) \in Z$. ∎

THEOREM 3.2. *Let X be Baire and Y be metric σ-compact. If $F : X \to Y$ has closed graph, then there is a dense representation $\{f_r : r \in \omega\}$ of F on X (i.e., $F(x) = \overline{\bigcup_{r\in\omega}\{f_r(x)\}}$ for any $x \in X$) such that any function f_r is quasi-continuous everywhere except on a closed nowhere dense set.*

Proof. Since Y is σ-compact, Y is separable. Let $\{y_s : s \in \omega\}$ be a dense set in Y and let $Y = \bigcup_{n\in\omega} C_n$ with C_n compact. For any $s, n, k \in \omega$ set $Z_{s,n,k} = S(y_s, 1/k) \cap C_n$, where d is a metric in Y and $S(y_s, 1/k) = \{y : d(y, y_s) \leq 1/k\}$. So, $Z_{s,n,k}$ is compact and without loss of generality we can suppose it is nonempty. By Theorem 3.1, for any $s, n, k \in \omega$ for which $A_{s,n,k} := F^-(Z_{s,n,k})$ is nonempty, there is a selection $f_{s,n,k} : X \to Y$ of F such that $f_{s,n,k}(x) \in Z_{s,n,k}$ for $x \in A_{s,n,k}$ and $f_{s,n,k}$ is quasi-continuous at any point of X except for a closed nowhere dense set. We will show that $\{f_{s,n,k} : s, n, k \in \omega\}$ is a dense representation of F on X. Let $x \in X$, $y \in F(x)$, and $\varepsilon > 0$. Then there are $k, s \in \omega$ such that $1/k < \varepsilon/2$ and $y_s \in S(y, 1/k)$. Since $Y = \bigcup_{n\in\omega} C_n$, there is $n \in \omega$ such that $y \in C_n$, so $y \in S(y_s, 1/k) \cap C_n = Z_{s,n,k}$. Since $f_{s,n,k}(x) \in Z_{s,n,k}$, we have $d(f_{s,n,k}(x), y) \leq 2/k < \varepsilon$. ∎

Now we can state the main theorem of our paper. If we identify a multifunction with its graph, we prove that for a sequence of lower quasi-continuous multifunctions, the graph upper Kuratowski limit is equal to the pointwise upper Kuratowski limit on a dense G_δ-set.

THEOREM 3.3. *Let X be Baire, Y be metric σ-compact and $\{F_n : n \in \omega\}$ be a sequence of lower quasi-continuous multifunctions from X to Y. Then there is a dense G_δ-subset A of X such that $[\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = [\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x$ for any $x \in A$ (in general $[\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ and $[\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x$ may be empty).*

Proof. Let $D := \{x \in X : [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x \neq \emptyset\}$ and $G := D^\circ$. Suppose $G \neq \emptyset$. Since the multifunction $F_0 : G \to Y$ defined as $F_0(x) = [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ for $x \in G$ has closed graph (in $G \times Y$), by Theorem 3.2 there is a dense representation $\{f_k : k \in \omega\}$ of F_0 on G such that any $f_k : G \to Y$ is quasi-continuous at any $x \notin A_k \subset G$ and A_k is closed nowhere dense in G. Since $G \setminus A_k$ is dense in G and $f_k(x) \in [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = F_0(x)$ for any $x \in G$, by Theorem 2.4, for any $k \in \omega$, there is a set $C_k \subset G$ which is a dense and G_δ-set in G such that $f_k(x) \in \mathrm{Ls}\,F_n(x)$ for any $x \in C_k$. Let $x \in A_0 := \bigcap_{k\in\omega} C_k$ and $B(x) := \bigcup_{k\in\omega}\{f_k(x)\} \subset \mathrm{Ls}\,F_n(x)$. Since $B(x)$ is dense in $F_0(x)$ (from dense representation) and from the inclusions $B(x) \subset \mathrm{Ls}\,F_n(x) \subset F_0(x)$ and the closedness of $\mathrm{Ls}\,F_n(x)$ and $F_0(x)$ it follows that $\overline{B(x)} = \mathrm{Ls}\,F_n(x) = F_0(x) = [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$, so $[\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x = [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ for any $x \in A_0$.

Since Y is σ-compact and $\mathrm{Ls}\,\mathrm{Gr}\,F_n$ is closed in $X \times Y$, the set D is F_σ. Let $D = \bigcup_{n\in\omega} K_n$ with K_n closed. By the inclusions

$$D \setminus G = \bigcup_{n\in\omega} K_n \setminus \Big(\bigcup_{n\in\omega} K_n\Big)^\circ \subset \bigcup_{n\in\omega} K_n \setminus \bigcup_{n\in\omega} K_n^\circ \subset \bigcup_{n\in\omega} (K_n \setminus K_n^\circ),$$

$D \setminus G$ is of first category and for $x \notin D$ evidently $\emptyset = [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = [\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x$. Moreover, for $x \in A := A_0 \cup (X \setminus D)$ we have $[\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = [\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x$, and A is G_δ and dense in X.

Finally, if $G = D^\circ = \emptyset$, then D is F_σ and of first category. That means $[\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x \subset [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x = \emptyset$ for any $x \in X \setminus D$, so $\emptyset = [\mathrm{Gr}\,\mathrm{Ls}\,F_n]_x = [\mathrm{Ls}\,\mathrm{Gr}\,F_n]_x$ holds on a dense G_δ-set. ∎

We finish with a few comments concerning closed graph theorems. Let us recall a few results. A multifunction $F : X \to Y$ is *upper* [*lower*] *semicontinuous* at a point x if for any open set G such that $F(x) \subset G$ [$F(x) \cap G \neq \emptyset$] there is an open set containing x such that $F(z) \subset G$ [$F(z) \cap G \neq \emptyset$] for any $z \in G$. By [6, Corollaries 3.4 and 3.5], if $F : X \to Y$ is a compact valued multifunction with closed graph and Y is Čech-complete [locally compact], then the set of points of upper semicontinuity of F is a G_δ-subset [an open subset] of X (see also [6, Theorem 2.3 and Corollary 2.2]). The closedness of the graph is related to so-called c-upper semicontinuity. A multifunction X is *c-upper semicontinuous* if $F^+(G)$ is open whenever G is open with compact complement. By [6, Propositions 1.4 and 1.5], if Y is locally compact (or X and Y are first countable) and $F : X \to Y$ is c-upper semicontinuous with closed values, then F has closed graph. The opposite implication holds trivially. Similarly we can define the notion of c-lower semicontinuity. Moreover (see [3]), if X is Baire and Y is Hausdorff and σ-compact, then for any c-lower semicontinuous multifunction $F : X \to Y$ such that any value $F(x)$ is a subset of some C_n, where $Y = \bigcup_{n\in\omega} C_n$ and C_n is compact, the set of points for which F is not lower semicontinuous is a nowhere dense set.

Our last theorem completes the results mentioned above. It guarantees only the lower quasi-continuity on a dense G_δ-set and it may be of importance in connection with upper Kuratowski limits.

A multifunction $F : X \to Y$ is called *minimal usco* (see [5]) if F is compact valued, upper semicontinuous and it does not contain properly any other compact valued upper semicontinuous multifunction from X to Y; moreover, F is *locally bounded* if for any x there is an open set U containing x and a compact set C such that $F(c) \subset C$ for any $c \in U$. Note that a minimal usco multifunction is always lower quasi-continuous.

THEOREM 3.4. *Let X be Baire and Y be metric σ-compact. If $F : X \to Y$ is a multifunction with closed graph, then F is lower quasi-continuous at any point from a dense G_δ-set. Moreover, there is a dense open set $H \subset X$ and a multifunction $F_0 : X \to Y$ which is locally bounded minimal usco on H and $F_0(x) \subset F(x)$ for any $x \in X$.*

Proof. Since F has closed graph, by Theorem 3.2 there is a dense representation $\{f_n : n \in \omega\}$ of F on X and any f_n is quasi-continuous at any $x \in A_n$, where A_n is open dense in X. Then $A := \bigcap_{n\in\omega} A_n$ is a dense G_δ-set. Let $x \in A$, V, U be open, $x \in U$ and $F(x) \cap V \neq \emptyset$. From the dense representation, there is $n \in \omega$ such that $f_n(x) \in V$, and by quasi-continuity of f_n at x there exists a nonempty open set $H \subset U$ such that $f_n(H) \subset V$. Since f_n is a selection of F, $F(x) \cap V \neq \emptyset$ for any $x \in H$. So F is lower quasi-continuous at x. The second part follows from [8, Theorem 2]. ∎

To end this paper, we formulate a consequence of the theorem above and state an open problem. By Theorem 3.4, since $\operatorname{Ls}\operatorname{Gr} F_n$ is a closed subset of $X \times Y$, the multifunction $F : x \mapsto [\operatorname{Ls}\operatorname{Gr} F_n]_x$ is lower quasi-continuous at any point from a dense G_δ-set, for any sequence $\{F_n : n \in \omega\}$, provided $[\operatorname{Ls}\operatorname{Gr} F_n]_x \neq \emptyset$ for any $x \in X$. Since for a sequence $\{F_n : n \in \omega\}$ of lower quasi-continuous multifunctions the equality $[\operatorname{Ls}\operatorname{Gr} F_n]_x = [\operatorname{Gr}\operatorname{Ls} F_n]_x$ holds on a dense G_δ-set, an open question is: under which conditions is $F : x \mapsto [\operatorname{Ls}\operatorname{Gr} F_n]_x$ or $\operatorname{Ls} F_n$ lower (or upper) quasi-continuous at any point? This question arises from the results concerning sequences of equi-quasi-continuous (or densely equi-quasi-continuous and quasi-continuous) functions whose limit is quasi-continuous (see [2], [4], [7]). Our study yields

THEOREM 3.5. *Let X be Baire and Y be metric σ-compact. If $\{F_n : n \in \omega\}$ is a sequence of lower quasi-continuous multifunctions for which $[\operatorname{Ls}\operatorname{Gr} F_n]_x$ is nonempty for any $x \in X$, then there is a multifunction $F_0 : X \to Y$ which is locally bounded minimal usco on a dense open set and satisfies $\operatorname{Gr} F_0 \subset \operatorname{Ls}\operatorname{Gr} F_n$ and $F_0(x) \subset \operatorname{Ls} F_n(x)$ for any x from a dense G_δ-set.*

Also in [7], similar results were investigated for minimal usco/cosco maps. One can learn more about minimal usco/cusco maps from [5] and the references therein.

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Milan Matejdes
Department of Mathematics and Computer Science
Faculty of Education
Trnava University in Trnava
Priemyselná 4
918 43 Trnava, Slovakia
E-mail: milan.matejdes@truni.sk