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p-ADIC NUMBERS: FROM SUPERSTRINGS AND MOLECULAR MOTORS TO COGNITION AND PSYCHOLOGY

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Abstract. This is a brief review devoted to applications of p-adic numbers in physics, probability theory, theory of dynamical systems, cryptography, biology, cognitive science, and psychology.

1. Introduction. During recent 15 years p-adic numbers served as a powerful mathematical tool for various applications, physics: theory of p-adic superstrings and quantum mechanics, theory of complex disordered systems — spin glasses; dynamical systems: theory of ergodicity, structure of cycles and attractors; cryptography: p-adic stream ciphers; cognition: from mental space to a model of hierarchic unconscious processing of information and interaction between consciousness and unconsciousness; psychology: resistant-depression, Freud's psychoanalysis. In many applications there is no need to proceed with the prime basis p. It is more natural and fruitful to proceed with rings of m-adic numbers, where $m > 1$ is an arbitrary natural number. In applications to biology and cognitive science the language of general ultrametric spaces is more appropriate.

During these years numerous articles (including reviews) were published, see, e.g., [4], [8]–[9]. However, the majority of such presentations have very advanced mathematical and physical level and are very longly. The aim of this review is to present the basic achievements in all aforementioned fields in a compact way. It can be considered as a very primary introduction to the subject.

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2. *p*-adic numbers and theoretical physics

2.1. *p*-adic numbers. According to the well-known *Ostrowsky theorem*, any nontrivial valuation on the field $\mathbb{Q}$ is equivalent either to the real valuation $|\cdot|$ or to one of the p-adic valuations $|\cdot|_p$, where p is a prime number. This p-adic norm $|\cdot|_p$ is defined as follows: if an arbitrary rational number $x \neq 0$ is represented as $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers m, n are not divisible by p, then

$$|x|_p = p^{-\gamma}, \quad x \neq 0, \qquad |0|_p = 0.$$

The norm $|\cdot|_p$ satisfies the strong triangle inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p$.

2.2. Physical numbers. Thus there are two equal in rights universes: the real universe and the p-adic one. The latter has specific and unusual properties. Nevertheless, there are a lot of papers where different applications of p-adic analysis to physical problems, stochastics, cognitive sciences and psychology are studied. In view of the Ostrowsky theorem, such investigations are not only of great interest in itself, but lead to applications and better understanding of similar problems in *usual* mathematical physics. Here we can mention the adelic approach to string amplitudes. The "ordinary real amplitudes" can be represented as the products (with respect to all primes) of corresponding p-adic amplitudes. Thus the "ordinary real string" can be modelled as a combination of p-adic string. We remark that p-adic models are simpler than the real ones. Thus the adelic decomposition helps to solve the calculation problem.

Our ideology, the p-adic physics group at Steklov Mathematical Institute of Russian Academy of Science (Moscow, 1987)[1] was formulated as follows: only rational numbers are "physical numbers"; data cannot be measured with infinite precision, only finite number of digits is approachable by any physical device.

To analyze data, one has to use tools of analysis, i.e., consider a completion of $\mathbb{Q}$. Typically the completion $\mathbb{R}$ is used in physics, engineering, biology, economics, but the same data can be analyzed by using one of the $\mathbb{Q}_p$ completions of $\mathbb{Q}$.

2.3. Which value of *p* has to be used? There are infinitely many primes and the corresponding fields $\mathbb{Q}_p$ are not isomorphic. The question about an approriate choice of p arises in all applications.

Possible answers (resulting from various applications):

- any prime p;
- p is present in a theoretical model as a parameter;
- all information about data, real and p-adic (for all p) can be unified in the adelic model;
- very large $p \to \infty$.

[1] At that time the group's members were Vladimirov, Volovich, Aref'eva, Dragovich, Khrennikov, Zelenov.

2.4. Further properties of p-adic numbers. The canonical form of any p-adic number $x \neq 0$ is given by the following expansion:

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \ldots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_j = 0, 1, \ldots, p-1$, $x_0 \neq 0$, $j = 0, 1, \ldots$, the p-adic valuation (norm) on $\mathbb{Q}_p$ is given by

$$|x|_p = p^{-\gamma}, \quad x \in \mathbb{Q}_p.$$

The ball of radius $r = p^{\gamma}$ with the center at a point $a \in \mathbb{Q}_p$ is defined as

$$U_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\} \subset \mathbb{Q}_p.$$

The ball $\mathbb{Z}_p \equiv U_1(0)$ is closed with respect to the operations of addition, subtraction, and multiplication (but division is in general undefined). This is a ring. All integer numbers belong to this set, $\mathbb{Z} \subset \mathbb{Z}_p$. Moreover, $\mathbb{Z}$ is dense in $\mathbb{Z}_p$; even the set of natural numbers is dense, i.e., any $x \in \mathbb{Z}_p$ can be approximated by a sequence of natural numbers.

Therefore $\mathbb{Z}_p$ is called *the ring of p-adic integers.* Its elements can be represented as

$$x = x_0 + x_1 p + x_2 p^2 + \ldots, \quad x_j = 0, 1, \ldots, p-1.$$

The previous construction can be generalized for any natural number $m > 1$, the rings of m-adic integers $\mathbb{Z}_m$ consist of elements of the form:

$$x = x_0 + x_1 m + x_2 m^2 + \ldots, \quad x_j = 0, 1, \ldots, m-1,$$

and the norm is defined as

$$|x|_m = m^{-\gamma},$$

where $x_0 = \ldots = x_{m-1} = 0$ and $x_m \neq 0$.

In applications to biology and cognition, there are no reasons to restrict modeling to prime numbers, it is useful to explore all $\mathbb{Z}_m$.

2.5. p-adic dynamical systems

- Discrete dynamical system theory studies *trajectories*, i.e., sequences of iterations
$$x_0, x_1 = f(x_0),\ \ldots,\ x_{i+1} = f(x_i) = f^{(i+1)}(x_0),\ \ldots,$$
where $f^{(s)}(x) = \underbrace{f(f(\ldots f(x))\ldots)}_{s}$.
- Consider a *dynamical system* $\langle \mathbb{Z}_p, \mu_p, f \rangle$. The space $\mathbb{Z}_p$ is equipped with a natural probability measure, namely, the *Haar measure* μ_p normalized so that $\mu_p(\mathbb{Z}_p) = 1$. The balls $U_{p^{-r}}(a)$ of nonzero radii constitute the base of the corresponding σ-algebra of measurable subsets, $\mu_p(U_{p^{-r}}(a)) = p^{-r}$.
- A measurable mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called *measure-preserving* if $\mu_p(f^{-1}(S)) = \mu_p(S)$ for each measurable subset $S \subset \mathbb{Z}_p$.
- A measure-preserving mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called *ergodic* if $f^{-1}(S) = S$ implies either $\mu_p(S) = 0$ or $\mu_p(S) = 1$.

The detailed studies on the theory of p-adic dynamical systems were presented in two monographs [2], [9]. In particular, we obtained criteria of ergodicity and measure-preserving for 1-Lip functions: We recall that the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is 1-*Lipschitz* if

$$|f(x) - f(y)|_p \leq |x - y|_p, \quad \text{for all} \quad x, y \in \mathbb{Z}_p.$$

In other words, $f(x) \equiv f(y) \pmod{p^k}$ once $x \equiv y \pmod{p^k}$ for all $k \geq 1$.

For all $k \geq 1$ a 1-Lipschitz transformation f is the reduced mapping modulo p^k, i.e. $f_{k-1} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$, $z \mapsto f(z) \pmod{p^k}$. The mapping f_k is well defined (the f_{k-1} does not depend on the choice of representative z in the ball $z + p^k\mathbb{Z}_p$). We remark that $\mathbb{Z}/p^{k+1}\mathbb{Z}$ can be represented as $\{0, 1, \ldots, p^k - 1\}$; hence, the function f_k can be realized as a function from $\{0, 1, \ldots, p^k - 1\}$ into itself. Such functions are known in cryptography as T-functions [2].

We point to the following applications of theory of p-adic dynamical systems:

- Cryptography [2]: pseudo-random generators and p-adic stream ciphers.[2]
- Cognitive science and psychology: modeling information processing [9], [11], [12]: here the process of processing of unconscious mental information is described by a p-adic (more generally ultrametric) dynamical system.
- Population dynamics: in [10] it was shown that p-adic dynamics of the population grows based on the logistic map exhibit some basic features of quantum dynamics.

2.6. *p*-adic string theory. In 1980–90th *Super-string Theory* actively played with novel models of space-time, 26-dimensional real space, superspace (some coordinates are anti-commutative, i.e., belong to super-commutative Banach superalgebra).

Some groups of experts in mathematical and theoretical physics, Volovich, Vladimirov, Aref'eva, Dragovich, Witten, Olson, Frampton, …, see [4] for details and references, decided to play with the *p-adic space-time* (the pioneer paper of Volovich was published in 1987, see [4] for the list of citations).

One of physical motivations (besides the aforementioned principle of different analysis-possibilities for rational data) was violation of the *Archimedean axiom* (one of the basic axioms of real analysis):

In $\mathbb{R}$: given a unit of measurement 1, a meter, and quantity L to be measured, there always exists a natural number n such that

$$(n - 1)1 \leq L < n1.$$

We can measure (with the precision given by 1) any quantity L with the aid of this meter.

In standard string theory over real numbers, *Veneziano amplitude* $A(a, b)$ plays the crucial role. It describes the scattering of four tachyons in the 26-dimensional open string. For us it is just important that each string model leads to computation of some quantity $A(a, b)$ which is expressed as a special convolution on the space-time, where a string lives. Such convolutions can be defined even for in the p-adic string theory. We recall that $\mathbb{Q}_p$ is a locally compact group with respect to the operation of addition. Thus there exists

[2] A stream cipher is an encoding device in which text digits (typically 0,1) are combined with the output of a pseudo-random generator (key-stream). Here each text digit is encrypted with the aid of mod 2 addition with the corresponding digit of the key-stream, to produce a digit of the encoded stream.

a translation invariant measure on it, the Haar measure. One can integrate over $\mathbb{Q}_p$ and define convolution integrals similar to the real case. In particular, for each prime number $p > 1$, one can define the p-adic analog of the *Veneziano amplitude*, $A_p(a, b)$. At the beginning it was just a formal generalization, but then one found an interesting coupling between the standard real and p-adic amplitudes. This was done in the *adelic approach.*

In the adelic approach one unifies all possible p-adic and real "coordinates":

$$x = (x_\infty, x_2, x_3, \dots, x_{1997}, \dots, x_p, \dots), \quad x_\infty \in \mathbb{R},\ x_p \in \mathbb{Q}_p.$$

(This is just a very rough idea of the adelic viewpoint on unification of real and p-adic numbers; there are also some constraints on behavior of coordinates leading to a special topology on the adel.)

The amplitudes are coupled through the following product formula:

$$A_\infty(a, b) \prod_p A_p(a, b) = 1, \quad a, b \in \mathbb{Q},$$

where we set $A_\infty(a, b) \equiv A(a, b)$ for the amplitude of the real string theory, Thus

$$A_\infty(a, b) = 1/\prod_p A_p(a, b).$$

Thus (according to Dragovich) our *real world is the result of integration by our senses and the corresponding devices of the variety of "prime worlds".*

2.7. p-adic quantum mechanics. There are two p-adic quantum models corresponding to ranges of values of wave functions

- $\psi : \mathbb{Q}_p \to \mathbb{C}$, see [4];
- $\psi : \mathbb{Q}_p \to \mathbb{Q}_p$ (or some algebraic extension of $\mathbb{Q}_p$ or even the field of complex p-adic numbers $\mathbb{C}_p$), see [8].

One of the main distinguishing features of the $(\psi : \mathbb{Q}_p \to \mathbb{C})$-model is the behavior of the quantum harmonic oscillator. It has the spectrum $(\lambda_n = p^n)$, $n = 0, 1, \dots$, and all eigenvalues for $n > 0$ have infinite degeneration.

Open Problem. To find some physical or biological system with such a spectrum.

One of the main distinguishing features of the $(\psi : \mathbb{Q}_p \to \mathbb{Q}_p)$-model is the appearance of p-valued probabilities. This induces the complicated problems of definition and interpretation of such probabilities.

- All experimental data belong to the field of "physical numbers" $\mathbb{Q}$, in particular, the relative frequencies of events $\nu_N = n/N$.
- Relative frequencies of events $\nu_N = n/N$ can be embedded not only into $\mathbb{R}$ (as it is done in classical probability theory), but also in any $\mathbb{Q}_p$.

Suppose now that it happens that the limit of frequencies (ν_N) exists in $\mathbb{Q}_p$.

Remark. This implies that there is no limit in $\mathbb{R}$, so from the conventional viewpoint there is no even statistical law, totally erratic data.

Consider this limit $\lim \nu_N$ with respect to the p-adic metric, its value $P \in \mathbb{Q}_p$ is considered as probability (of the event E whose frequencies were observed).

Some toy biological models of the exponential population growth were considered in the book [8]. However, I did not find real data exhibiting such features. One of the problem is to find the concrete value of p for the possible p-adic probabilistic structure of the data. In the biological models presented in [8], the prime number p was a parameter of the model given from the very beginning.

On the other hand, theoretically I and collaborators were able to construct complete theory of p-adic probability, e.g., limit theorems, random variables, stochastic processes.

3. Ultrametricity, hierarchy and cognition

3.1. Ultrametric spaces. A metric ρ is an ultrametric if it satisfies the *strong triangle inequality*:

$$\rho(x,y) \leq \max\left[\rho(x,z), \rho(z,y)\right], \quad x,y,z \in X. \tag{1}$$

Here each triangle is isosceles. We set

$$U_r(a) = \{x \in X : \rho(x,a) \leq r\}, \quad U_r^-(a) = \{x \in X : \rho(x,a) < r\}.$$

These are balls of radius r with center a. The balls have the following properties:

1. Let U and V be two balls in X. Then there are only two possibilities: (a) balls are ordered by inclusion (i.e., $U \subset V$ or $V \subset U$); (b) balls are disjoint. (There is the third possibility in the Euclidean space. Therefore clusterization into balls is natural in ultrametric spaces and not in, e.g., Euclidean space.)
2. Each point of a ball may serve as a center.
3. Let U be a ball and let a point x not belong to U. Then distances from x to all points belonging to U are equal:

$$\rho(x,y) = \rho(x,z), \quad y,z \in U.$$

So, ultrametric geometry differs crucially from Euclidean geometry.

3.2. *m*-adic numbers as ultrametric spaces. Consider an ultrametric space $\mathbf{Z}_m$, where every point x has the infinite number of coordinates

$$x = (\alpha_1, \ldots, \alpha_n, \ldots), \quad \alpha_j \in A_m = (0, \ldots, m-1), \tag{2}$$

where $m > 1$ is a natural number, the base of the alphabet A_m. For two points $x = (\alpha_j)$, $y = (\beta_j)$ we set

$$\rho_m(x,y) = \frac{1}{m^k}$$

if $\alpha_j = \beta_j$, $j = 0, 1, \ldots, k-1$, and $\alpha_k \neq \beta_k$. This is a metric and even an ultrametric.

A point $x = (\alpha_j)$ can be identified with the m-adic number:

$$x = \alpha_0\alpha_1 \ldots \alpha_k \ldots \equiv \alpha_0 + \alpha_1 m + \ldots + \alpha_k m^k + \ldots. \tag{3}$$

The series converges in the metric space $(\mathbf{Z}_m, \rho_m)$.

3.3. *m*-adic modeling of cognition. We start with a discussion about differences in mathematical representations of the physical and mental worlds. Aristotle pointed to the following characterizations of physical and mental spaces, respectively:

- Physical space is continuous, infinite divisible, connected.
- Mental space is discrete, hierarchic, totally disconnected.

We present the mental space as discrete, hierarchic, and totally disconnected topological space. One of the basic models of such spaces is given by ultrametric spaces and more specially by m-adic trees.

We use dynamical systems in such spaces to model flows of unconscious information at different levels of mental representation hierarchy, see [1]–[14]:

- mental points,
- categories,
- ideas.

We remark that our model can be interpreted as an unconventional computational model: non-algorithmic hierarchic "computations" (identified with the process of thinking at the unconscious level).

Recently the ultrametric approach to unconsciousness was also developed in the works of psychologist Lauro-Grotto [18] and Murtagh [19], [20], cognitive computation and data analysis expert.

3.4. Mental space. Geometrically a mental space is a tree, representing the intrinsically hierarchic structure of mental information. Thus mathematically physics is based on the straight line geometry and cognition on the tree geometry.

For analysis, it is great (although not necessary) to have an arithmetic structure similar to the arithmetics of real numbers.

The rings of m-adic numbers $\mathbf{Z}_m$, where $m > 1$ is a natural number, are geometrically given by homogeneous trees, with m-branches leaving each vertex.

I proposed to choose m-adic trees as possible models of mental space X_{mental}: points of this space, mental points, are branches of the m-adic tree. It is possible to encode the branches by sequences of digits, $\alpha_j = 0, 1, \ldots, m-1$.

These numbers can be considered as the *mental coordinates* representing mental points, $x \in X_{\text{mental}}$.

By using mental coordinates we are able to embed into the space X_{mental} the mental analogs of physical rigid bodies — basic categories (special associative classes) and ideas. They are represented, respectively, by balls and *collections of balls* in the ultrametric mental space.

Category: Since in our model the mental hierarchy is encoded by the topology of the mental space, it represents the associative coupling of mental points into balls. A larger ball couples together more mental points. Thus it is a more general category.

IDEA: This is a "cognitive mental image" which is identified as a collection of categories. A category, say A, belonging to two different ideas, say I_1 and I_2 (so $A \in I_1 \cap I_2$), connects these two cognitive mental images. Each category, an ultrametric ball, can be split into disjoint categories, balls of smaller radii. It can be considered as an idea formed from those (less general) sub-categories.

3.5. Dynamical thinking: 3 coupled levels processing. In our model dynamical systems (DSs) in spaces of categories and ideas are induced by DS for mental points. There is a nonlinear relation between input and output

$$x_n + 1 = f(x_n), \quad x_n \in X_{\text{mental}}. \tag{4}$$

The description of functioning of the human brain by DSs feedback processes is a well established approach. The main difference is that conventional DSs work in the real physical space of electric potentials and in our DSs work in the m-adic mental space, a kind of information space.

In spite of the fact that dynamics of categories and ideas can be in principle reduced to dynamics of mental points composing those "mental bodies", those dynamics exhibit their own interesting properties which could not be seen on the level of point-wise dynamics.

3.5.1. *Consciousness⇋unconsciousness cooperation*:

- C formulates problems, and sends them to UC.
- The process of finding a solution by complex DSs, "thinking processors", is hidden in UC. Each processor is determined by a function $f : X_{\text{mental}} \to X_{\text{mental}}$. It produces iterations of points of mental space.
- C controls only some exceptional situations in the work of DSs in UC; attractors and cycles.
- DSs in X_{mental} induce DSs of ball-categories and ideas (collections of balls). It is crucial that behaviors of DSs in X_{mental} and its lifting to spaces of categories and ideas can be very different.
- Extremely cycling (chaotic) behavior at the level of mental points (and even categories) can imply nice stabilization to attractors on the level of ideas. It is profitable to formulate problems and solutions at the level of ideas and not mental points. Hierarchic representations regularize dynamics!

3.6. Integrative levels of procession of mental information. An integrative level, or level of organization, is a set of phenomena emerging on pre-existing phenomena of lower level. Our ultrametric model of DS thinking showed that the geometric structure of the mental space automatically generates a "vertical tower" of integrative levels of information processing.

This vertical hierarchy is based on the initial horizontal hierarchy in information presentation.

In the 2-adic case mental points have the form: $x = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$, $\alpha_j = 0, 1$ ("no"/"yes" coding), where the first coordinate is the most important and so on; e.g., the coordinates can be characteristics of decreasing weight in representing an individual; e.g., α_0 (age): young/not, α_1 (gender): female/male, α_2 (hair): blond/not, α_3 (eyes): blue/not, ...

Take the ball given by the constraint $\alpha_0 = 0$: $U_{1/2}(0)$. It represents the category of young people; a sharper associative category is given by the fixing of two first mental coordinate, e.g., $\alpha_0 = 0$, $\alpha_1 = 0$, this is $U_{1/4}(0)$. It represents the category of young girls. The ball $U_{1/4}(2)$ (here $2 = (0, 1, 0, \ldots)$) represents the category of young boys. We remark that $U_{1/2}(0) = U_{1/4}(0) \cup U_{1/4}(2)$. This is the simplest idea, of young people, the union of the categories of young boys and girls.

3.7. Neuronal realization and coupling with Edelman's theory of neuronal group selection. Consider so-called hierarchic chain of neurons,

$$\mathbf{n} = (n_0, n_1, \dots, n_k, \dots)$$

(in the ideal model the chain is infinite). Suppose that the states of each neuron can be encoded by numbers $\alpha \in A_m = (0, 1, \dots, m-1)$. Then $\mathbf{n}$ produces mental points with coordinates from the alphabet A_m; e.g., just firing/not coding, in the previous example: n_0 represents age, n_1 — gender, n_2 — hair color, ..., see the paper [13].

Of course, the hierarchy in the neuronal chain and, hence, in the mental point depends on the context. Thus mental state space is context dependent.

One of the basic features of the model is that the same set of neurons can be combined in hierarchic chains in different ways by generating different mental representations. Moreover, in accordance with Edelman's TNGS [6], the same neuron can serve for different purposes in different contexts, see also [7]. This feature of neurons was emphasized by Edelman who pointed out that neuronal elements which are structurally very different can perform the same mental function. However, Edelman did not consider the hierarchic neuronal chains. We also remark that, to characterize this property of neurons, Edelman used the terminology "degeneracy" [6] which is has some degree of ambiguity.

3.8. Applications. The main problem in development of real applications is that we cannot determine the transformation laws for DSs working in UC. So, our predictions are of only qualitative nature. Moreover, real mental spaces have complex tree structures; hence, the homogeneous m-adic mental space is just a toy model, but analytical results for DSs were obtained only in this case.

Nevertheless, the following theoretical models can play some role in applications:

- Model of production and processing of information which hierarchically very complex by neuronal structures.
- Mathematical model of treatment-resistant depression. The model says: simpler basic coding implies higher probability of depressive states; so 2-adic Alice easier fall to such depression than say 11-adic Kate.
- Mathematical model for Freud's psychoanalysis, it provides a detailed DS description of creation and psychoanalytic treatment of symptoms. Thus psychoanalysis got a rigorous math grounds.

3.9. Genetics and molecular biology. The 2-adic plane representation of the genetic code was constructed in the paper [15], [16]; a model of functioning of molecular motors based on the hierarchic ultrametric representation was developed in the paper [17].

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