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INTRODUCTION

ARITHMETIC METHODS IN MATHEMATICAL PHYSICS AND BIOLOGY

GRZEGORZ BANASZAK

Department of Mathematics and Computer Science
Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
E-mail: banaszak@amu.edu.pl

JAN MILEWSKI

Institute of Mathematics, Poznań University of Technology
Piotrowo 3A, 60-965 Poznań, Poland
E-mail: jsmilew@wp.pl

PRZEMYSŁAW WALISZEWSKI

Department of Urology, Justus-Liebig University
Rudolf-Buchheim-Strasse 7, 35392 Giessen, Germany
E-mail: complexityresearch@yahoo.com

Abstract. The First International Conference *Arithmetic Methods in Mathematical Physics and Biology* was held on August 3–8, 2014, in the Mathematical Research and Conference Center of the Institute of Mathematics of Polish Academy of Sciences, at Będlewo, Poland. We organized this conference with the aim of bringing classical areas of mathematics, such as arithmetic, algebra, algebraic number theory, fractal geometry to the attention of mathematical physicists and biologists to establish some novel tools for research on physical and biological complexity. The state-of-the-art lectures defined complexity as a consequence of interactions within a complex system. Some general principles, such as the Complexity Correspondence Principle, were proposed starting from the formal mathematics underlying the theory of many-body physical systems. This principle suggests that the system can only be designed and controlled by a second system of equal or greater complexity. A number of auspicious arithmetic approaches to a variety of complexity issues were presented. For example, growth and self-organization of cells

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was modeled by cellular automata. The universal circular fractal model of adenocarcinomas enabled the stratification of prostate carcinomas into the classes of equivalence defined by the cut-off values of the global capacity fractal dimension D0 and, in consequence, the quantitative evaluation of tumor aggressiveness by complexity measures. Both Galois theory and algebraic number theory were shown to be useful for Bethe Ansatz, the quantum mechanics problem with important applications to Quantum Computing. p-adic numbers were applied to solve problems in probability theory, dynamical systems, cryptography, and cognitive science. Arithmetic occurred to be very useful in Biometry. The next Conference is scheduled on August 5–11, 2018 at Będlewo.

"*If there is an equation for a curve like a bell, there must be an equation for one like a bluebell, and if a bluebell, why not a rose? Do we believe nature is written in numbers?*"

Thomasina, in: Tom Stoppard, *Arcadia*

The First International Conference *Arithmetic Methods in Mathematical Physics and Biology* evolved from discussions on physical and biological complexity initiated during the 8th European Congress on Mathematical and Theoretical Biology in Cracow in 2011 [4]. A dominating opinion on both fractal geometry and time series analysis, an effective language for pattern recognition in contemporary biology and medicine, was that the advanced mathematical study could bring some novel mathematical tools or computer algorithms useful in research on physical or biological complexity, and vice versa, that some physical or biological issues may give an impulse to advanced arithmetic research.

There is a long tradition of interplay between advanced mathematical research and different areas of science. Let us mention mathematicians and physicists who contributed to this field (the list is far from being complete) starting from the bigwigs of the modern mathematical analysis, such as Isaac Newton, Gottfried Wilhelm Leibniz, Leonhard Euler and Carl Friedrich Gauss to Georg Friedrich Bernhard Riemann (Riemann geometry), Oliver Heaviside (complex numbers and electrical circuits, vector analysis, differential equations), Joseph-Louis Lagrange (Lagrangian operator) or William Rowan Hamilton (Hamilton operator) to Stefan Banach (Banach spaces, theorem on contracting mappings), Sophus Lie (Lie algebra), Andrey Kolmogorov (theory of probability), Wacław Sierpiński (number theory and fractal geometry), Benoit Mandelbrot (fractal geometry), Paul Dirac (quantum theory), Albert Einstein (theory of relativity), Werner Heisenberg (quantum theory), James Clerk Maxwell (electromagnetic field equations), Ilya Prigogine (deterministic chaos and thermodynamics), Henri Poincaré (dynamical systems), John von Neumann (computer theory), Alan Turing (computer algorithm complexity), and Stanisław Ulam (computing, Monte Carlo method, branching processes). The last, but not least, we must also mention three Polish mathematicians of the University of Poznań, Poland, Marian Rejewski, Henryk Zygalski, and Jerzy Różycki, a team who pioneered the application of group theory to cryptography and who deciphered the Enigma code in the 1930s. Their work was passed along to Alan Turing during World War II whose team at Bletchley Park extended the work of the Polish cryptologists and built up Britain's codebreaking center. In that way, the codebreaking efforts helped the Allies significantly to win World War II [6].

Let us present some of the topics discussed in the lectures given during the Conference and presented in these proceedings.

Complexity and fractal geometry played an important part at the Conference. Some novel perspectives for future research were shown. The issue of complexity was discussed for the first time during the "1988 Complex Systems Summer School" in Santa Fe, New Mexico. Participants of that School outlined in a systematic manner some ideas, introduced basic notions, and defined issues for interdisciplinary research on complexity [7]. In particular, it was concluded that most biological problems were of great complexity. There were no mathematical tools that would enable a synthetic description of complex spatial patterns and underlying dynamic processes, nor unifying theories for physical and biological complex systems. Most important, there was no common language between biologists, chemists, physicists and mathematicians. A success of that School was, however, the discovery of a novel philosophy for the holistic research of complex systems, as opposed to a reductionist, analytical approach, called since that time on science of complexity. In particular, it was noticed that similar spatial or temporal patterns may emerge from seemingly dissimilar systems at vastly different scales. Starting from the formal mathematics underlying the theory of many-body physical systems, Andrei Kirilyuk defined complexity as a result of nonlinear interactions between elements of a complex system and proposed some general principles, such as the Complexity Correspondence Principle. This principle suggests that a system can only be designed and controlled by a second system of equal or greater complexity (see also Principle of Complexity Management [10]).

The lecture given by Andreas Deutsch confirmed that computer simulations often remain the only avenue along which both spatial and temporal evolution of complex systems, spanning a variety of species from bacteria through fish to birds, can be studied. Tissue growth is a fundamental biological phenomenon. Modeling of growth of malignant tumors (cancers) plays a significant role in clinical research. Computer simulations with the application of cellular automata provide findings that are more important than results of classical mathematical analysis of growth models (see the article by M. Tanase in this volume, pages 167–182) [4, 8].

Fractal geometry provides tools for holistic synthesis rather than reductionist analysis. Indeed, one of the most important parameters used in the statistical models of cancer to calculate the risk of progression is tumor aggressiveness. This parameter is evaluated subjectively by a comparison of the tumor structure with the structure of the normal-appearing tissue. The intra- and inter-observer variability is very high, approaching values of 40%–80%. The application of both complexity measures and the circular fractal model of adenocarcinomas for evaluation of the spatial distribution of cancer cell nuclei enabled a stratification of prostate carcinoma cases into equivalence classes and a quantitative evaluation of tumor aggressiveness (see the article by P. Waliszewski, pages 183–196) [8, 9].

The lectures on image analysis by Helmut Ahammer, Herbert Jelinek, Konradin Metze, and Martin Obert summarized current research on the application of fractal geometry in analysis of tissue patterns, such as cancer, vascular network in retinopathy and biological signals in neurological disorders or radiological images.

In recent years, research has been conducted on the application of both Galois symmetries and algebraic number theory to the well-known Bethe Ansatz problem. It turns out that these new methods can successfully be applied to quantum mechanics. This application opens a new way to investigating a spectrum of Heisenberg Hamiltonians and the related parameters as well as the configuration space of magnetic ring states. Both Galois theory and algebraic number theory bring to quantum mechanics methods that are different from the previous ones based on real and functional analysis. The arithmetic tools enabled Jan Milewski to introduce the notions of Galois qubits and qudits. Galois qubits and qudits are elementary memory units of a hypothetical quantum computer. Research concerning these topics was presented in the lectures by Grzegorz Banaszak and Tadeusz Lulek (see the related articles [1, 2, 3]).

Research in biometry was presented in the lectures by Preda Mihailescu and Benjamin Tams. They discussed some imminent vulnerabilities of biometric systems, which have not been taken into consideration before. They explained why a comparison of the lower bounds of security of the biometric systems with the lower bounds known from cryptography is a natural necessity to avoid the case when insufficient security in complex hybrid systems compromises the stronger components in the complex. P. Mihailescu and B. Tams also discussed the challenges of modern IT security induced by proliferation of password usage and the advent of biometry. This is an important application of biometry, which appears in the governmental use, such as biometric ID's, biometric control at customs and many more similar applications.

The classical Lissajous curves are useful for describing oscillations in two perpendicular directions. They are of common use in various mechanical and electrical systems. They are closely connected to the roots of corresponding quadratic polynomials. In the paper by P. Krasoń, J. Milewski, W. Bondarewicz and A. Wojtaszek (pages 83–98), the authors consider the situation where the potential is given by a polynomial of degree four. The natural parametrization of Lissajous curves may be applied in this situation and it has an additional advantage that it incorporates the initial conditions of a movement. The authors give a detailed study of these generalized Lissajous curves in terms of the type of roots of the quartics involved. The study may have various applications ranging from the Duffing oscillator to the study of geodesics in the Kerr metric.

A connection between statistical mechanics and a natural number partition problem is presented in the article by A. Rovenchak (pages 155–166). The author investigates general unrestricted linear and plane partitions. He shows how statistical quantum mechanics can be used to obtain some partition formulas. In computations, A. Rovenchak uses harmonic oscillators that obey the Bose statistics. Restricted partitions are also discussed. The author states a conjecture concerning the asymptotic formula for the intermediate number of parts of the restricted plane partitions.

The field of p-adic numbers, a basic tool of algebraic number theory, is used in the research of A. Khrennikov. The author presents applications of p-adic numbers in many areas of science: theoretical physics, probability theory, dynamical systems, cryptography, biology, cognitive science, and psychology. The main purpose of A. Khrennikov's article (pages 47–56) is the presentation of basic results concerning applications of p-adic numbers in a compact and user friendly way [5].

The conference gave an impulse to organize an international institute for complexity research in Poland. That institute would be a kind of a network co-ordinating qualified research on arithmetic methods and their application in studying physical or biological complexity. It will also organize the next conference scheduled on August 5–11, 2018. This conference will take place in the Conference Center of the Polish Academy of Sciences at Będlewo, about 35 km south-west of Poznań, Poland. On behalf of the Scientific Committee, we invite you to participate in that event.

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