APPLICATIONES MATHEMATICAE
44,1 (2017), pp. 15–31

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# ANALYSIS OF AN ADHESIVE CONTACT PROBLEM FOR ELASTIC MATERIALS

*Abstract.* The goal of this paper is to study a mathematical model which describes the adhesive frictionless contact between a deformable body and a foundation. The body consists of an elastic material and the process is assumed to be quasistatic. The adhesive contact condition on the normal direction is modeled by a version of normal compliance condition with unilateral constraint. The adhesion is modeled with a surface variable, the bonding field, whose evolution is described by a first-order differential equation. We present a variational formulation of the mechanical problem and prove the existence and uniqueness of a weak solution. Also, we study a penalized contact problem which admits a unique solution. We prove that when the penalization parameter converges to zero, the solution converges to the solution of the original model. The technique of the proof is based on time-dependent variational inequalities, differential equations and the Banach fixed-point theorem.

**1. Introduction.** Contact mechanics has many applications in industry as well as in daily life, such as combustion engines, coupling devices, braking systems, metalworking, and many others. It plays an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Because of the usefulness of these processes, modelling, mathematical analysis and numerical simulations of various contact processes are extensively studied. Many mathematical tools are employed

2010 *Mathematics Subject Classification*: 35J85, 47J20, 49J40, 74M15.
*Key words and phrases*: elastic, normal compliance, adhesion, frictionless, unilateral, weak solution.
Received 21 February 2016; revised 1 July 2016.
Published online 27 December 2016.

 

in the theory, such as nonlinear inclusions, variational inequalities, etc. An early attempt to study contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [19]. In that book we find a detailed analysis of adhesive contact problems with mathematical and numerical studies. Moreover, existence results were established in [1, 6, 8, 16, 20–23] in the study of unilateral contact for elastic materials.

The aim of this paper is to study a mathematical model which describes a frictionless contact problem between a nonlinear elastic body and a foundation. As in [13] the contact is modelled with a version of normal compliance and unilateral constraint. Moreover, the adhesion of contact surfaces is taken into account. Analysis of models for frictionless adhesive contact can be found in [4, 5, 9, 18–22] and the references therein. Also as in [10, 11], in this paper, we introduce the surface internal variable, the *bonding field,* which describes the fractional density of active bonds on the contact surface and which is denoted by $\beta$. This variable is restricted to values $0 \leq \beta \leq 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$, the adhesion is partial and only a fraction $\beta$ of the bonds are active. We refer the reader to the extensive bibliography on the subject in [3, 10, 11, 12, 14, 15, 17–19].

In this work we continue the investigation of adhesive frictionless contact problems begun in [21, 22]. There, models for quasistatic process of frictionless contact between a deformable body and a foundation have been analyzed; the contact was described by normal compliance, adhesion and unilateral constraint; the existence of a unique weak solution to the models has been obtained by using time-dependent variational inequalities, differential equations and the Banach fixed-point theorem. We derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain a regularity result for the solution. Moreover, we study a penalized problem which we consider as a frictionless contact problem and adhesion with unlimited penetration. We prove its unique weak solvability and show that the solution of the original model is obtained by passing to the limit as the penalization parameter converges to zero.

The paper is structured as follows. In Section 2 we describe the mechanical model. In Section 3, we introduce some notation, list the assumptions on the problem's data and give the variational formulation. In Section 4 we state and prove our main existence and uniqueness result, Theorem 4.1. Finally, in Section 5, we prove a convergence result for a penalized problem, Theorem 5.2.

**2. Problem statement.** A nonlinear elastic body occupies a regular domain $\Omega$ of $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\partial\Omega = \Gamma$. We assume that $\Gamma$ is partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $\mathrm{meas}(\Gamma_1) > 0$. Let $[0, T]$ denote the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$, and therefore the displacement vanishes there. It is acted upon by a volume force of density $f_1$ on $\Omega \times (0, T)$ and surface tractions of density $f_2$ on $\Gamma_2 \times (0, T)$. The body is in unilateral and adhesive contact with a deformable foundation over the potential contact surface $\Gamma_3$.

Thus, the classical formulation of the mechanical problem is as follows.

PROBLEM $P_1$. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\operatorname{div} \sigma(u) = -f_1 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma(u) = F\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu \le g, \quad \sigma_\nu + p_\nu(\beta, u_\nu) \le 0 \\ (\sigma_\nu + p_\nu(\beta, u_\nu))(u_\nu - g) = 0 \\ \sigma_\tau = 0 \end{array}\right\} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\dot\beta = H_{\mathrm{ad}}(\beta, u_\nu) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.7}$$

Equation (2.1) represents the equilibrium equation where $\sigma = \sigma(u)$ denotes the stress tensor. Equation (2.2) represents the elastic constitutive law of the material in which $F$ is a given function and $\varepsilon(u)$ denotes the linearized strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which $\nu$ denotes the unit outward normal vector on $\Gamma$ and $\sigma\nu$ represents the Cauchy stress vector. Condition (2.5) represents unilateral frictionless contact with adhesion in which $p_\nu$ is the normal contact function. The usual choice of the function $p_\nu$ is (see [19])

$$p_\nu(\beta, u_\nu) = p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu),$$

where $p$ is a normal compliance function which satisfies the following assumptions:

$$\begin{cases} \text{(a) } p : \mathbb{R} \to \mathbb{R}; \\ \text{(b) there exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \text{ for all } r_1, r_2 \in \mathbb{R}; \\ \text{(c) } p(r) = 0 \text{ for all } r \le 0. \end{cases}$$

$c_\nu$ is a given adhesion coefficient and $R_\nu$ is a truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [19]). Equation (2.6) is an ordinary differential equation which describes the evolution of the bonding field. The function $H_{\mathrm{ad}}$ (see [20, 21, 22]) is the adhesion evolution rate function and it depends on the bonding field and on normal displacement. In the study of our model, we assume that $H_{\mathrm{ad}}$ may change sign, and this fact allows for rebonding after debonding took place, or even allows for cycles of debonding and rebonding. An example of such a function is

$$H_{\mathrm{ad}}(\beta, r) = -\gamma_0 \frac{\beta_+}{1+\beta_+}(R_\nu(r))^2,$$

where $\gamma_0 > 0$ is a bonding coefficient. For more details on adhesive models, we refer to [19] and the references therein. Finally, (2.7) is the initial condition, in which $\beta_0$ denotes the initial bonding field and a dot above a variable represents its derivative with respect to time.

**3. Variational formulation.** We denote by $S_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$) and $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^d$ and $S_d$. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{1/2}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{1/2}$. Here and below, the indices $i$ and $j$ run between 1 and $d$ and the summation convention over repeated indices is adopted.

To proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \quad H_1 = (H^1(\Omega))^d, \quad Q = \{\sigma = (\sigma_{ij});\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$Q_1 = \{\sigma \in Q;\ \operatorname{div}\sigma \in H\}.$$

Note that $H$ and $Q$ are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \qquad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij}\, dx.$$

The linearized strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \left(\tfrac{1}{2}(u_{i,j} + u_{j,i})\right),$$

and $\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of $\sigma$. For $v \in H_1$ we still write $v$ for the trace of $v$ and we denote by $v_\nu$ and $v_\tau$ the normal and tangential components of $v$ on the boundary $\Gamma$ given by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu \nu.$$

We also denote by $\sigma_\nu$ and $\sigma_\tau$ the normal and the tangential traces of a function $\sigma \in Q_1$, and when $\sigma$ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and the following Green formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div} \sigma, v)_H = \int_\Gamma \sigma\nu.v\, da \qquad \forall v \in H_1,$$

where $da$ is the surface measure element. Now, let $V$ be the closed subspace of $H_1$ defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements be given by

$$K = \{v \in V;\ v_\nu \leq g \text{ a.e. on } \Gamma_3\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, the following Korn inequality holds [7]:

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \qquad \forall v \in V, \tag{3.1}$$

where $c_\Omega > 0$ is a constant which depends only on $\Omega$ and $\Gamma_1$. We equip $V$ with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (3.1) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on $V$. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \qquad \forall v \in V. \tag{3.2}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0,T;V)$. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to $X$; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$f_1 \in W^{1,\infty}(0,T;H), \qquad f_2 \in W^{1,\infty}(0,T;(L^2(\Gamma_2))^d) \tag{3.3}$$

and denote by $f(t)$ the element of $V$ defined by

$$(f(t),v)_V = \int_\Omega f_1(t).v\,dx + \int_{\Gamma_2} f_2(t).v\,da \qquad \forall v \in V, t \in [0,T]. \tag{3.4}$$

Using (3.3) and (3.4) yields

$$f \in W^{1,\infty}(0,T;V).$$

In the study of the mechanical problem $P_1$ we assume that the nonlinear elasticity operator $F : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} \text{(a) there exists } M > 0 \text{ such that} \\ \quad |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \le M|\varepsilon_1 - \varepsilon_2| \text{ for all } \varepsilon_1, \varepsilon_2 \in S_d \text{ and} \\ \quad \text{a.e. } x \in \Omega; \\ \text{(b) there exists } m > 0 \text{ such that} \\ \quad (F(x,\varepsilon_1) - F(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge m|\varepsilon_1 - \varepsilon_2|^2 \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \in S_d \text{ a.e. } x \in \Omega; \\ \text{(c) } x \mapsto F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \varepsilon \in S_d; \\ \text{(d) } F(x, 0_{S_d}) = 0_{S_d} \text{ for a.e. } x \in \Omega. \end{cases} \tag{3.5}$$

Next we define the functional $j : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j(\beta,u,v) = \int_{\Gamma_3} p_\nu(\beta,u_\nu)v_\nu\,da \ \forall(\beta,u,v) \in L^2(\Gamma_3) \times V \times V.$$

We assume that the normal function $p_\nu : \Gamma_3 \times [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) there exists } L_\nu > 0 \text{ such that} \\ \quad |p_\nu(x,\theta_1,r_1) - p_\nu(x,\theta_2,r_2)| \le L_\nu(|\theta_1 - \theta_2| + |r_1 - r_2|) \\ \quad \text{for all } \theta_1, \theta_2 \in [0,1], r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_3; \\ \text{(b) } x \mapsto p_\nu(x,\theta,r) \text{ is measurable on } \Gamma_3 \text{ for any } \theta \in [0,1], r \in \mathbb{R}; \\ \text{(c) } p_\nu(\cdot,\theta,r) \in L^\infty(\Gamma_3) \text{ for all } \theta \in [0,1], r < 0 \text{ and a.e. } x \in \Gamma_3; \\ \text{(d) } p_\nu(x,\theta,0) = 0 \text{ for all } \theta \in [0,1] \text{ and a.e. } x \in \Gamma_3. \end{cases} \tag{3.6}$$

The adhesive evolution rate function $H_{\mathrm{ad}} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy

$$(3.7)\quad \begin{cases} \text{(a) there exists } L_{\mathrm{ad}} > 0 \text{ such that} \\ \quad |H_{\mathrm{ad}}(x,b_1,r_1) - H_{\mathrm{ad}}(x,b_2,r_2)| \le L_{\mathrm{ad}}(|b_1-b_2| + |r_1-r_2|) \\ \quad \text{for all } b_1,b_2,r_1,r_2 \in \mathbb{R} \text{ and a.e. } x \in \mathit{\Gamma}_3; \\ \text{(b) } x \mapsto \mathrm{H}_{\mathrm{ad}}(x,b,r) \text{ is Lebesgue measurable on } \mathit{\Gamma}_3 \\ \quad \text{for all } b,r \in \mathbb{R}; \\ \text{(c) } (b,r) \to \mathrm{H}_{\mathrm{ad}}(x,b,r) \text{ is continuous on } \mathbb{R}\times\mathbb{R} \text{ for a.e. } x \in \mathit{\Gamma}_3; \\ \text{(d) } H_{\mathrm{ad}}(x,0,r) = 0, H_{\mathrm{ad}}(x,b,r) \ge 0 \text{ for all } b \le 0,\ r \in \mathbb{R} \\ \quad \text{and a.e. } x \in \mathit{\Gamma}_3, \text{ and} \\ \quad H_{\mathrm{ad}}(x,b,r) \le 0 \text{ for all } b \ge 1, r \in \mathbb{R} \text{ and a.e. } x \in \mathit{\Gamma}_3. \end{cases}$$

We assume that the initial bonding field satisfies

$$(3.8)\qquad \beta_0 \in L^2(\mathit{\Gamma}_3), \quad 0 \le \beta_0 \le 1 \quad \text{a.e. on } \mathit{\Gamma}_3,$$

and the gap function satisfies

$$(3.9)\qquad g \in L^\infty(\mathit{\Gamma}_3) \quad \text{and} \quad g \ge 0 \quad \text{a.e. on } \mathit{\Gamma}_3.$$

We also need to introduce the set

$$B = \{\theta : [0,T] \to L^2(\mathit{\Gamma}_3);\ 0 \le \theta(t) \le 1, \forall t \in [0,T],\ \text{a.e. on } \mathit{\Gamma}_3\}.$$

Finally, assuming the solution to be sufficiently regular and applying Green's formula, we deduce the following variational formulation of the mechanical problem $P_1$.

Problem $P_2$. Find a displacement field $u : [0,T] \to V$ and a bonding field $\beta : [0,T] \to L^2(\mathit{\Gamma}_3)$ such that

$$(3.10)\qquad \begin{aligned} &u(t) \in K, \\ &\big(F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t))\big)_Q + j\big(\beta(t), u(t), v - u(t)\big) \\ &\qquad\qquad \ge (f(t), v - u(t))_V \quad \forall v \in K,\ t \in [0,T], \end{aligned}$$

$$(3.11)\qquad \dot\beta(t) = H_{\mathrm{ad}}(\beta(t), u_\nu(t)) \quad \text{a.e. } t \in (0,T),$$

$$(3.12)\qquad \beta(0) = \beta_0.$$

**4. An existence and uniqueness result.** Our main result in this section is the following.

Theorem 4.1. *Let* (3.3) *and* (3.5)–(3.9) *hold. Then Problem* $P_2$ *has a unique solution which satisfies*

$$(4.1)\qquad u \in W^{1,\infty}(0,T;V) \cap C([0,T];K),$$

$$(4.2)\qquad \beta \in W^{1,\infty}(0,T;L^2(\mathit{\Gamma}_3)) \cap B,$$

*if*

$$(4.3)\qquad L_\nu d_\Omega^2 < m.$$

The proof of Theorem 4.1 is carried out in several steps. In the first step, let $k > 0$ and consider the closed subset $Y$ of $C([0,T]; L^2(\Gamma_3))$ defined as

$$Y = \{\theta \in C([0,T]; L^2(\Gamma_3)) \cap B;\ \theta(0) = \beta_0\},$$

where $C([0,T]; L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_Y = \max_{t\in[0,T]} [\exp(-kt)\|\theta(t)\|_{L^2(\Gamma_3)}] \quad \text{for } \theta \in C([0,T]; L^2(\Gamma_3)).$$

Next for a given $\beta \in Y$, we consider the following variational problem.

Problem $P_{1\beta}$. Find a displacement field $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &u_\beta(t) \in K,\\ &\big(F\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t))\big)_Q + j\big(\beta(t), u_\beta(t), v - u_\beta(t)\big)\\ &\qquad \geq (f(t), v - u_\beta(t))_V \quad \forall v \in K,\ t \in [0,T]. \end{aligned} \tag{4.4}$$

We have the following result.

Proposition 4.2. *Problem $P_{1\beta}$ has a unique solution which satisfies*

$$u_\beta \in C([0,T]; K). \tag{4.5}$$

*Proof.* Let the operator $A_{\beta(t)} : V \to V$ be defined by

$$(A_{\beta(t)}u, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + j(\beta(t), u, v) \quad \forall u, v \in V.$$

We use (3.2), (3.5)(a), (3.6)(a) to show that $A_{\beta(t)}$ is Lipschitz continuous. Also $A_{\beta(t)}$ is strongly monotone. In fact, in view of (3.6)(a), we have

$$|(p_\nu(\theta, r_1) - p_\nu(\theta, r_2))(r_1 - r_2)| \leq L_\nu |r_1 - r_2|^2$$

for all $\theta \in [0,1]$ and $r_1$, $r_2 \in \mathbb{R}$, and

$$(p_\nu(\theta, r_1) - p_\nu(\theta, r_2))(r_1 - r_2) \geq -L_\nu |r_1 - r_2|^2$$

for all $\theta \in [0,1]$ and $r_1, r_2 \in \mathbb{R}$. Then from (3.2) and (3.5)(b), we get

$$(A_{\beta(t)}u - A_{\beta(t)}v, u - v)_V \geq (m - L_\nu d_\Omega^2)\|u - v\|_V^2 \ \text{ for all } u, v \in V.$$

Hence, $A_{\beta(t)}$ is strongly monotone if (4.3) holds. Then, by a standard existence and uniqueness result for elliptic quasivariational inequalities (see [2]), it follows that there exists a unique $u_\beta(t) \in K$ which satisfies the inequality (4.4) since $K$ is a non-empty, closed convex subset of $V$. ∎

Next, as in [19], to prove (4.5), it suffices to see that if (4.3) holds then there exists a constant $c > 0$ such that

$$\begin{aligned} &\|u_\beta(t_1) - u_\beta(t_2)\|_V\\ &\leq \frac{c}{m - L_\nu d_\Omega^2}\big(\|f(t_1) - f(t_2)\|_V + \|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)}\big) \quad \forall t_1, t_2 \in [0,T]. \end{aligned} \tag{4.6}$$

Therefore, from (4.6), as $f \in C([0,T]; V)$ and $\beta \in C([0,T]; L^2(\Gamma_3))$, we immediately get (4.5). ∎

In the second step, we consider the following initial value problem.

Problem $P_{2\beta}$. Find a bonding field $\theta_\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\theta}_\beta(t) = H_{\mathrm{ad}}(\theta_\beta(t), u_\nu(t)) \quad \text{a.e. } t \in (0,T), \tag{4.7}$$

$$\theta_\beta(0) = \beta_0. \tag{4.8}$$

We obtain the following result.

Lemma 4.3. *Problem $P_{2\beta}$ has a unique solution $\theta_\beta$ which satisfies*

$$\theta_\beta \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B.$$

*Proof.* Define the mapping $F_\beta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ by

$$F_\beta(t,\theta) = H_{\mathrm{ad}}(\theta, u_{\beta\nu}(t)) \qquad \text{for all } t \in [0,T] \text{ and } \theta \in L^2(\Gamma_3).$$

It follows from the assumption (3.7) on $H_{\mathrm{ad}}$ that $F_\beta$ is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\beta(t,\theta)$ belongs to $L^\infty(0,T; L^2(\Gamma_3))$. Then, from a version of the Cauchy–Lipschitz theorem, we deduce the existence of a unique $\theta_\beta \in W^{1,\infty}(0,T; L^2(\Gamma_3))$ which satisfies (4.7) and (4.8). The regularity $\theta_\beta \in B$, follows from (4.7), (4.8) and (3.8) (see [19, 20]). ∎

Therefore, from Lemma 4.3, we deduce that for all $\beta \in Y$, the solution $\theta_\beta$ of Problem $P_{2\beta}$ belongs to $Y$. Hence we can define the mapping $\Psi : Y \to Y$ by

$$\Psi\beta = \theta_\beta.$$

The third step is the following lemma.

Lemma 4.4. *The mapping $\Psi$ has a unique fixed point $\beta^*$.*

*Proof.* We have

$$\Psi\beta(t) = \beta_0 - \int_0^t H_{\mathrm{ad}}(\theta_\beta(s), u_{\beta\nu}(s))\, ds,$$

where $u_\beta$ is the solution of Problem $P_{1\beta}$. Then for $\beta_1$, $\beta_2 \in Y$, using (3.7)(a) (see [20]), we get

$$|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)| \le L_{\mathrm{ad}} \int_0^t \big(|\theta_{\beta_1}(s) - \theta_{\beta_2}(s)| + |u_{\beta_1\nu}(s) - u_{\beta_2\nu}(s)|\big)\, ds.$$

This inequality implies

$$\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_{L^2(\Gamma_3)} \le c_1 \int_0^t \big(\|\theta_{\beta_1}(s) - \theta_{\beta_2}(s)\|_{L^2(\Gamma_3)} + \|u_{\beta_1\nu}(s) - u_{\beta_2\nu}(s)\|_{L^2(\Gamma_3)}\big)\, ds$$

for some positive constant $c_1$. Applying Gronwall's inequality and using (3.2), it follows that there exists a constant $c_2 > 0$ such that

$$\|\theta_{\beta_1}(t) - \theta_{\beta_2}(t)\|_{L^2(\Gamma_3)} \le c_2 \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V ds.$$

Now, using (4.4), (3.5), (3.7), (3.8) and (4.3), we deduce after some calculations that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \le c_3 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}$$

for some constant $c_3 > 0$. Hence, there exists a constant $c_4 > 0$ such that

$$\|\Psi\beta_1(t) - \Psi\beta_2(t)\|_{L^2(\Gamma_3)} \le c_4 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds \quad \forall t \in [0,T].$$

Therefore,

$$\|\Psi\beta_1 - \Psi\beta_2\|_Y \le \frac{c_4}{k} \|\beta_1 - \beta_2\|_Y \quad \forall \beta_1, \beta_2 \in Y.$$

Thus, for $k > c_4$, $\Psi$ is a contraction. Then it has a unique fixed point $\beta^*$ which satisfies (4.7) and (4.8). On the other hand, from (4.6) we deduce that $u_{\beta^*} \in W^{1,\infty}(0,T;V)$.

*Proof of Theorem 4.1.* Let $\beta = \beta^*$ and let $u_{\beta^*}$ be the solution to Problem $P_{1\beta}$. We conclude by (4.4), (4.7) and (4.8) that $(u_{\beta^*}, \beta^*)$ is a solution of Problem $P_2$. Now to prove the uniqueness of the solution, suppose that $(u, \beta)$ is a solution of Problem $P_2$ which satisfies (3.10)–(3.12). It follows from (3.10) that $u$ is a solution of Problem $P_{1\beta}$, and by Proposition 3.2 we get $u = u_\beta$. Taking $u = u_\beta$ in (3.11) and using the initial condition (3.12), we deduce that $\beta$ is a solution of Problem $P_{2\beta}$. Finally, using Lemma 4.3, we obtain $\beta = \beta^*$ and so $(u_{\beta^*}, \beta^*)$ is a unique solution to Problem $P_2$ which satisfies (4.1), (4.2).

**5. A convergence result.** In this section we consider a frictionless contact problem with normal compliance and adhesion with unlimited penetration. The unilateral contact condition (2.5) is replaced by the contact condition

$$-\sigma_\nu = p_\delta(u_\nu) + p_\nu(u_\nu, \beta) \quad \text{on } \Gamma_3 \times (0,T),$$

where the penalized function $p_\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$p_\delta(r) = \left(\frac{r-g}{\delta}\right)_+.$$

We recall that $\delta > 0$ is the penalization parameter and $1/\delta$ is interpreted as the stiffness coefficient of the foundation. We understand that when $\delta$ is small, the reaction of the foundation to the penetration is important; when $\delta$ is large, the reaction is smaller. We study the behaviour of the solution as $\delta \to 0$ and prove that in the limit we obtain a solution of the adhesive frictionless contact problem with normal compliance and unilateral constraint. Next we define $j_\delta : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_\delta(\beta, u, v) = \int_{\Gamma_3} (p_\delta(u_\nu) + p_\nu(u_\nu, \beta))v_\nu \, da \quad \forall \beta \in L^2(\Gamma_3),\ \forall u, v \in V.$$

With this notation, the formulation of the penalized problem with frictionless contact and unilateral constraint with adhesion is the following.

PROBLEM $P_{1\delta}$. Find a displacement field $u_\delta : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta_\delta : \Gamma_3 \times [0,T] \to [0,1]$ such that

$$\operatorname{div} \sigma(u_\delta) = -f_1 \quad \text{in } \Omega \times (0,T), \tag{5.1}$$

$$\sigma(u_\delta) = F\varepsilon(u_\delta) \quad \text{in } \Omega \times (0,T), \tag{5.2}$$

$$u_\delta = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{5.3}$$

$$\sigma_\delta \nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{5.4}$$

$$-\sigma_{\delta\nu} = p_\delta(u_{\delta\nu}) + p_\nu(u_{\delta\nu}, \beta_\delta) \quad \text{on } \Gamma_3 \times (0,T), \tag{5.5}$$

$$\dot\beta_\delta = H_{\mathrm{ad}}(\beta_\delta, u_{\delta\nu}) \quad \text{on } \Gamma_3 \times (0,T), \tag{5.6}$$

$$\beta_\delta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{5.7}$$

In this probem we denote $\sigma(u_\delta) = \sigma_\delta$. With this notation, the variational problem of Problem $P_{1\delta}$ reads as follows:

PROBLEM $P_{2\delta}$. Find a displacement field $u_\delta : [0,T] \to V$ and a bonding field $\beta_\delta : [0,T] \to L^2(\Gamma_3)$ such that

$$\big(F\varepsilon(u_\delta(t)), \varepsilon(v)\big)_Q + j_\delta\big(\beta_\delta(t), u_\delta(t), v\big) = (f(t), v)_V \quad \forall v \in V,\ t \in [0,T], \tag{5.8}$$

$$\dot\beta_\delta(t) = H_{\mathrm{ad}}(\beta_\delta(t), u_{\delta\nu}(t)) \quad \text{a.e. } t \in (0,T), \tag{5.9}$$

$$\beta_\delta(0) = \beta_0. \tag{5.10}$$

We have the following result.

PROPOSITION 5.1. *Problem $P_{2\delta}$ has a unique solution which satisfies*

$$(u_\delta, \beta_\delta) \in W^{1,\infty}(0,T;V) \times W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B. \tag{5.11}$$

*Proof.* As in [20], the proof is similar to that of Theorem 4.1 and it is carried out in several steps. We omit the details and just recall the main steps.

(i) For any $\beta \in Y$, we prove that there exists a unique $u_{\beta\delta} \in C([0,T];V)$ such that

$$\big(F\varepsilon(u_{\beta\delta}(t)), \varepsilon(v)\big)_Q + j_\delta\big(\beta(t), u_{\beta\delta}(t), v\big) = (f(t), v)_V \quad \forall v \in V,\ t \in [0,T]. \tag{5.12}$$

To see this, for all $t \in [0,T]$ we consider the operator $A_t : V \to V$ defined by

$$(A_t u, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + j_\delta(\beta(t), u, v) \quad \forall u, v \in V.$$

We use (3.2), (3.5)–(3.9), (4.3) and the fact that for $a, b \in \mathbb{R}$ we have $|a_+ - b_+| \le |a - b|$ and $(a - b)(a_+ - b_+) \ge (a_+ - b_+)^2$ to see that the operator $A_t$ is strongly monotone and Lipschitz continuous, and therefore invertible.

(ii) For a given $\beta \in Y$, there exists a unique element $\beta_\delta$ such that

$$\beta_\delta \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B, \tag{5.13}$$

$$\dot\beta_\delta(t) = H_{\mathrm{ad}}(\beta_\delta(t), u_{\beta\delta\nu}(t)) \quad \text{a.e. } t \in (0,T), \tag{5.14}$$

$$\beta_\delta(0) = \beta_0. \tag{5.15}$$

(iii) Let $\beta_\delta$ defined in (ii) and denote again by $u_\delta$ the function obtained in step (i) for $\beta = \beta_\delta$. Then, by using (5.13)–(5.15) it is easy to see that $(u_\delta, \beta_\delta)$ is the unique solution to Problem $P_{2\delta}$ and it satisfies (5.11). ∎

Now, as in [20, 21], we study the convergence of the solution $(u_\delta, \beta_\delta)$ as $\delta \to 0$ in the following theorem.

THEOREM 5.2. *Assume that* (3.5)–(3.9) *hold. Then we have the following convergences:*

$$\lim_{\delta\to 0} \|u_\delta(t) - u(t)\|_V = 0 \quad \textit{for all } t \in [0,T], \tag{5.16}$$

$$\lim_{\delta\to 0} \|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0 \quad \textit{for all } t \in [0,T]. \tag{5.17}$$

The proof is carried out in several steps. In the first step, we prove the lemma below.

LEMMA 5.3. *For each $t \in [0,T]$, there exists $\bar u(t) \in K$ such that after passing to a subsequence, still denoted $(u_\delta(t))$, we have*

$$u_\delta(t) \to \bar u(t) \quad \textit{weakly in } V. \tag{5.18}$$

*Proof.* Let $t \in [0,T]$. Taking $v = u_\delta(t)$ in (5.8), we find

$$\big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t))\big)_Q + j_\delta\big(\beta_\delta(t), u_\delta(t), u_\delta(t)\big) = (f(t), u_\delta(t))_V. \tag{5.19}$$

Now, we have

$$j_\delta\big(\beta_\delta(t), u_\delta(t), u_\delta(t)\big) = \int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ u_{\delta\nu}(t)\, da + \int_{\Gamma_3} p_\nu(\beta_\delta(t), u_{\delta\nu}(t)) u_{\delta\nu}(t)\, da.$$

As

$$\int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ u_{\delta\nu}(t)\, da \ge 0,$$

it follows that

$$\big(F\varepsilon(u_\delta(t)),\varepsilon(u_\delta(t))\big)_Q+\int_{\Gamma_3} p_\nu(\beta_\delta(t),u_{\delta\nu}(t))u_{\delta\nu}(t)\,da\le (f(t),u_\delta(t))_V.$$

Hence, using (3.2), (3.5)(b) and (3.6)(a), we get

$$(m-L_\nu d_\Omega^2)\|u_\delta(t)\|_V^2\le (f(t),u_\delta(t))_V.$$

Then from (4.3), this inequality implies that

$$\|u_\delta(t)\|_V\le \frac{\|f\|_{C([0,T];V)}}{m-L_\nu d_\Omega^2}.$$

Next denote

$$\frac{\|f\|_{C([0,T];V)}}{m-L_\nu d_\Omega^2}=C.$$

Since the sequence $(u_\delta(t))$ is bounded in $V$, there exists an element $\bar u(t)\in V$ and a subsequence again denoted $(u_\delta(t))$ such that (5.18) holds. Also from (5.19) we deduce that

$$j_\delta\big(\beta_\delta(t),u_\delta(t),u_\delta(t)\big)\le (f(t),u_\delta(t))_V.$$

Then it follows from the definition of $j_\delta$ that

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))u_{\delta\nu}(t)\,da\le (f(t),u_\delta(t))_V-\int_{\Gamma_3} p_\nu(\beta_\delta(t),u_{\delta\nu}(t))u_{\delta\nu}(t)\,da.$$

On the other hand,

$$\begin{aligned}\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))u_{\delta\nu}(t)\,da=&\int_{\Gamma_3}\Big(\frac{u_{\delta\nu}(t)-g}{\delta}\Big)_+(u_{\delta\nu}(t)-g)\,da\\&+\int_{\Gamma_3} g\Big(\frac{u_{\delta\nu}(t)-g}{\delta}\Big)_+\,da,\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\int_{\Gamma_3}\Big(\frac{u_{\delta\nu}(t)-g}{\delta}\Big)_+(u_{\delta\nu}(t)-g)\,da\\&\qquad\le (f(t),u_\delta(t))_V-\int_{\Gamma_3} p_\nu(\beta_\delta(t),u_{\delta\nu}(t))u_{\delta\nu}(t)\,da.\end{aligned}$$

If we now use (3.2) and (3.6)($a$), this inequality implies that

$$\|(u_{\delta\nu}(t)-g)_+\|^2_{L^2(\Gamma_3)}\le \delta(\|f\|_{C([0,T];V)}C+L_\nu d_\Omega^2C^2).$$

Hence, using (5.18), we deduce that

$$\|(\bar u_\nu(t)-g)_+\|_{L^2(\Gamma_3)}\le\liminf_{\delta\to 0}\|(u_{\delta\nu}(t)-g)_+\|_{L^2(\Gamma_3)}=0. \tag{5.20}$$

Therefore it follows from (5.20) that $(\bar{u}_\nu(t) - g)_+ = 0$, i.e. $\bar{u}_\nu(t) \le g$ a.e. on $\Gamma_3$, which implies that

$$\bar{u}(t) \in K. \tag{5.21}$$

Now, we state the following problem.

Problem $P_3$. Find a bonding field $\beta_a : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_a(t) = H_{\mathrm{ad}}(\beta_a(t), \overline{u}_\nu(t)) \quad \text{a.e. } t \in (0,T), \tag{5.22}$$

$$\beta_a(0) = \beta_0. \tag{5.23}$$

As in [20, Lemma 3.2] we have the following result.

Lemma 5.4. *Problem $P_3$ has a unique solution which satisfies*

$$\beta_a \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B.$$

Now, we show the convergence result below.

Lemma 5.5. *We have*

$$\lim_{\delta\to 0} \|\beta_\delta(t) - \beta_a(t)\|_{L^2(\Gamma_3)} = 0 \quad \textit{for all } t \in [0,T]. \tag{5.24}$$

*Proof.* In view of (3.6)(a), we have

$$\|\beta_\delta(t) - \beta_a(t)\|_{L^2(\Gamma_3)} \le c_5 \int_0^t (\|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} + \|\beta_\delta(s) - \beta_a(s)\|_{L^2(\Gamma_3)})\, ds,$$

where $c_5 > 0$. Hence, by Gronwall's inequality, we deduce

$$\|\beta_\delta(t) - \beta_a(t)\|_{L^2(\Gamma_3)} \le c_6 \int_0^t \|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} ds \tag{5.25}$$

for some positive constant $c_6$. Note that the weak convergence (5.18) combined with the compactness of the trace operator implies that $u_{\delta\nu}(t) \to \bar{u}_\nu(t)$ strongly in $L^2(\Gamma_3)$ as $\delta \to 0$. On the other hand,

$$\|u_{\delta\nu}(t) - \bar{u}_\nu(t)\|_{L^2(\Gamma_3)} \le d_\Omega(C + \|\bar{u}(t)\|_V).$$

Then it follows from Lebesgue's convergence theorem that

$$\lim_{\delta\to 0} \int_0^t \|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} ds = 0. \tag{5.26}$$

The convergence result (5.24) is now a consequence of (5.25) and (5.26). ∎

Next, we prove the lemma below.

Lemma 5.6. *For each $t \in [0,T]$, we have $(\bar{u}(t), \beta_a(t)) = (u(t), \beta(t))$, where $(u, \beta)$ is a solution of Probem $P_1$.*

*Proof.* Let $v \in K$. Taking $v - u_\delta(t)$ in (5.12) yields

$$(5.27)\qquad \big(F\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t))\big)_Q + j_\delta\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big) = (f(t), v - u_\delta(t))_V \quad \forall v \in K.$$

We have

$$j_\delta\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big) = \int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\, da + j\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big).$$

But since

$$\int_{\Gamma_3} p_\delta(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\, da \geq 0,$$

it follows that

$$(5.28)\qquad \big(F\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t))\big)_Q + j\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big) \geq (f(t), v - u_\delta(t))_V \quad \forall v \in K.$$

We conclude from the above that

$$(5.29)\qquad \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - v)\big)_Q \leq j\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big) + (f(t), u_\delta(t) - v)_V \quad \forall v \in K.$$

Note that (5.18), (5.24) and (3.6)(a) imply that

$$(5.30)\qquad \lim_{\delta\to 0} j\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big) = j\big(\beta_a(t), \bar{u}(t), v - \bar{u}(t)\big).$$

Therefore, taking $v = \bar{u}(t)$ in (5.29), passing to the upper limit as $\delta \to 0$ in the resulting inequality and using equality (5.30), we get

$$(5.31)\qquad \limsup_{\delta\to 0} \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - \bar{u}(t))\big)_Q \leq 0.$$

On the other hand, the pseudomonotonicity of $F$ implies that

$$(5.32)\qquad \liminf_{\delta\to 0} \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - v)\big)_Q \geq \big(F\varepsilon(\bar{u}(t)), \varepsilon(\bar{u}(t) - v)\big)_Q \quad \forall v \in V.$$

Now, passing to the lower limit as $\delta \to 0$ in (5.29) and using again (5.30) yields

$$(5.33)\qquad \liminf_{\delta\to 0} \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - v)\big)_Q \leq j\big(\beta_a(t), \bar{u}(t), v - \bar{u}(t)\big) + (f(t), \bar{u}(t) - v)_V \quad \forall v \in K.$$

By combining (5.32) and (5.33), we get

$$(5.34)\qquad \big(F\varepsilon(\bar{u}(t)), \varepsilon(\bar{u}(t) - v)\big)_Q \leq j\big(\beta_a(t), \bar{u}(t), v - \bar{u}(t)\big) + (f(t), \bar{u}(t) - v)_V \quad \forall v \in K.$$

It follows from (5.21)–(5.23) and (5.34) that $(\overline{u}, \beta_a)$ is a solution of Problem $P_2$. Lemma 5.6 is now a consequence of the uniqueness part in Theorem 4.1. ∎

LEMMA 5.7. *Let $(u, \beta)$ be the solution of Problem $P_2$. Then for each $t \in [0, T]$, the whole sequence $(u_\delta(t), \beta_\delta(t))$ converges weakly to $(u(t), \beta(t))$.*

*Proof.* Let $t \in [0, T]$. From Lemma 5.6 we see that $(\bar{u}(t), \beta_a(t))$ is the unique weak limit in $V \times L^2(\Gamma_3)$ of any weakly convergent subsequence of the sequence $\{(u_\delta(t), \beta_\delta(t))\}$ and therefore the whole sequence $\{(u_\delta(t), \beta_\delta(t))\}$ converges weakly to the element $(\bar{u}(t), \beta_a(t))$. An application of Lemma 5.6 concludes the proof. ∎

Now, we have all the ingredients to prove Theorem 5.2.

*Proof of Theorem 5.2.* To prove (5.2), let $\delta > 0$ and take $v = \overline{u}(t)$ in (5.32) to see that

$$\liminf_{\delta \to 0} \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - \bar{u}(t))\big)_Q \geq 0.$$

By combining this inequality with (5.31), we find

$$\lim_{\delta \to 0} \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - \bar{u}(t))\big)_Q = 0.$$

Hence, using the equality $\overline{u}(t) = u(t)$, we get

$$\lim_{\delta \to 0} \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - u(t))\big)_Q = 0. \tag{5.35}$$

On the other hand, from (5.18), we have

$$\lim_{\delta \to 0} \big(F\varepsilon(u(t)), \varepsilon(u_\delta(t) - u(t))\big)_Q = 0. \tag{5.36}$$

Now, using the strong monotonicity of the operator $F$, we obtain

$$\begin{aligned} m\|u_\delta(t) - u(t)\|_V^2 &\leq \big(F\varepsilon(u_\delta(t)) - F\varepsilon(u(t)), \varepsilon(u_\delta(t) - u(t))\big)_Q \\ &= \big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t) - u(t))\big)_Q - \big(F\varepsilon(u(t)), \varepsilon(u_\delta(t) - u(t))\big)_Q. \end{aligned} \tag{5.37}$$

Consequently, from (5.35)–(5.36) and (5.37), we obtain the strong convergence (5.16). To prove (5.17), it suffices to use Lemma 5.6 and (5.24).

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