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ON A GENERALIZATION OF LISSAJOUS CURVES AND ITS APPLICATIONS

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Abstract. In the paper we consider a generalization of classical Lissajous curves to the situation where corresponding differential forms involve square roots of quartics. We give a new interesting parametrization of these curves and fully analyze their behaviour in terms of roots of the quartics. We indicate natural applications of our method to the analysis of a Duffing oscillator where the Higgs potential is described by a quartic. We also describe an application to the study of movement of a test body in an axially symmetric gravitational field described by the Kerr metric.

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1. Introduction. In the paper we introduce a generalization of Lissajous curves to the case when potential is given by a polynomial of degree four. Classical Lissajous curves describe oscillations in two orthogonal directions and yield square roots of quadratic polynomials. We notice that natural parametrization of these curves gives rise to a parametrization of generalized Lissajous curves. Our parametrization has the advantage that in physical situations it incorporates initial conditions of the movement. We study dynamics of systems described by the differential forms that involve square roots of quartics in denominators. This requires a detailed study of dependence of our generalized Lissajous curves in terms of the nature of roots of corresponding quartics. This is done in Sections 3.1 and 3.2. In the "generic" cases (i.e. in the open regions of an appropriate Euclidean space) one obtains elliptic functions, whereas in the degenerate cases (corresponding to the boundaries of these regions)—quasi-elliptic or elementary functions.

A natural application of the curves we introduce is an analysis of a Duffing oscillator (cf. [2], [3], [4], [20]), where Higgs potential is given by a quartic. Our approach may also be used for a new description of the motion in (r, θ)-plane of a test body in an axially symmetric gravitational field described by the Kerr metric (see Section 4). This may be useful in study of black holes [5], [6], [10], [13].

2. Lissajous curves and generalization. Consider the differential equation

$$\frac{\omega_1^{-1}\,dx}{\sqrt{A^2 - x^2}} = \frac{\omega_2^{-1}\,dy}{\sqrt{B^2 - y^2}}\,. \tag{1}$$

which describes oscillations in two perpendicular directions x and y, with angular frequencies ω_1 and ω_2, and amplitudes A and B respectively. Making an appropriate choice of signs of the square roots in (1) we obtain the following parametric description of the oscillations

$$x = A \sin \omega_1(t - t_1), \quad y = B \sin \omega_2(t - t_2). \tag{2}$$

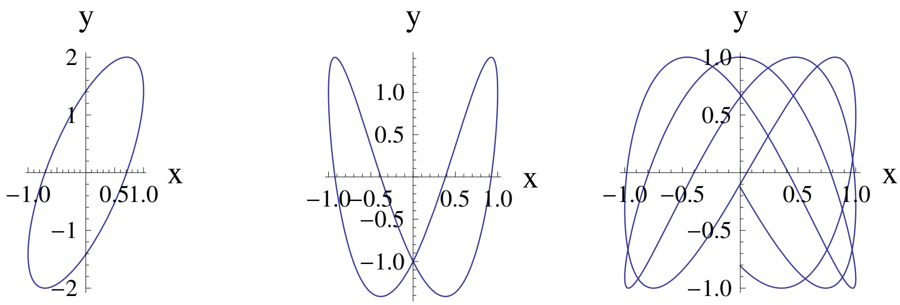
This is a parametric description of a Lissajous curve of the motion.

Remark 2.1. Notice that the parameter t is not an affine parameter of the Lissajous curve (2). It is very convenient because mathematically dt is equal to the both sides of (1). The constants of integration t_1 and t_2 determine initial conditions of the motion. We also assume that the signs of the square roots in (1) and in the sequel, i.e. (9) can be taken $\pm$. This is to assure that our generalized Lissajous curves are continuous.

In Figures 1, 2 classical Lissajous curves for various values of $A, B, \omega_1, \omega_2, t_1, t_2$ are depicted.

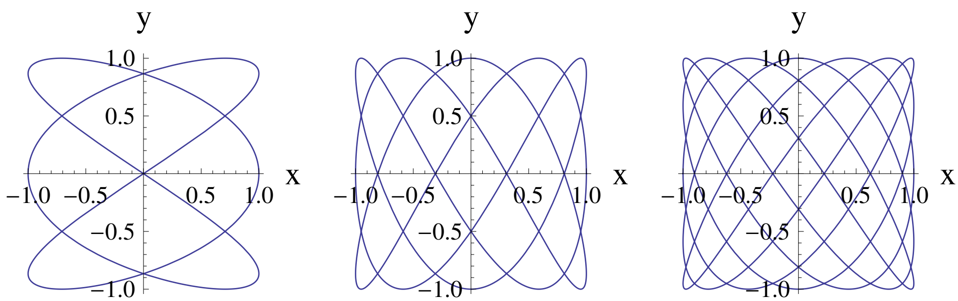
— in Figure 1a) $A = 1,\ t_1 = 0,\ \omega_1 = 1,\ B = 2,\ t_2 = \pi/4,\ \omega_2 = 1;$
— in Figure 1b) $A = 1,\ t_1 = 0,\ \omega_1 = 1,\ B = \sqrt{2}, t_2 = \pi/8,\ \omega_2 = 2;$
— in Figure 1c) $A = 1,\ t_1 = 0,\ \omega_1 = 1,\ B = 1,\ t_2 = -3\pi/4,\ \omega_2 = \sqrt{3};$
— in Figure 2a) $A = 1,\ t_1 = 0,\ \omega_1 = 3,\ B = 1,\ t_2 = \pi/2,\ \omega_2 = 2;$
— in Figure 2b) $A = 1,\ t_1 = \pi/2,\ \omega_1 = 3,\ B = 1,\ t_2 = 0,\ \omega_2 = 5;$
— in Figure 2c) $A = 1,\ t_1 = \pi/2,\ \omega_1 = 5,\ B = 1,\ t_2 = 0,\ \omega_2 = 7.$

Notice that in the case 1a) it is an ellipse. In case 1c) the ratio $\frac{\omega_1}{\omega_2}$ is irrational and the Lissajous curve is not closed.



a) b) c)

Figure 1



a) b) c)

Figure 2

2.1. Motion in one-dimensional potential field. The law of conservation of energy

$$\frac{1}{2}mv^2 + U(x) = E, \quad v = \frac{dx}{dt} \tag{3}$$

yields the differential equation with separated variables (cf. [18])

$$\sqrt{\frac{m}{2}}\,\frac{dx}{\sqrt{E - U(x)}} = dt. \tag{4}$$

2.2. Motion in two-dimensional separated potential. Consider the motion in the (x, y)-plane in the potential field of the form

$$U(x, y) = U_1(x) + U_2(y). \tag{5}$$

Then both the energy E_1 of the motion in the x-direction and the energy E_2 of the motion in the y-direction are conserved.

$$\frac{1}{2}mv_x^2 + U_1(x) = E_1, \quad \frac{1}{2}mv_y^2 + U_2(y) = E_2, \quad E_1 + E_2 = E. \tag{6}$$

One obtains similarly to (1) an equality of forms

$$\sqrt{\frac{m}{2}}\,\frac{dx}{\sqrt{E_1-U_1(x)}}=\sqrt{\frac{m}{2}}\,\frac{dy}{\sqrt{E_2-U_2(y)}}=dt. \tag{7}$$

A particularly interesting situation arises when one wishes to describe the character of motion of a test particle in a field whose potential is a superposition of two perpendicular Higgs potentials

$$U_1(x)=\alpha(x^2-x_0^2)^2,\quad U_2(x)=\beta(y^2-y_0^2)^2. \tag{8}$$

We will study the following more general equality of forms

$$\frac{dx}{\sqrt{\alpha(x-x_1)(x-x_2)(x-x_3)(x-x_4)}}=\frac{dy}{\sqrt{\beta(y-y_1)(y-y_2)(y-y_3)(y-y_4)}}. \tag{9}$$

3. Integration of differential forms (9). In this section we will be interested in finding a solution of (9) similar to (2). So let us consider the following equality of forms

$$\frac{dz}{\sqrt{\hat{a}(z-z_1)(z-z_2)(z-z_3)(z-z_4)}}=du. \tag{10}$$

We can approach the integration of (10) in the following ways

(P1) Assuming that only coefficients of a quartic are given, find the Legendre normal form of the integral corresponding to the left hand side of (10). Then compute z. This leads to Jacobi elliptic functions or elementary functions depending on the nature of the roots of the quartic.

(P2) Use the cross-ratio of roots to find a map which pulls back the form (10) to its normal form. Then integrate and compute z. This procedure works nicely for simple roots and requires knowledge of them.

We begin with description of (P1).

3.1. Roots of a quartic with real coefficients. Now let us recall the nature of roots of a quartic with real coefficients (cf. [22])

$$az^4+bz^3+cz^2+dz+e. \tag{11}$$

Let

$$\begin{aligned}\mathrm{Disc}=256a^3e^3-192a^2bde^2-128a^2c^2e^2+144a^2cd^2e-27a^2d^4+144ab^2ce^2\\ -6ab^2d^2e-80abc^2de+18abcd^3+16ac^4e-4ac^3d^2-27b^4e^2\\ +18b^3cde-4b^3d^3-4b^2c^3e+b^2c^2d^2\end{aligned} \tag{12}$$

be the discriminant of (11). Let further

$$P=8ac-3b^2, \tag{13}$$

$$D_0=c^2-3bd+12ae, \tag{14}$$

and

$$D=64a^3e-16a^2c^2+16ab^2c-16a^2bd-3b^4. \tag{15}$$

Then according to [22] the following cases fully describe all possible roots of the quartic (11).

1. If Disc > 0 then all roots of (11) are either real or complex:

a) if $P < 0$ and $D < 0$ then all roots are real and distinct,
b) if $P > 0$ and $D > 0$ then (11) has two pairs of complex conjugate roots.

2. If Disc < 0 then (11) has two real and two complex roots.

3. If Disc $= 0$ then either (11) has a multiple root, or is a square of a quadratic polynomial:

a) if $P < 0$ and $D < 0$ and $D_0 \neq 0$ then (11) has a real double root and two simple real roots,
b) if either $P > 0$ and $D \neq 0$ or $D > 0$, then (11) has a real double root and a pair of complex conjugate roots,
c) if $D_0 = 0$ and $D \neq 0$ there is a triple real root and a simple real root,
d) if $D = 0$ then

    i) if $P < 0$ then (11) has two real double roots,

    ii) if $P > 0$ then (11) has two complex conjugate double roots,

    iii) if $D_0 = 0$ then all roots of (11) are equal to $-\frac{b}{4a}$.

If we assume that in (11) $a \neq 0$ (the quartic is of general form) then it is possible to find all the roots by the formulas:

$$z_{1,2} = -\frac{b}{4a} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}\,, \tag{16}$$

$$z_{3,4} = -\frac{b}{4a} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}\,, \tag{17}$$

where

$$p = \frac{8ac - 3b^2}{8a^2}\,, \qquad q = \frac{b^3 - 4abc + 8a^2 d}{8a^3} \tag{18}$$

and

$$S = \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3a}\Big(Q + \frac{D_0}{Q}\Big)}\,, \qquad Q = \sqrt[3]{\frac{D_1 + \sqrt{D_1^2 - 4D_0^3}}{2}}\,. \tag{19}$$

In (19) D_0 is given by (14) and

$$D_1 = 2c^3 - 9bcd + 27b^2 e + 27ad^2 - 72ace. \tag{20}$$

Remark 3.1. One has $D_1^2 - 4D_0^3 = -27\,\mathrm{Disc}$.

Remark 3.2. If Disc $\neq 0$ and $D_0 = 0$ the sign of $\sqrt{D_1^2 - 4D_0^3}$ has to be chosen to ensure $Q \neq 0$. If $S = 0$, then one has to change the cubic root in (19) to ensure $S \neq 0$. This is always possible unless a quartic has a real root of order 4, i.e. we are in the case 3d) iii).

After finding the roots of the quartic we can transform (10) to the Legendre normal form.

3.2. Integrals of (10). Now we transform the elliptic integral to the canonical form. This procedure is due to Legendre and it is well known (cf. [15]). One uses the homographic transformation

$$z = \frac{\mathbf{a}t + \mathbf{b}}{\mathbf{c}t + \mathbf{d}} \tag{21}$$

and gets under the root sign

$$\hat{a}(g_0 t^2 + g_1 t + g_2)(h_0 t^2 + h_1 t + h_2), \tag{22}$$

where

$$\begin{gathered} g_0 = (\mathbf{a} - \mathbf{c}z_1)(\mathbf{a} - \mathbf{c}z_2), \quad g_1 = (\mathbf{a} - \mathbf{c}z_1)(\mathbf{b} - \mathbf{d}z_2) + (\mathbf{a} - \mathbf{c}z_2)(\mathbf{b} - \mathbf{d}z_1), \\ g_2 = (\mathbf{b} - \mathbf{d}z_1)(\mathbf{b} - \mathbf{d}z_2). \end{gathered} \tag{23}$$

Formulas for h_0, h_1, and h_2 are obtained from (23) by substituting z_3 for z_1 and z_4 for z_2. The homographic transformation is chosen in such a way that $g_1 = h_1 = 0$. This leads to the following system of linear equations for $\frac{\mathbf{d}}{\mathbf{b}} + \frac{\mathbf{c}}{\mathbf{a}}$ and $\frac{\mathbf{d}}{\mathbf{b}} \cdot \frac{\mathbf{c}}{\mathbf{a}}$:

$$\begin{aligned} (z_1 + z_2)\left\{\frac{\mathbf{d}}{\mathbf{b}} + \frac{\mathbf{c}}{\mathbf{a}}\right\} - 2z_1 z_2 \left\{\frac{\mathbf{d}}{\mathbf{b}} \cdot \frac{\mathbf{c}}{\mathbf{a}}\right\} = 2, \\ (z_3 + z_4)\left\{\frac{\mathbf{d}}{\mathbf{b}} + \frac{\mathbf{c}}{\mathbf{a}}\right\} - 2z_3 z_4 \left\{\frac{\mathbf{d}}{\mathbf{b}} \cdot \frac{\mathbf{c}}{\mathbf{a}}\right\} = 2. \end{aligned} \tag{24}$$

The equation (10) has now the form

$$\frac{(\mathbf{ad} - \mathbf{bc})\, dt}{\sqrt{\hat{a}(g_0 t^2 + g_2)(h_0 t^2 + h_2)}} = du. \tag{25}$$

After the substitution $x = \sqrt{\frac{|g_0|}{|g_2|}}\, t$ we obtain the following cases of equalities of real forms

$$\frac{C\, dx}{\sqrt{\pm(1 \pm x^2)(1 \pm k^2 x^2)}} = du, \tag{26}$$

where

$$C = \frac{(\mathbf{ad} - \mathbf{bc})}{\sqrt{|g_0 h_2 \hat{a}|}} \quad \text{and} \quad k = \sqrt{\frac{|h_0 g_2|}{|h_2 g_0|}}\,. \tag{27}$$

Thus we have to analyze the real integrals

$$\int \frac{dx}{\sqrt{\pm(1 \pm x^2)(1 \pm k^2 x^2)}} = \int \frac{du}{C}\,. \tag{28}$$

Expressions (28) can always be transformed to the canonical form

$$\int \frac{d\theta}{\sqrt{1 - l^2 \sin^2\theta}}\,. \tag{29}$$

Remark 3.3. Notice that in the integral (29), in the non-degenerate case, one can assume that the modulus $l \in (0, 1)$. Indeed for $l > 1$ we have

$$\int \frac{d\theta}{\sqrt{1 - l^2 \sin^2\theta}} = \frac{1}{l} \int \frac{d\vartheta}{\sqrt{1 - l^{-2} \sin^2\vartheta}}\,, \qquad l \sin\theta = \sin\vartheta. \tag{30}$$

In what follows we assume that $l \in (0, 1)$. If this is not the case one has to use (30) and change the modulus to l^{-2} and also divide by l the constant C of (27).

Let

$$w := \int_0^\phi \frac{d\theta}{\sqrt{1 - l^2 \sin^2\theta}}\,. \tag{31}$$

In the sequel we will use the Jacobi elliptic functions (cf. [19], [15], [1]), defined as:

$$\begin{gathered}\operatorname{sn}(w,l) = \sin\phi, \quad \operatorname{cn}(w,l) = \cos\phi, \quad \operatorname{dn}(w,l) = \sqrt{1 - l^2\sin^2\phi}\,,\\ \operatorname{sc}(w,l) = \frac{\operatorname{sn}(w,l)}{\operatorname{cn}(w,l)} = \tan\phi, \quad \operatorname{nc}(w,l) = \frac{1}{\operatorname{cn}(w,l)}\,.\end{gathered} \tag{32}$$

(A) The roots are all real and simple. This corresponds to the case 1a) of Section 3.1.

We have two possibilities $\int \frac{dx}{\sqrt{\pm(1-x^2)(1-k^2x^2)}}$, $k \in (0,1)$. The case

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,, \quad x = \sin\theta, \tag{33}$$

is canonical and

$$x = \operatorname{sn}\Big(\frac{1}{C}(u-u_0), k\Big). \tag{34}$$

In the second case we can express x by the function dn (cf. [23] p. 490):

$$\int \frac{dx}{\sqrt{-(1-x^2)(1-k^2x^2)}} = \int \frac{du}{C}\,, \quad x = \frac{1}{k}\operatorname{dn}\Big(\frac{u-u_0}{C}, k'\Big), \quad k' = \sqrt{1-k^2}\,. \tag{35}$$

Finally,

$$z = \frac{\mathbf{a}t + \mathbf{b}}{\mathbf{c}t + \mathbf{d}} \quad \text{where} \quad t = \sqrt{\frac{|g_2|}{|g_0|}}\,x. \tag{36}$$

(B) Two distinct real roots and a conjugate pair of (distinct) complex roots. This corresponds to case 2 of Section 3.1.

Here, we have also two possibilities $\int \frac{dx}{\sqrt{\pm(1-x^2)(1+k^2x^2)}}$, $k \in \mathbb{R}$. Substituting in the first case

$$\int \frac{dx}{\sqrt{(1-x^2)(1+k^2x^2)}} = -\int \frac{d\theta}{\sqrt{(1+k^2) - k^2\sin^2\theta}}\,, \quad x = \cos\theta, \tag{37}$$

we obtain

$$x = \operatorname{cn}\Big(\frac{1}{C_1}\,(u-u_0), l\Big), \quad l = \frac{k}{\sqrt{1+k^2}}\,, \quad C_1 = \frac{-C}{\sqrt{1+k^2}}\,. \tag{38}$$

Finally, we use (36) to compute z.

Notice that the following substitution for the second case

$$\int \frac{dx}{\sqrt{-(1-x^2)(1+k^2x^2)}} = -\int \frac{dv}{\sqrt{(1-v^2)(k^2+v^2)}}\,, \quad v = \frac{1}{x}\,, \tag{39}$$

yields (cf. [23], p. 490)

$$x = \operatorname{nc}\Big(\frac{1}{C_1}\,(u-u_0), l\Big), \quad l = \frac{1}{\sqrt{1+k^2}}\,, \quad C_1 = \frac{C}{\sqrt{1+k^2}}\,. \tag{40}$$

(C) The roots are imaginary and distinct, i.e. there are two different pairs of complex conjugate roots and the quartic has the form $\hat{a}(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)$. This corresponds to the case 1b) of Section 3.1.

Here, we have only one possibility (minus sign under the square root is not permitted) and we obtain

$$\int \frac{dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \int \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}}, \quad x=\tan\theta, \quad k\in(0,1). \tag{41}$$

Thus,

$$x = \operatorname{sc}\Big(\frac{1}{C}(u-u_0), k'\Big), \quad k' = \sqrt{1-k^2}. \tag{42}$$

Again we use (36) to get z.

REMARK 3.4. In the original Legendre transformations (cf. [15], [8]) in cases (B) and (C) there are complex parameters and one obtains complex transformations. For physical reasons we preferred to stay with real integrals and substitutions. Therefore we departed from the original procedure at formula (25).

(D) There is a pair of real or conjugate complex double roots, i.e. the quartic is of the form $\hat{a}(z-z_1)^2(z-z_2)^2$, $z_1, z_2 \in \mathbb{R}$ or $z_2 = \bar{z_1}$. This corresponds to the cases 3d) i) and 3d) ii).

In this case (25) takes the form

$$\frac{D\,dt}{g_2+g_0t^2} = du, \qquad \text{where} \qquad D = \frac{\mathbf{ad}-\mathbf{bc}}{\sqrt{\hat{a}}}. \tag{43}$$

For the real roots the signs of g_0, g_2 are different, and for the complex ones $g_0, g_2 > 0$. The real case yields

$$\begin{gathered}\int \frac{D\,dt}{g_2+g_0t^2} = \frac{\epsilon D}{\sqrt{|g_0g_2|}} \operatorname{ar\,tanh}\Bigg(\sqrt{\frac{|g_0|}{|g_2|}}t\Bigg) + u_0 = u, \\ t = \sqrt{\frac{|g_2|}{|g_0|}} \cdot \tanh\Bigg(\frac{\sqrt{|g_0g_2|}}{\epsilon D}(u-u_0)\Bigg), \quad \text{where} \quad \epsilon = \operatorname{sign} g_2,\end{gathered} \tag{44}$$

and the complex one gives

$$\int \frac{D\,dt}{g_2+g_0t^2} = \frac{D}{\sqrt{g_0g_2}} \arctan\Bigg(\sqrt{\frac{g_0}{g_2}}t\Bigg) + u_0 = u, \quad t = \sqrt{\frac{g_2}{g_0}} \cdot \tan\Bigg(\frac{\sqrt{g_0g_2}}{D}(u-u_0)\Bigg). \tag{45}$$

We can now easily express $z = \frac{\mathbf{a}t+\mathbf{b}}{\mathbf{c}t+\mathbf{d}}$ in terms of the parameter u.

(E) The quartic has a real double root and a pair of simple roots. This corresponds to the cases 3a) and b) of Section 3.1.

Now (25) is of the form

$$\int \frac{D\,dt}{t\sqrt{g_2+g_0t^2}} = -\int \frac{D\,dv}{\sqrt{g_0+g_2v^2}}, \quad v = \frac{1}{t}, \tag{46}$$

where $D = \frac{\mathbf{ad}-\mathbf{bc}}{\sqrt{\hat{a}h_0}}$, $g_0, g_2 \neq 0$.

We have three possibilities for the signs of g_0 and g_2. The first two correspond to real roots:

$$\int \frac{-D\,dv}{\sqrt{|g_0|-|g_2|v^2}} = \frac{D}{\sqrt{|g_2|}}\arccos\sqrt{\frac{|g_2|}{|g_0|}}\,v+u_0,\ t=\frac{\sqrt{|g_2|}}{\sqrt{|g_0|}\cos\frac{\sqrt{|g_2|}}{D}(u-u_0)}\,, \tag{47}$$

$$\int \frac{-D\,dv}{\sqrt{-|g_0|+|g_2|v^2}} = -\frac{D}{\sqrt{|g_2|}}\operatorname{ar\,cosh}\sqrt{\frac{|g_2|}{|g_0|}}\,v+u_0,\ t=\frac{\sqrt{|g_2|}}{\sqrt{|g_0|}\cosh\frac{\sqrt{|g_2|}}{D}(u-u_0)}\,. \tag{48}$$

The third corresponds to a pair of simple complex roots

$$\int \frac{-D\,dv}{\sqrt{|g_0|+|g_2|v^2}} = -\frac{D}{\sqrt{|g_2|}}\operatorname{ar\,sinh}\sqrt{\frac{|g_2|}{|g_0|}}\,v+u_0,\quad t=\frac{-\sqrt{|g_2|}}{\sqrt{|g_0|}\sinh\frac{\sqrt{|g_2|}}{D}(u-u_0)}\,. \tag{49}$$

Then we use (21) to get z.

(F) The quartic has a triple real root and a simple real root — case 3c) of Section 3.1.

In this case the integral of (10) becomes

$$\int \frac{dz}{(z-r_0)\sqrt{\hat{a}(z-r_0)(z-r_1)}} = \frac{2}{r_1-r_0}\sqrt{\frac{1}{\hat{a}}\frac{(z-r_1)}{(z-r_0)}}+u_0=u. \tag{50}$$

Hence

$$z=\frac{\hat{a}r_0(r_1-r_0)^2(u-u_0)^2-4r_1}{\hat{a}(r_1-r_0)^2(u-u_0)^2-4}\,. \tag{51}$$

(G) The quartic has a real root of order four — case 3d) iii) of Section 3.1.

In this case the integral of (10) is

$$\int \frac{dz}{\sqrt{\hat{a}}\,(z-r_0)^2} = \frac{-1}{\sqrt{\hat{a}}\,(z-r_0)}+u_0=u, \tag{52}$$

$$z=r_0-\frac{1}{\sqrt{\hat{a}}(u-u_0)}\,. \tag{53}$$

Now we will discuss the procedure (P2).

3.3. Cross-ratio of the roots and pull-back of the form (10). Define the cross-ratio of the complex numbers $z_1,\dots,z_4$

$$(z_1,z_2;z_3,z_4)=\frac{(z_3-z_1)(z_4-z_2)}{(z_4-z_1)(z_3-z_2)}\,. \tag{54}$$

The expression is not symmetric. Let $\lambda := (z_1,z_2;z_3,z_4)$. The symmetric group acting on these numbers changes the cross-ratio:

$$(z_{\sigma(1)},z_{\sigma(2)};z_{\sigma(3)},z_{\sigma(4)})=\lambda,\ \lambda^{-1},\ 1-\lambda,\ 1-\lambda^{-1},\ (1-\lambda)^{-1},\ (1-\lambda^{-1})^{-1} \tag{55}$$

(if μ is a value of the cross-ratio (55) then μ^{-1} and $1-\mu$ are also its values). Moreover the isotropy group of $(z_1,z_2;z_3,z_4)$ is generated by $(1,2)(3,4)$ and $(1,3)(2,4)$, i.e. it is the Klein four group. Let S be a Riemann sphere and (a_1,a_2,a_3) a sequence of pairwise different complex numbers. Define

$$f_{a_1,a_2,a_3}: S\longrightarrow S,\quad f_{a_1,a_2,a_3}(z):=(z,a_1;a_2,a_3), \tag{56}$$

The following lemma and Corollary 3.6 describe the change of variables in (10). Various parts of the lemma are present in the literature (cf. [11], [16], [21]) but we decided to include it for convenience.

Lemma 3.5. *For any sequences* $\overline{\mathbf{a}} = (a_1, a_2, a_3)$ *and* $\overline{\mathbf{b}} = (b_1, b_2, b_3)$ *of pairwise different complex numbers there exists a unique holomorphic map of a Riemann sphere*:

$$g_{\overline{\mathbf{a}},\overline{\mathbf{b}}} : S \to S \quad \textit{such that} \quad g_{\overline{\mathbf{a}},\overline{\mathbf{b}}}(a_i) = b_i, \quad i = 1, 2, 3. \tag{57}$$

Furthermore, for any sequences $\overline{\mathbf{c}} = (c_1, c_2, c_3, c_4)$ *and* $\overline{\mathbf{d}} = (d_1, d_2, d_3, d_4)$ *of pairwise different complex numbers with the same cross-ratio there exists a unique holomorphic map of a Riemann sphere*:

$$h_{\overline{\mathbf{c}},\overline{\mathbf{d}}} : S \to S \quad \textit{such that} \quad h_{\overline{\mathbf{c}},\overline{\mathbf{d}}}(c_i) = d_i, \quad i = 1, 2, 3, 4. \tag{58}$$

Proof. Notice that f_{a_1,a_2,a_3} is a holomorphic bijection. In fact it is given by a homographic map. Notice that since S is compact any meromorphic function of S is a rational function of $\mathbb{C} \cup \infty \cong S$. Thus any biholomorphic map is given by a homographic map. We have

$$f_{a_1,a_2,a_3}(a_1) = 1, \quad f_{a_1,a_2,a_3}(a_2) = 0, \quad f_{a_1,a_2,a_3}(a_3) = \infty. \tag{59}$$

Define $g_{\overline{\mathbf{a}},\overline{\mathbf{b}}} = f^{-1}_{b_1,b_2,b_3} \cdot f_{a_1,a_2,a_3}$. Since any homographic map with three points fixed is an identity we see that $g_{\overline{\mathbf{a}},\overline{\mathbf{b}}}$ is a unique holomorphic map satisfying condition (57). Now assume that $\overline{\mathbf{c}}$ and $\overline{\mathbf{d}}$ are two sequences with the same cross-ratio. Define

$$h_{\overline{\mathbf{c}},\overline{\mathbf{d}}} = g_{(c_1,c_2,c_3),(d_1,d_2,d_3)} = g_{(c_1,c_2,c_4),(d_1,d_2,d_4)} = \dots . \tag{60}$$

The equalities (60) show that the condition in (58) is satisfied. ∎

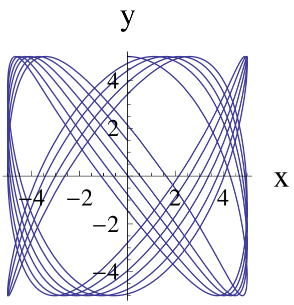
For a holomorphic map f of S and a differential form ω let $f^*\omega$ be a pullback of ω via f.

Corollary 3.6. *We have*

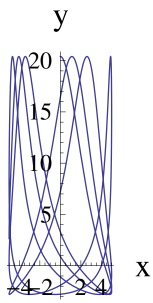
$$h^*_{\overline{\mathbf{w}},\overline{\mathbf{z}}} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} = \frac{\sqrt{\frac{w_1-w_2}{z_1-z_2}\frac{w_3-w_4}{z_3-z_4}}\, dw}{\sqrt{(w-w_1)(w-w_2)(w-w_3)(w-w_4)}}. \tag{61}$$

Remark 3.7. Lemma 3.5 and Corollary 3.6 show that for the generic situation (i.e. no multiple roots of a quartic) a cross-ratio λ of roots is a good quantity for describing integrals of (10) and therefore generalized Lissajous curves.

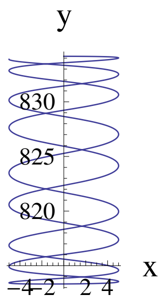
3.4. Numerical results. We obtain various Lissajous curves by integrating forms in the equality (9) as it was described in Section 3.2. Several sample curves are depicted in Figures 3–6. The letters indicate the type of a quartic (9) whereas in parenthesis we give the number of the formula used, e.g. A(34)–B(38) means that the integral of the left hand side of (9) is of type (A) and is given by (34), whereas the right hand side is of type (B) and is given by (38).



A(34)–B(38)

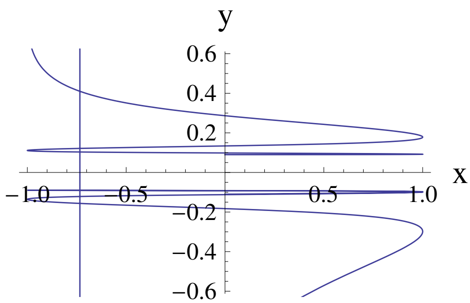


A(34)–B(38)

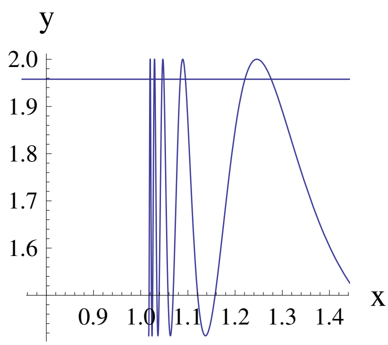


A(34)–A(35)

Figure 3

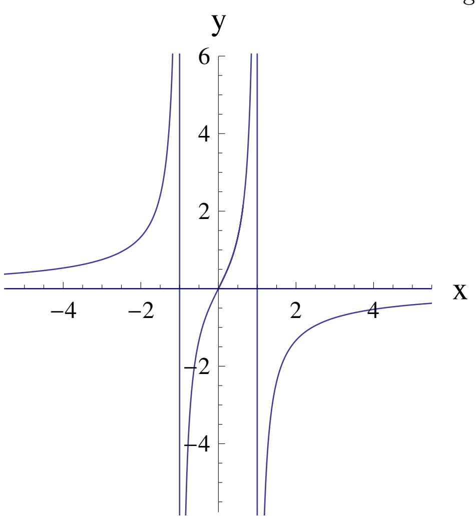


A(34)–E(47)

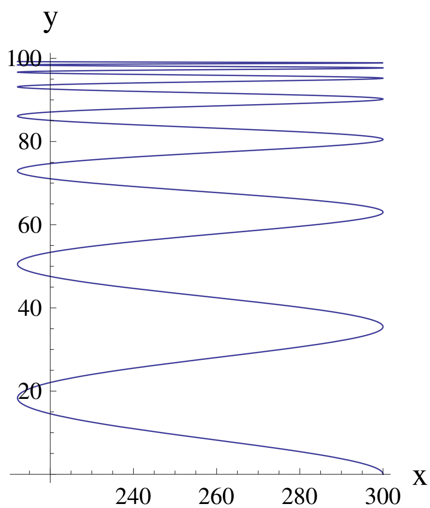


F(51)–A(35)

Figure 4

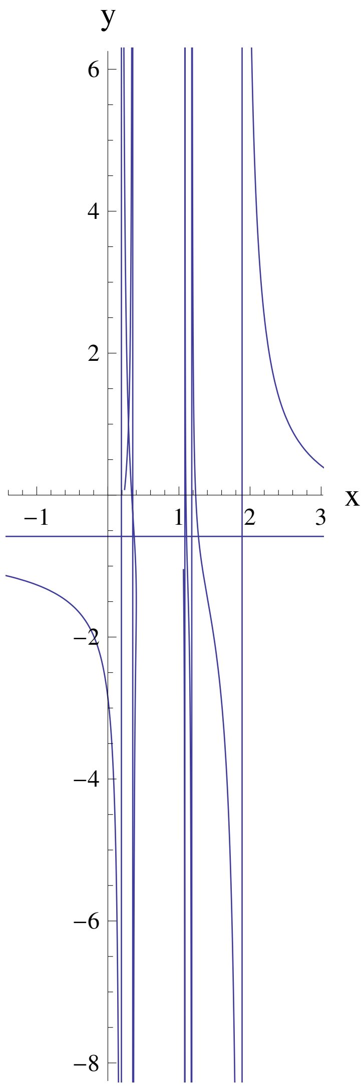


D(45)–D(45)

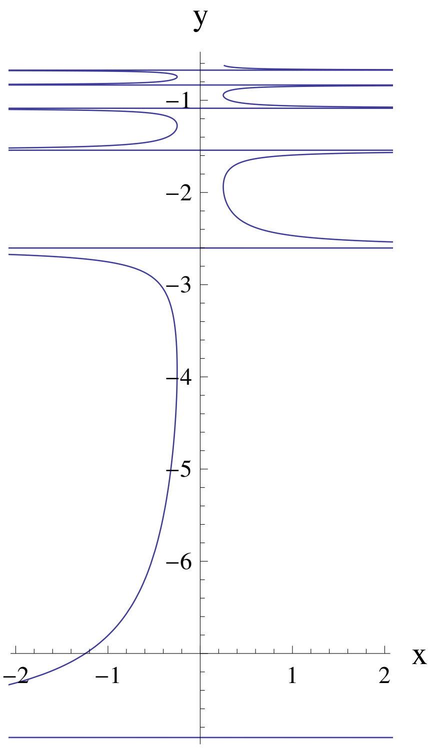


A(35)–D(44)

Figure 5



F(51)–D(45)



E(47)–G(53)

Figure 6

REMARK 3.8. Notice that in the cases where a quartic has multiple roots we have special behaviour of generalized Lissajous curves. There might be oscillations with increasing frequency, as in Figure 5 (cf. formulas (35), (36) versus z_2 computed from (44)), or some areas in the phase space (z_1, z_2) may be excluded, as in the left picture of Figure 6 (this is due to the fact that the function (51) is not a surjection on $\mathbb{R}$).

4. Geodesics in Kerr metric. Stationary axially symmetric gravitational fields are of particular interest in astrophysics. In the case of axial symmetry Kerr found an exact solution to the Einstein equations of general relativity [10], [6], [12], [17]. This solution describes the geometry, i.e. the metric tensor in the vacuum. This remarkable solution is a generalization of the Schwarzschild's solution for spherically symmetric gravitational field and describes the gravitational field of uncharged rotating black hole. Much of research have been devoted to describe the motion of a particle in presence of a rotating black hole [5], [6], [10], [13]. The equations of geodesics have the following form [12]:

$$dt = \frac{[r^2\Delta + 2Mr(r^2+a^2)]E - 2Mra\Phi}{\Delta\sqrt{(B(r))}}\,dr + \frac{a^2E\cos^2\theta}{\sqrt{A(\theta)}}\,d\theta, \tag{62}$$

$$d\phi = \frac{(r^2-2Mr)\Phi + 2MraE}{\Delta\sqrt{(B(r))}}\,dr + \frac{\Phi\cot^2\theta}{\sqrt{A(\theta)}}\,d\theta, \tag{63}$$

$$\frac{dr}{\sqrt{(B(r))}} = \frac{d\theta}{\sqrt{A(\theta)}}\,, \tag{64}$$

where

$$A(\theta) = Q + a^2(E^2-m^2)\cos^2\theta - \Phi^2\cot^2\theta, \tag{65}$$

$$B(r) = [r^2\Delta + 2Mr(r^2+a^2)]E^2 - 4MraE\Phi - (r^2-2Mr)\Phi^2 - \Delta(m^2r^2+Q) \tag{66}$$

and

$$\Delta = r^2 - 2Mr + a^2. \tag{67}$$

Equations (62) and (63) will be treated in a forthcoming paper. Notice, however, that one can use equation (64) and describe the motion in the (r,θ)-plane.

4.1. Movement in the (r,θ)-phase space. A simple substitution $x = \cos\theta$ transforms the right hand side of (3) to the form

$$\frac{dx}{\sqrt{a''x^4 + c''x^2 + e''}}\,, \tag{68}$$

where

$$a'' = -a^2(-1+E^2), \qquad c'' = a^2(-1+E^2) - \Phi^2 - Q \quad \text{and} \quad e'' = Q. \tag{69}$$

Expanding (66) with respect to the variable r we obtain the left hand side of (64):

$$\int \frac{dr}{\sqrt{a'r^4 + b'r^3 + c'r^2 + d'r + e'}}\,, \tag{70}$$

$$a' = E^2 - 1, \quad b' = 2M, \quad c' = a^2(-1+E^2) - \Phi^2 - Q, \tag{71}$$
$$d' = 2a^2E^2M - 4aEM\Phi + 2M\Phi^2 + 2MQ, \quad e' = -a^2Q.$$

Let us write (68) and (70) in the form

$$\frac{dz}{\sqrt{\hat{a}(z-z_1)(z-z_2)(z-z_3)(z-z_4)}}\,. \tag{72}$$

Thus (64) becomes an equation of the form (9).

Remark 4.1. Since we did not want to change notation in formulas in (62)–(67) where a is a constant, we had to introduce $'$ (resp. $''$) for the coefficients of the corresponding quartics in formulas (68)–(71).

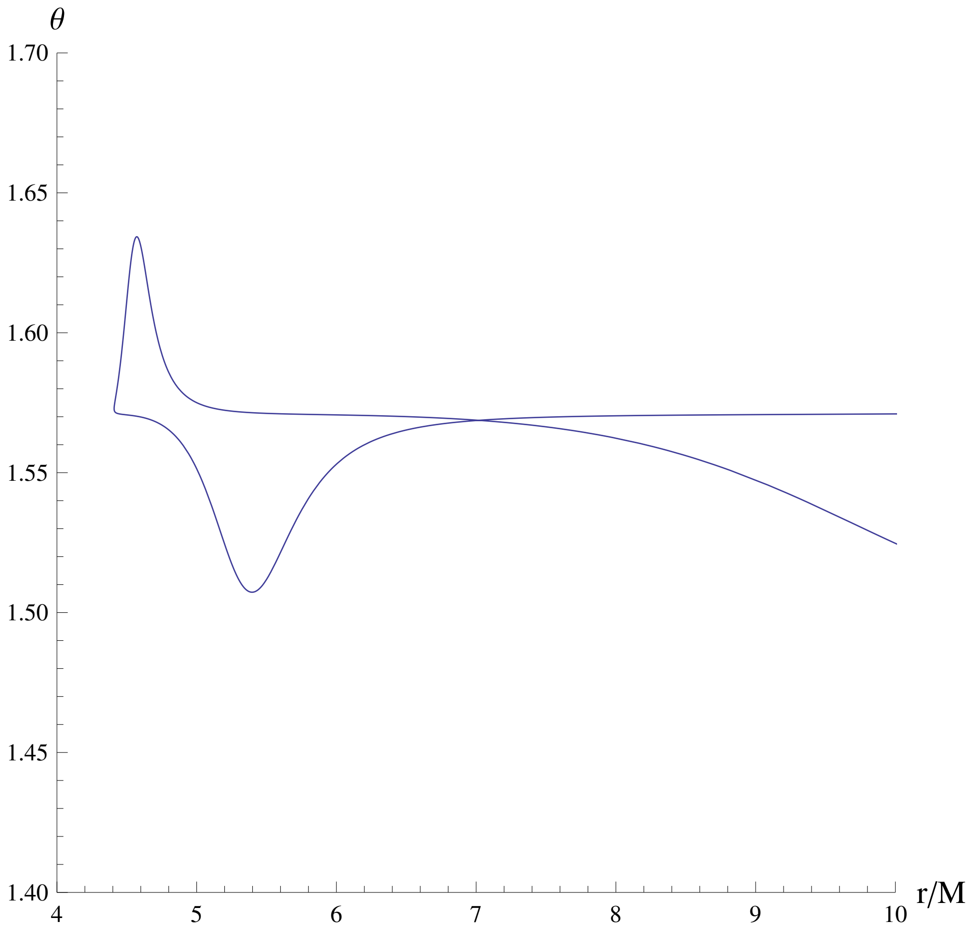


Figure 7. Movement of a particle in $(r/M, \theta)$-phase space

Remark 4.2. The quartic appearing in the right hand side of (64) is biquadratic, so finding roots and the analysis of the corresponding cases simplifies. One readily sees that cases (F) and (G) are impossible. In Figure 7 we plotted the movement of a mass particle in $(r/M, \theta)$-phase space in the presence of a rotating black hole for the following data: $\hat{a} = 0.8$, $\hat{\Phi} = 5$, $\hat{E} = 1.2$, $\hat{Q} = 0.1$. We used natural units and related them to M as in [12], p. 31, i.e. $\hat{a} = a/M$, $\hat{\Phi} = \Phi/(mM)$, $\hat{E} = E/m$, $\hat{Q} = Q/(M^2m^2)$. We set $m = 1$.

We solve this equation using methods described in Section 3.

Remark 4.3. Notice that after finding the parametric solution of the equation corresponding to (9) we have to apply back substitution, i.e. arccos to the right hand side of our parametric solution to obtain θ.

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