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ON HAMILTON–POINCARÉ FIELD EQUATIONS

SANTIAGO CAPRIOTTI

Departamento de Matemática, Universidad Nacional del Sur
8000 Bahía Blanca, Argentina
E-mail: santiago.capriotti@uns.edu.ar

JUAN CARLOS MARRERO

ULL-CSIC Geometría Diferencial y Mecánica Geométrica
Departamento de Matemáticas, Estadística e IO
Sección de Matemáticas, Facultad de Ciencias, Universidad de La Laguna
La Laguna, Tenerife, Canary Islands, Spain
E-mail: jcmarrer@ull.edu.es

Dedicated to Janusz Grabowski on the occasion of his 60th birthday

Abstract. We introduce the prolongation, to the reduced extended multimomentum bundle, of a vertical vector field (in the total space of the corresponding configuration bundle) which is invariant under the action of the symmetry Lie group. Using this construction, we present a geometric description of the Hamilton–Poincaré field equations associated with a symmetric Hamiltonian field theory. Finally, we discuss an example: the theory of minimal submanifolds of a Riemannian manifold.

1. Introduction. The geometric formulation of classical field theories has been an active area of research in recent years, especially by the use of new tools which come from differential geometry. In fact, following the ideas of Tulczyjew, the Polish group in Warsaw has developed a recent and intensive research in the field (see [13, 14, 16, 19, 20, 21]). Other different approaches to the study of classical field theories of first order, by using multisymplectic geometry, have been discussed by several authors (see [2, 4, 6, 7, 8, 9, 25, 26, 27, 28]).

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On the other hand, as in classical mechanics, the study of classical field theories in the presence of a symmetry Lie group is very interesting (see [5, 10, 11, 12]). In particular, the geometric description of the reduction by a symmetry Lie group of a Lagrangian field theory has been developed by some authors (see [10] and the references therein). Comparatively speaking, less attention has been devoted to the study of the reduction of a Hamiltonian field theory by a symmetry Lie group (see [5]).

The aim of this note is to give the first steps to fill this gap.

In fact, using the multisymplectic structure of the extended multimomentum bundle, we introduce the prolongation to this space of a vertical vector field in the corresponding configuration bundle. This construction allows us to give a geometric description of the Hamilton–De Donder–Weyl equations for a Hamiltonian field theory. In addition, in the presence of a symmetry Lie group, we will see that an invariant vertical vector field in the configuration bundle may be also lifted to a vector field on the reduced extended multimomentum bundle. As in the unreduced case, using this construction, we will present a suitable description of the evolution equations for the reduced Hamiltonian field theory, that is, a description of the Hamilton–Poincaré field equations.

We remark that the theory of lifts and prolongations of tensor fields to the Lagrangian and Hamiltonian phase spaces has been a tool commonly used by J. Grabowski and his collaborators in the study of the geometric structures which arise in physics, particularly in classical mechanics (see, for instance, [15, 17, 18, 22, 23, 24]).

The note is organized as follows. In Section 2, we will introduce the prolongation, to the extended multimomentum bundle, of a vertical vector field in the total space of the configuration bundle. We also see that if the vertical vector field is invariant under the action of a symmetry Lie group then it may be also lifted to the reduced extended multimomentum bundle. In Sections 3 and 4, using the previous constructions, we will present a geometric description of the Hamilton–De Donder–Weyl equations and the Hamilton–Poincaré field equations for a Hamiltonian field theory and the reduction of a symmetric Hamiltonian field theory, respectively. Finally, in Section 5, we will discuss an example: the theory of minimal submanifolds of a Riemannian manifold Q, as a Hamiltonian field theory, and its reduction by a Lie subgroup of the isometry Lie group of Q.

Throughout the paper, summation over repeated indices is understood.

2. Prolongations of (invariant) vertical vector fields to the (reduced) extended multimomentum bundle. In this section, we will introduce the prolongation of a (invariant) vector field to the (reduced) extended multimomentum bundle associated with (the reduction of) a configuration fiber bundle. For this purpose, we will use the multisymplectic structure on the extended multimomentum bundle (see, for instance, [3]).

2.1. The multisymplectic structure on the extended multimomentum bundle. Let $\pi : P \to M$ be a fibration, that is, a surjective submersion. We will assume that the manifold M is orientable with fixed orientation which is determined by a volume form Ω.

Then, we will consider the following spaces:

1. The affine dual bundle $(J^1\pi)^+$ to the 1-jet bundle $\pi_{1,0} : J^1\pi \to P$ associated with the fibration π.

2. The dual bundle $(J^1\pi)^0$ to the vector bundle $V(J^1\pi)$ ($V(J^1\pi)$ is the vector bundle associated to the affine bundle $\pi_{1,0} : J^1\pi \to P$).

$(J^1\pi)^+$ may be identified with the vector bundle (over P)

$$\mathcal{M}\pi = \Lambda^m_2 P = \{\gamma \in \Lambda^m T^*P : i_u i_v \gamma = 0, \, \forall u, v \in V\pi\},$$

where m is the dimension of M and $V\pi$ is the vertical bundle to π.

$(J^1\pi)^0$ may be identified with the quotient vector bundle (over P)

$$\mathcal{M}^0\pi = \frac{\Lambda^m_2 P}{\Lambda^m_1 P},$$

where $\Lambda^m_1 P$ is the vector bundle

$$\Lambda^m_1 P = \{\gamma \in \Lambda^m T^*P : i_u \gamma = 0, \, \forall u \in V\pi\}.$$

$\mathcal{M}\pi$ (respectively, $\mathcal{M}^0\pi$) is the extended multimomentum bundle (respectively, the restricted multimomentum bundle) associated with π. We will denote by $\nu : \mathcal{M}\pi \to P$ and $\nu^0 : \mathcal{M}^0\pi \to P$ the canonical projections.

$\mathcal{M}\pi$ is a principal $\mathbb{R}$-bundle over $\mathcal{M}^0\pi$ and the canonical projection $\mu : \mathcal{M}\pi \to \mathcal{M}^0\pi$ is just the principal bundle projection.

We take local coordinates (x^i) and (x^i, u^I) on M and P, respectively, which are adapted to the fibration π and such that

$$\Omega = d^m x = dx^1 \wedge \cdots \wedge dx^m.$$

Then, an element γ of $\mathcal{M}\pi$ is of the form

$$\gamma = p \, d^m x + p^i_I \, du^I \wedge d^{m-1}x_i,$$

where $d^{m-1}x_i = i(\frac{\partial}{\partial x^i}) d^m x$. Thus, (x^i, u^I, p, p^i_I) (respectively, (x^i, u^I, p^i_I)) are local coordinates on $\mathcal{M}\pi$ (respectively, on $\mathcal{M}^0\pi$) in such a way that

$$\mu(x^i, u^I, p, p^i_I) = (x^i, u^I, p^i_I).$$

$\mathcal{M}\pi$ admits a canonical multisymplectic structure of order $m+1$ which may be defined as follows.

Let $\lambda_{\mathcal{M}\pi}$ be the multimomentum Liouville m-form on $\mathcal{M}\pi$ given by

$$\lambda_{\mathcal{M}\pi}(\gamma)(X_1, \ldots, X_m) = \gamma((T_\gamma \nu)(X_1), \ldots, (T_\gamma \nu)(X_m)), \tag{1}$$

for $\gamma \in \mathcal{M}\pi$ and $X_1, \ldots, X_m \in T_\gamma \mathcal{M}\pi$.

Then, the multisymplectic structure $\omega_{\mathcal{M}\pi}$ on $\mathcal{M}\pi$ is defined by

$$\omega_{\mathcal{M}\pi} = -d\lambda_{\mathcal{M}\pi} \tag{2}$$

(for the definition and properties of multisymplectic structures see, for instance, [3]).

In local coordinates,

$$\lambda_{\mathcal{M}\pi} = p d^m x + p^i_I du^I \wedge d^{m-1}x_i, \quad \omega_{\mathcal{M}\pi} = -dp \wedge d^m x - dp^i_I \wedge du^I \wedge d^{m-1}x_i. \tag{3}$$

2.2. Prolongation of a vertical vector field to the extended multimomentum bundle. Let $\pi : P \to M$ be a fibration and $\mathcal{M}\pi$ be the extended multimomentum bundle associated with $\pi : P \to M$ as in the previous section.

Suppose that U is a vector field on P which is vertical with respect to the projection π. Then, we will define a vector field U^{1*} on $\mathcal{M}\pi$ which is ν-projectable on U.

For this purpose, we consider the $(m-1)$-form $\widetilde{U}$ on $\mathcal{M}\pi$ given by

$$\widetilde{U}(\gamma)(Y_1,\dots,Y_{m-1}) = \gamma(U(\nu(\gamma)),(T_\gamma\nu)(Y_1),\dots,(T_\gamma\nu)(Y_{m-1})) \tag{4}$$

for $\gamma\in\mathcal{M}\pi$ and $Y_1,\dots,Y_{m-1}\in T_\gamma\mathcal{M}\pi$.

It is clear that $\widetilde{U}$ is a section of the vector bundle over $\mathcal{M}\pi$

$$\textstyle\bigwedge_1^{m-1}(\mathcal{M}\pi) = \{\Theta\in\bigwedge^{m-1}(\mathcal{M}\pi) : i(X)\Theta = 0,\ \forall X\in V(\pi\circ\nu)\}.$$

Thus, $d\widetilde{U}$ is a section of the vector bundle $\bigwedge_2^m(\mathcal{M}\pi)\to\mathcal{M}\pi$ (as we know, $\bigwedge_2^m(\mathcal{M}\pi)$ is the affine dual bundle to the 1-jet bundle $(\pi\circ\nu)_{1,0} : J^1(\pi\circ\nu)\to\mathcal{M}\pi$).

If (x^i,u^I,p,p_I^i) are local coordinates on an open set $\mathcal{M}V$ of $\mathcal{M}\pi$ and $U = U^I\partial/\partial u^I$, then

$$\widetilde{U} = U^I p_I^i d^{m-1}x_i,\ d\widetilde{U} = \left(\frac{\partial U^I}{\partial x^i}p_I^i\right)d^mx + p_I^i\frac{\partial U^I}{\partial u^J}du^J\wedge d^{m-1}x_i + U^I dp_I^i\wedge d^{m-1}x_i. \tag{5}$$

Now, we may introduce the vector field U^{1*}.

Proposition 2.1. *Let U be a vector field on P which is vertical with respect to the projection $\pi : P\to M$. Then, there exists a unique vector field U^{1*} on $\mathcal{M}\pi$ which is characterized by the condition*

$$i_{U^{1*}}\omega_{\mathcal{M}\pi} = d\widetilde{U}. \tag{6}$$

U^{1*} is the prolongation to $\mathcal{M}\pi$ of the vector field U.

From (3), (5) and (6), it follows that

$$U^{1*} = U^I\frac{\partial}{\partial u^I} - \left(\frac{\partial U^I}{\partial x^i}p_I^i\frac{\partial}{\partial p} + p_I^i\frac{\partial U^I}{\partial u^J}\frac{\partial}{\partial p_J^i}\right). \tag{7}$$

2.3. Prolongation of an invariant vertical vector field to the reduced extended multimomentum bundle. Let $\pi : P\to M$ be a fibration over an orientable manifold M with volume form Ω and a free proper left action of a Lie group G on P such that

$$\pi(gp) = \pi(p),\qquad \text{for } g\in G \text{ and } p\in P. \tag{8}$$

This implies that there exists a fibration

$$\bar{\pi} : P/G\to M$$

from the space of orbits P/G on M such that $\pi = \bar{\pi}\circ\pi_P$, where $\pi_P : P\to P/G$ is the principal G-bundle projection.

The action of G on P may be lifted, in a natural way, to an action of G on the extended multimomentum bundle $\mathcal{M}\pi = \Lambda_2^m P$. In fact, if $\alpha\in\mathcal{M}\pi$ and $g\in G$ then

$$(g\alpha)(v_1,\dots,v_m) = \alpha(g^{-1}v_1,\dots,g^{-1}v_m) \tag{9}$$

for $v_1,\dots,v_m\in T_{gp}P$ and $\alpha\in\mathcal{M}_p\pi$, with $p\in P$. Then $\mathcal{M}\pi$ becomes a principal G-bundle over the space of orbits $\widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G$ with principal bundle projection $\pi_{\mathcal{M}\pi} : \mathcal{M}\pi\to\widehat{\mathcal{M}\pi}$. Moreover, the projection $\nu : \mathcal{M}\pi\to P$ is G-equivariant.

Note that

$$(g\Omega)(p) = \Omega(p),\qquad \text{for } p\in P,$$

and, thus, the previous action induces a free and proper action of G on $\mathcal{M}^0\pi$ such that the canonical projection $\mu : \mathcal{M}\pi\to\mathcal{M}^0\pi$ is G-equivariant.

Moreover, using (1) and the fact that ν is G-equivariant, we deduce that $\lambda_{\mathcal{M}\pi}$ is G-invariant. Therefore, we have the following result

Proposition 2.2. *The multimomentum Liouville m-form $\lambda_{\mathcal{M}\pi}$ is G-invariant. Thus, the canonical multisymplectic structure $\omega_{\mathcal{M}\pi}$ also is G-invariant.*

Using again the fact that ν is G-equivariant, (1), (4) and Propositions 2.1 and 2.2, we may prove that

Proposition 2.3. *If $\xi_P \in \mathfrak{X}(P)$ (respectively, $\xi_{\mathcal{M}\pi} \in \mathfrak{X}(\mathcal{M}\pi)$) is the infinitesimal generator associated with $\xi \in \mathfrak{g}$ and $\lambda_{\mathcal{M}\pi}$ is the Liouville multimomentum m-form then*

$$i_{\xi_{\mathcal{M}\pi}}\lambda_{\mathcal{M}\pi} = \widetilde{\xi_P} \quad \text{and} \quad \xi_{\mathcal{M}\pi} = (\xi_P)^{1*}.$$

The vertical bundle of π is invariant under the tangent action of G on TP and the space of orbits $\widehat{V\pi} = V\pi/G$ is a vector bundle over $\widehat{P} = P/G$. The sections of this vector bundle are the vector fields on P which are π-vertical and G-invariant.

One may prolong such vector fields to the reduced extended multimomentum bundle $\widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G$. In fact, using (4), (6), Proposition 2.3 and the fact that ν is G-equivariant, we deduce that

Theorem 2.4. *Let U be a π-vertical G-invariant vector field on P. Then*

(i) *The $(m-1)$-form $\widetilde{U}$ on $\mathcal{M}\pi$ is basic with respect to the projection $\pi_{\mathcal{M}\pi} : \mathcal{M}\pi \to \widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G$ and, thus, there exists a $(m-1)$-form $\widehat{\widetilde{U}}$ on $\widehat{\mathcal{M}\pi}$ such that $\pi^*_{\mathcal{M}\pi}(\widehat{\widetilde{U}}) = \widetilde{U}$.*

(ii) *The prolongation U^{1*} to $\mathcal{M}\pi$ of U is G-invariant and, therefore, it is $\pi_{\mathcal{M}\pi}$-projectable over a vector field $\widehat{U^{1*}}$ on $\widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G$.*

$\widehat{U^{1*}}$ is called the prolongation to $\widehat{\mathcal{M}\pi}$ of the section U of the vector bundle $\widehat{V\pi} = V\pi/G \to \widehat{P} = P/G$.

The vector field U is π_P-projectable over a vector field $\widehat{U}$ on $\widehat{P} = P/G$. It is clear that $\widehat{U}$ is vertical with respect to the projection $\overline{\pi} : P/G \to M$ and, thus, one may consider the prolongation $\widehat{U}^{1*}$ to the extended multimomentum bundle $\mathcal{M}\overline{\pi} = \bigwedge_2^m(P/G)$.

In fact, we have the following relation between the vector fields $\widehat{U^{1*}}$ and $\widehat{U}^{1*}$.

Theorem 2.5.

(i) *The multimomentum bundle $\mathcal{M}\overline{\pi} = \bigwedge_2^m(P/G)$ is a vector subbundle of the vector bundle $\nu/G : \widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G \to \widehat{P} = P/G$.*

(ii) *The restriction to $\mathcal{M}\overline{\pi}$ of the vector field $\widehat{U^{1*}}$ on $\widehat{\mathcal{M}\pi}$ is tangent to $\mathcal{M}\overline{\pi}$ and*

$$\widehat{U^{1*}}\big|_{\mathcal{M}\overline{\pi}} = \widehat{U}^{1*}.$$

2.4. More local expressions in the presence of the symmetry Lie group. As we will proceed locally, we will assume, without the loss of generality, that

$$P = P/G \times G, \quad P/G = M \times N$$

and, in addition:

1. The action of G on P is the left translation on the second factor.
2. The map $\overline{\pi}$ is the projection on the first factor.

Then, $\mathcal{M}\pi$ (respectively, $\mathcal{M}^0\pi$) may be identified with $\mathbb{R} \times TM \otimes (G \times \mathfrak{g}^* \times T^*N)$ (respectively, $TM \otimes (G \times \mathfrak{g}^* \times T^*N)$). In fact, the map $\Psi : \mathbb{R} \times TM \otimes (G \times \mathfrak{g}^* \times T^*N) \to \mathcal{M}\pi$ (respectively, $\Psi^0 : TM \otimes (G \times \mathfrak{g}^* \times T^*N) \to \mathcal{M}^0\pi$) given by

$$\Psi(p, u_x \otimes (g, \kappa, \alpha_n)) = p\Omega(x) + (\overleftarrow{\kappa}(g) + \alpha_n) \wedge i(u_x)\Omega(x)$$
$$(\text{respectively, } \Psi^0(u_x \otimes (g, \kappa, \alpha_n)) = [(\overleftarrow{\kappa}(g) + \alpha_n) \wedge i(u_x)\Omega(x)])$$

for $p \in \mathbb{R}$, $u_x \in T_xM$ and $(g, \kappa, \alpha_n) \in G \times \mathfrak{g}^* \times T^*_nN$, is an isomorphism. Here,

$$\overleftarrow{\kappa}(g) = (T^*_g l_{g^{-1}})(\kappa),$$

$l_{g^{-1}} : G \to G$ being the left translation by $g \in G$.

Under the previous identifications, the actions of G on $\mathcal{M}\pi$ and $\mathcal{M}^0\pi$ are the left translations on the G-factor.

Now, we take local coordinates (x^i) (respectively, (g^α) and (u^a)) on M (respectively, G and N) such that $\Omega = d^m x$ and we will choose a basis $\{\xi_\alpha\}$ of the Lie algebra $\mathfrak{g}$ of G.

Then, we can consider local coordinates

$$(x^i, g^\alpha, u^a, p, \overleftarrow{p}^i_\alpha, p^i_a)$$

on $\mathcal{M}\pi \simeq \mathbb{R} \times TM \otimes (G \times \mathfrak{g}^* \times T^*N)$, where

$$\overleftarrow{p}^i_\alpha(p, u_x \otimes (g, \kappa, \alpha_n)) = u_x(x^i)\kappa(\xi_\alpha).$$

In a similar way, we can consider local coordinates $(x^i, g^\alpha, u^a, \overleftarrow{p}^i_\alpha, p^i_a)$ on $\mathcal{M}^0\pi$.

From (3), we deduce that the local expression of the Liouville multimomentum m-form is

$$\lambda_{\mathcal{M}\pi} = p d^m x + (\overleftarrow{p}^i_\gamma \overleftarrow{\xi}^\gamma + p^i_a du^a) \wedge d^{m-1}x_i, \tag{10}$$

where $\{\xi^\alpha\}$ is the dual basis of $\{\xi_\alpha\}$ and $\overleftarrow{\xi}^\alpha$ is the left-invariant 1-form on G whose value at the identity of G is ξ^α.

Thus, if $c^\gamma_{\alpha\beta}$ are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_\alpha\}$, it follows that the local expression of the canonical multisymplectic structure on $\mathcal{M}\pi$ is

$$\begin{aligned}\omega_{\mathcal{M}\pi} = -dp \wedge d^m x - d\overleftarrow{p}^i_\gamma \wedge \overleftarrow{\xi}^\gamma \wedge d^{m-1}x_i \\ + \tfrac{1}{2}\overleftarrow{p}^i_\gamma c^\gamma_{\alpha\beta} \overleftarrow{\xi}^\alpha \wedge \overleftarrow{\xi}^\beta \wedge d^{m-1}x_i - dp^i_a \wedge du^a \wedge d^{m-1}x_i.\end{aligned} \tag{11}$$

Now, since the vector field U is π-vertical and G-invariant, we have

$$U = U^\alpha(x, u)\overleftarrow{\xi}_\alpha + U^a(x, u)\frac{\partial}{\partial u^a}$$

where $\overleftarrow{\xi}_\alpha$ is the left-invariant vector field on G induced by $\xi_\alpha \in \mathfrak{g}$.

This implies that (see (5))

$$\begin{aligned}\widehat{\widetilde{U}} &= (\overleftarrow{p}^i_\alpha U^\alpha + p^i_a U^a) d^{m-1}x_i, \\ d\widehat{\widetilde{U}} &= \left(\overleftarrow{p}^i_\alpha \frac{\partial U^\alpha}{\partial x^i} + p^i_a \frac{\partial U^a}{\partial x^i}\right) d^m x + \left(\overleftarrow{p}^i_\alpha \frac{\partial U^\alpha}{\partial u^b} + p^i_a \frac{\partial U^a}{\partial u^b}\right) du^b \wedge d^{m-1}x_i \\ &\quad + U^\alpha d\overleftarrow{p}^i_\alpha \wedge d^{m-1}x_i + U^a dp^i_a \wedge d^{m-1}x_i.\end{aligned} \tag{12}$$

Therefore, from (6) and (11), we deduce that

$$
\begin{aligned}
U^{1*} = U^{\alpha}\overleftarrow{\xi}_{\alpha} + U^{a}\frac{\partial}{\partial u^{a}} - \left(\overleftarrow{p}^{i}_{\alpha}\frac{\partial U^{\alpha}}{\partial x^{i}} + p^{i}_{a}\frac{\partial U^{a}}{\partial x^{i}}\right)\frac{\partial}{\partial p} \\
+ U^{\alpha}\overleftarrow{p}^{i}_{\gamma}c^{\gamma}_{\alpha\beta}\frac{\partial}{\partial \overleftarrow{p}^{i}_{\beta}} - \left(\overleftarrow{p}^{i}_{\alpha}\frac{\partial U^{\alpha}}{\partial u^{b}} + p^{i}_{a}\frac{\partial U^{a}}{\partial u^{b}}\right)\frac{\partial}{\partial p^{i}_{b}}
\end{aligned}
$$

where $c^{\gamma}_{\alpha\beta}$ are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\xi_{\alpha}\}$.

Consequently, we have

$$
\begin{aligned}
\widehat{U^{1*}} = U^{a}\frac{\partial}{\partial u^{a}} - \left(\overleftarrow{p}^{i}_{\alpha}\frac{\partial U^{\alpha}}{\partial x^{i}} + p^{i}_{a}\frac{\partial U^{a}}{\partial x^{i}}\right)\frac{\partial}{\partial p} \\
+ U^{\alpha}\overleftarrow{p}^{i}_{\gamma}c^{\gamma}_{\alpha\beta}\frac{\partial}{\partial \overleftarrow{p}^{i}_{\beta}} - \left(\overleftarrow{p}^{i}_{\alpha}\frac{\partial U^{\alpha}}{\partial u^{b}} + p^{i}_{a}\frac{\partial U^{a}}{\partial u^{b}}\right)\frac{\partial}{\partial p^{i}_{b}}.
\end{aligned} \tag{13}
$$

3. Hamilton–De Donder–Weyl equations in terms of prolongations of vertical vector fields. In this section, we will present a description of the Hamilton–De Donder–Weyl equations using the theory of prolongations of vertical vector fields to the extended multimomentum bundle.

First of all, we will review the expression of the Hamilton–De Donder–Weyl equations using the multisymplectic structure on the extended multimomentum bundle and the homogeneous extended Hamiltonian function associated with a Hamiltonian section (for more details, see [6]).

3.1. Hamilton–De Donder–Weyl equations. Let $\pi : P \to M$ be a surjective submersion over an orientable manifold with volume form Ω as in Section 2.

Moreover, suppose that $h : \mathcal{M}^0\pi \to \mathcal{M}\pi$ is a Hamiltonian section, that is, h is a section of the canonical projection $\mu : \mathcal{M}\pi \to \mathcal{M}^0\pi$. Then, since $\mu : \mathcal{M}\pi \to \mathcal{M}^0\pi$ defines a principal $\mathbb{R}$-bundle, we may define the extended homogeneous Hamiltonian function $F_h : \mathcal{M}\pi \to \mathbb{R}$ given by

$$F_h(\gamma)\Omega = h(\mu(\gamma)) - \gamma, \qquad \text{for } \gamma \in \mathcal{M}\pi. \tag{14}$$

If the local expression of h is

$$h(x^i, u^I, p^i_I) = (x^i, u^I, -H(x^i, u^I, p^i_I), p^i_I) \tag{15}$$

we have

$$F_h(x^i, u^I, p, p^i_I) = -p - H(x^i, u^I, p^i_I). \tag{16}$$

A (local) section s_0 of the canonical projection $\pi \circ \nu^0 : \mathcal{M}^0\pi \to M$ is a solution of the Hamilton–De Donder–Weyl equations for h if

$$(h \circ s_0)^*(i_X(\omega_{\mathcal{M}\pi} - d\mathcal{H})) = 0, \tag{17}$$

for any $(\pi\circ\nu)$-vertical vector field X on $\mathcal{M}\pi$, where $\mathcal{H} = F_h\Omega$ is the extended Hamiltonian density.

If

$$s_0(x^i) = (x^i, u^I(x), p_I^i(x))$$

then s_0 is a solution of the Hamilton–De Donder–Weyl equations if

$$\frac{\partial u^I}{\partial x^i} = \frac{\partial H}{\partial p_I^i}, \qquad \frac{\partial p_I^i}{\partial x^i} = -\frac{\partial H}{\partial u^I}. \tag{18}$$

3.2. The description in terms of the prolongation of vertical vector fields to the extended multimomentum bundle. From (5), (7), (15), (16) and (18), we deduce the following result

THEOREM 3.1. *A (local) section s_0 of the canonical projection $\pi \circ \nu^0 : \mathcal{M}^0\pi \to M$ is a solution of the Hamilton–De Donder–Weyl equations for h if and only if*

$$(h \circ s_0)^*(d\widetilde{U}) = (U^{1*}(F_h) \circ h \circ s_0)\Omega \tag{19}$$

for every π-vertical vector field U on P.

4. A derivation of the Hamilton–Poincaré field equations. In this section, we will present two derivations of the Hamilton–Poincaré field equations. The first one is a local derivation from the Hamilton–De Donder–Weyl equations for an equivariant Hamiltonian section. In the second one, we will obtain an intrinsic expression of the Hamilton–Poincaré field equations using the theory of prolongations of invariant vertical vector fields to the reduced extended multimomentum bundle (see Section 2.3).

4.1. A local derivation. We will use the same notation as in Section 2.4.

Let $h : \mathcal{M}^0\pi \to \mathcal{M}\pi$ be a Hamiltonian section. Denote by $F_h : \mathcal{M}\pi \to \mathbb{R}$ the extended homogeneous Hamiltonian function and by $\mathcal{H}$ the extended Hamiltonian density. Then,

$$\mathcal{H} = -(p + H(x^i, g^\alpha, u^a, \overleftarrow{p}_\alpha^i, p_a^i))d^m x \tag{20}$$

where $H : \mathcal{M}^0\pi \to \mathbb{R}$ is the local restricted Hamiltonian function.

Therefore, using (11), (17) and (20), we deduce that a (local) section s_0 of the canonical projection $\pi \circ \nu^0 : \mathcal{M}^0\pi \to M$

$$s_0(x^i) = (x^i, g^\alpha(x), u^a(x), \overleftarrow{p}_\alpha^i(x), p_a^i(x))$$

is a solution of the Hamilton–De Donder–Weyl equations for h if

$$\overleftarrow{\xi}^\alpha\left(\frac{\partial}{\partial g^\mu}\right)\frac{\partial g^\mu}{\partial x_i} = \frac{\partial H}{\partial \overleftarrow{p}_\alpha^i}, \qquad \frac{\partial \overleftarrow{p}_\alpha^i}{\partial x^i} = -\overleftarrow{p}_\gamma^i c_{\alpha\beta}^\gamma \overleftarrow{\xi}^\beta\left(\frac{\partial}{\partial g^\mu}\right)\frac{\partial g^\mu}{\partial x^i} - \overleftarrow{\xi}_\alpha(H) \tag{21}$$

and

$$\frac{\partial u^a}{\partial x^i} = \frac{\partial H}{\partial p_a^i}, \qquad \frac{\partial p_a^i}{\partial x^i} = -\frac{\partial H}{\partial u^a}. \tag{22}$$

Now, suppose that h is G-equivariant, Then, since the action of G on $\mathcal{M}^0\pi$ and $\mathcal{M}\pi$ is the left translation on the G-factor, we infer that H is G-invariant which implies that

$$\overleftarrow{\xi}_\alpha(H) = 0, \qquad \text{for all } \alpha. \tag{23}$$

On the other hand, the orbit spaces $\widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G$ and $\widehat{\mathcal{M}^0\pi} = \mathcal{M}^0\pi/G$ may be identified with

$$\mathbb{R} \times TM \otimes (\mathfrak{g}^* \times T^*N) \quad \text{and} \quad TM \otimes (\mathfrak{g}^* \times T^*N),$$

respectively (see Section 2.4). Thus, we can consider local coordinates

$$(x^i, u^a, p, \overleftarrow{p}^i_\alpha, p^i_a) \quad \text{and} \quad (x^i, u^a, \overleftarrow{p}^i_\alpha, p^i_a)$$

on $\widehat{\mathcal{M}\pi}$ and $\widehat{\mathcal{M}^0\pi}$, respectively.

Moreover, from (23), it follows that there exists a real function

$$\hat{H} : \widehat{\mathcal{M}^0\pi} \to \mathbb{R}$$

(the local reduced restricted Hamiltonian function), such that

$$H = \hat{H} \circ \pi_{\mathcal{M}^0\pi},$$

where $\pi_{\mathcal{M}^0\pi} : \mathcal{M}^0\pi \to \widehat{\mathcal{M}^0\pi}$ is the canonical projection.

In addition, if $s_0 : M \to \mathcal{M}^0\pi$ is a solution of the Hamilton–De Donder–Weyl equations and $\hat{s}_0 = \pi_{\mathcal{M}^0\pi} \circ s_0$ is the projection of s_0 on $\widehat{\mathcal{M}^0\pi}$,

$$\hat{s}_0(x^i) = (x^i, u^a(x), \overleftarrow{p}^i_\alpha(x), p^i_a(x)),$$

then, using (21), (22) and (23), we deduce that

$$\frac{\partial u^a}{\partial x^i} = \frac{\partial \hat{H}}{\partial p^i_a}, \qquad \frac{\partial \overleftarrow{p}^i_\alpha}{\partial x^i} = -c^\gamma_{\alpha\beta} \overleftarrow{p}^i_\gamma \frac{\partial \hat{H}}{\partial \overleftarrow{p}^i_\beta}, \qquad \frac{\partial p^i_a}{\partial x^i} = -\frac{\partial \hat{H}}{\partial u^a}. \tag{24}$$

(24) are the Hamilton–Poincaré field equations. Note that in the particular case when $P/G = M$ and $\bar{\pi} : P/G \to M$ is the identity map, we recover the Lie–Poisson field equations which were discussed in [5].

4.2. An intrinsic expression. Under the same hypotheses as in Section 2.3, we see that the free and proper action of G on $\mathcal{M}\pi$ is compatible with the principal action of $\mathbb{R}$ on $\mathcal{M}\pi$ in such a way that

$$g(\gamma + p) = g\gamma + p, \qquad \text{for } g \in G,\ \gamma \in \mathcal{M}\pi \text{ and } p \in \mathbb{R}. \tag{25}$$

So, a free and proper action of G on $\mathcal{M}^0\pi$ is induced such that the canonical projection $\mu : \mathcal{M}\pi \to \mathcal{M}^0\pi$ is G-equivariant.

Thus, μ induces a smooth map $\widehat{\mu} : \widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G \to \widehat{\mathcal{M}^0\pi} = \mathcal{M}^0\pi/G$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}\pi & \xrightarrow{\ \mu\ } & \mathcal{M}^0\pi \\ {\scriptstyle \pi_{\mathcal{M}\pi}} \downarrow & & \downarrow {\scriptstyle \pi_{\mathcal{M}^0\pi}} \\ \widehat{\mathcal{M}\pi} = \mathcal{M}\pi/G & \xrightarrow{\ \widehat{\mu}\ } & \widehat{\mathcal{M}^0\pi} = \mathcal{M}^0\pi/G \end{array}$$

is commutative. In fact, using (25), we deduce that the principal action of $\mathbb{R}$ on $\mathcal{M}\pi$ induces a principal action of $\mathbb{R}$ on $\widehat{\mathcal{M}\pi}$ such that the space of orbits $\widehat{\mathcal{M}\pi}/\mathbb{R}$ is just $\widehat{\mathcal{M}^0\pi} = \mathcal{M}^0\pi/G$. In other words, $\widehat{\mu}$ is a principal $\mathbb{R}$-bundle projection.

On the other hand, the canonical projection $\nu_0 : \mathcal{M}^0\pi \to P$ is G-equivariant and it induces a projection $\nu^0\big|G : \widehat{\mathcal{M}^0\pi} \to \widehat{P} = P/G$. In addition, $\widehat{\mathcal{M}^0\pi}$ is a vector bundle over $\widehat{P}$ with vector bundle projection $\nu^0\big|G : \widehat{\mathcal{M}^0\pi} \to \widehat{P}$.

$\widehat{h}$ is the reduced Hamiltonian section and the diagram

$$\begin{array}{ccc} \mathcal{M}^0\pi & \xrightarrow{\;h\;} & \mathcal{M}\pi \\ {\scriptstyle \pi_{\mathcal{M}^0\pi}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{M}\pi}} \\ \widehat{\mathcal{M}^0\pi} & \xrightarrow{\;\widehat{h}\;} & \widehat{\mathcal{M}\pi} \end{array}$$

is commutative.

Let $F_h : \mathcal{M}\pi \to \mathbb{R}$ be the extended Hamiltonian function. Then, from (14), (25) and since h and μ are G-equivariant, it follows that F_h is G-equivariant. Therefore, there exists a unique real smooth function $\widehat{F_h} : \widehat{\mathcal{M}\pi} \to \mathbb{R}$ satisfying

$$\widehat{F_h} \circ \pi_{\mathcal{M}\pi} = F_h.$$

$\widehat{F_h}$ is the reduced extended Hamiltonian function.

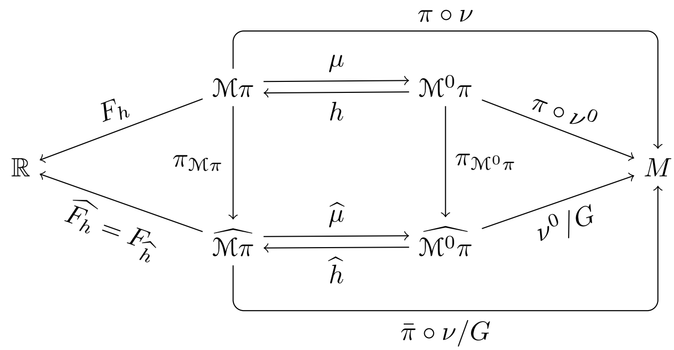
On the other hand, $\widehat{h}$ induces a real smooth function $F_{\widehat{h}} : \widehat{\mathcal{M}\pi} \to \mathbb{R}$ which is given by

$$F_{\widehat{h}}(\widehat{\alpha}) = \widehat{h}(\widehat{\mu}(\widehat{\alpha})) - \widehat{\alpha}$$

for $\widehat{\alpha} \in \widehat{\mathcal{M}\pi}$.

It is easy to see that $F_{\widehat{h}} \circ \pi_{\mathcal{M}\pi} = F_h$ and, therefore, $\widehat{F_h} = F_{\widehat{h}}$.

The following diagram illustrates the previous situation:



Using the same notation as in Section 4.1, we have

$$\widehat{h}(x^i, u^a, \overleftarrow{p}^i_\alpha, p^i_a) = (x^i, u^a, -\hat{H}(x^i, u^a, \overleftarrow{p}^i_\alpha, p^i_a), \overleftarrow{p}^i_\alpha, p^i_a) \tag{26}$$

and

$$F_{\widehat{h}}(x^i, u^a, p, \overleftarrow{p}^i_\alpha, p^i_a) = -p - \hat{H}(x^i, u^a, \overleftarrow{p}^i_\alpha, p^i_a). \tag{27}$$

Next, for a π-vertical G-invariant vector field U on P, we will denote by $\widehat{\widehat{U}}$ the corresponding $(m-1)$-form on $\widehat{\mathcal{M}\pi}$ and by $\widehat{U^{1*}}$ the prolongation to $\widehat{\mathcal{M}\pi}$ (see Section 2.3, Theorem 2.4).

Then, using (12), (13), (24), (26) and (27), we may prove the following result.

Theorem 4.1. *A section $\widehat{s_0}$ of the canonical projection $\pi \circ \nu^0 \big| G : \widehat{\mathcal{M}^0\pi} \to M$ is a solution of the Hamilton–Poincaré field equations for h if and only if*

$$(\widehat{h} \circ \widehat{s_0})^*(d\widehat{\widehat{U}}) = (\widehat{U^{1*}}(F_{\widehat{h}}) \circ \widehat{h} \circ \widehat{s_0})\Omega \tag{28}$$

for every G-invariant π-vertical vector field U on P.

5. Example: Minimal immersions. We will try to apply the previous setting to an example discussed in [10], dealing with minimal immersions from a manifold X into a Riemannian manifold Q with metric g. The Lagrangian of these systems is very simple: For $\eta : X \to Q$ an immersion, the underlying variational problem consists into minimize its area, which is nothing but the integral extended to $\operatorname{Im}\eta$ of the canonical volume form $dV_{\tilde{g}}$, where $\tilde{g}$ is the metric induced on X by the immersion

$$\tilde{g} := \eta^* g.$$

Thus the bundle underlying this problem as field theory is the trivial bundle $\pi : P = X \times Q \to X$. Introducing local coordinates x^i for X and u^I on Q such that $\tilde{g} = \tilde{g}_{ij} dx^i \otimes dx^j$ denotes the metric on X induced by the immersion, we have

$$\tilde{g}_{ij} = g_{IJ} \frac{\partial \eta^I}{\partial x^i} \frac{\partial \eta^J}{\partial x^j}.$$

Then, the volume form reads

$$dV_{\tilde{g}} = \sqrt{\det \tilde{g}}\, dx^1 \wedge \cdots \wedge dx^m,$$

so an expression for the Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m X$ would be

$$\mathcal{L}(x^i, u^I, u^J_j) := \sqrt{\det(g_{IJ}(u) u^I_i u^J_j)} dx^1 \wedge \cdots \wedge dx^m,$$

and the (local) Lagrangian function determined by this data is

$$L(j^1\sigma) := (\det(g_{IJ}(u) u^I_i u^J_j))^{1/2}.$$

5.1. Multimomentum formalism. The extended multimomentum space $\mathcal{M}\pi := \Lambda^m_2 P$ is locally given by

$$\mathcal{M}\pi|_{(x^i, u^I)} = \{p d^m x + p^i_I du^I \wedge d^{m-1} x_i\}$$

where, as usual, $d^m x := dx^1 \wedge \cdots \wedge dx^m$ and $d^{m-1} x_i := i(\partial/\partial x^i) d^m x$; in these coordinates the restricted multimomentum space $\mathcal{M}\pi^0$ admits the description

$$\mathcal{M}\pi^0|_{(x^i, u^I} = \{p^i_I du^I \wedge d^{m-1} x_i\},$$

and the Hamiltonian section $h : \mathcal{M}^0\pi \to \mathcal{M}\pi$ associated to $\mathcal{L}$ becomes

$$h(x^i, u^I, p^j_J) = (x^i, u^I, L(x^i, u^I, u^J_i) - p^i_K u^K_i, p^j_J),$$

with u^J_i being determined through the equation

$$p^j_I = \frac{\partial L}{\partial u^I_j}.$$

Lemma 5.1. *The conjugate momentum variables are related to velocities via*

$$p^j_I = M^{ij} g_{IJ} u^J_i L(x^i, u^I, u^J_j),$$

where M^{ij} are the components of the inverse of the matrix M with components given by

$$M_{ij} := g_{IJ} u^I_i u^J_j.$$

Then

$$p^j_K u^K_j = M^{ij} g_{JI} u^I_i u^J_j L(x^i, u^I, u^J_j) = M^{ij} M_{ji} L(x^i, u^I, u^J_j) = m L(x^i, u^I, u^J_j),$$

so the Hamiltonian section $h : \mathcal{M}^0\pi \to \mathcal{M}\pi$ is

$$h(x^i, u^I, p^j_J) = (x^i, u^I, (1-m) L(x^i, u^I, u^J_i), p^j_J)$$

with u^J_j such that

$$p^j_I = M^{ij} g_{IJ} u^J_i L(x^i, u^I, u^J_j). \tag{29}$$

Then the Hamiltonian density $\mathcal{H} : \mathcal{M}\pi \to \Lambda^m X$ is given by

$$\mathcal{H}(x^i, u^I, p, p^i_I) = (-p - (m-1) L(x^i, u^I, u^I_i))\, d^m x \tag{30}$$

with u^J_j determined by (29).

Now, by using the projection $\pi \circ \nu : \mathcal{M}\pi \to X$ (where $\nu : \mathcal{M}\pi \to P$ is the canonical projection restricted to the extended multimomentum space) we can consider the Hamiltonian density as an m-form $(\pi \circ \nu)^* \mathcal{H}$ on $\mathcal{M}\pi$.

Let us define the map

$$N : (x^i, u^I, p^j_J) \mapsto (g^{IJ} p^i_I p^j_J) \in M_m(\mathbb{R});$$

therefore by assuming p- and u-variables related through (29), it results that

$$N^{ij} = L^2 M^{ij},$$

and so

$$\det N = L^{2(m-1)}.$$

It means that we can express $\mathcal{H}$, defined in (30), in terms of the (x^i, u^I, p, p^i_J)-variables, namely

$$\mathcal{H}(x^i, u^I, p, p^j_J) = [-p - (m-1)(\det N)^{1/(2(m-1))}] d^m x.$$

Therefore the local expression for the Hamiltonian reads

$$H(x^i, u^I, p^j_J) = (m-1)(\det N)^{1/(2(m-1))}. \tag{31}$$

In particular

$$\frac{\partial H}{\partial u^I} = \frac{H}{2(m-1)}\, N_{jl} p^j_J p^l_K \frac{\partial g^{JK}}{\partial u^I}, \tag{32}$$

$$\frac{\partial H}{\partial p^k_I} = \frac{H}{(m-1)}\, g^{IJ} N_{kl} p^l_J. \tag{33}$$

5.2. Hamilton–De Donder–Weyl equations for minimal immersions. We will find the Hamilton–De Donder–Weyl equations of motion for mininal immersions. Recall from Section 3.1 that a section $s_0 : X \to \mathcal{M}^0\pi$ is a solution of the Hamilton equations of motion corresponding to the Hamiltonian density $\mathcal{H}$ if and only if equation (19) holds.

LEMMA 5.2. *If a section $s_0 : X \to \mathcal{M}^0\pi$ locally given by*

$$s_0(x^i) = (x^i, u^I(x), p^k_J(x))$$

is a solution of equation (19), *then*

$$\frac{\partial u^I}{\partial x^i} = \frac{H}{m-1}\, g^{IJ} N_{ik} p^k_J.$$

Proof. It follows by using (18) and (33). ∎

Let us introduce a coordinate chart (u^i, u^A) on $U \subset Q$ such that

$$i_U(X) := U \cap (pr_2 \circ \nu^0 \circ s_0)(X) = \{(u^i, u^A) : u^A = 0\}.$$

In particular, it means that

$$\frac{\partial u^I}{\partial x^i} = \begin{cases} \delta_i^j & \text{for } I = j, \\ 0 & \text{if } I = A. \end{cases}$$

Therefore

$$p_I^i = L M^{ji} g_{Ij};$$

in particular

$$p_j^i = L M^{ki} g_{kj}.$$

Because in the chosen coordinates $M_{ij} = g_{ij}$, it implies that

$$p_A^i = L g^{ij} g_{Aj}, \quad p_j^i = L \delta_j^i. \tag{34}$$

Now let us define the basis

$$\left\{ X_i := \frac{\partial}{\partial u^i},\, Z_A := \frac{\partial}{\partial u^A} - g_{Aj} g^{jk} \frac{\partial}{\partial u^k} \right\}$$

adapted to the decomposition $TU = T(i_U(X)) \oplus [T(i_U(X))]^{\perp}$.

Additionally, we will suppose that $s_0(X)$ is in the open set defined by the inequality

$$H \neq 0.$$

Then, using (32), (33), (34) and the fact that

$$L = \frac{H}{m-1},$$

we deduce the following result.

Lemma 5.3. *If a section* $s_0 : X \to \mathcal{M}^0\pi$ *locally given by*

$$s_0(x^i) = (x^i, u^I(x), p_J^k(x))$$

is a solution of equation (19), *then*

$$(h \circ s_0)^*(d\widetilde{X}_i) = -\frac{1}{2} \frac{H}{(m-1)} g_{jk} \frac{\partial M^{jk}}{\partial u^i} d^m x.$$

Now, we may obtain the following characterization of the Hamilton equations.

Theorem 5.4. *The equations* (19) *for the Hamiltonian* (31) *are equivalent to the fact that the immersion* $pr_2 \circ \nu^0 \circ s_0 : X \to Q$ *is minimal.*

Proof. Using (5), (7), (32), (33) and (34), we deduce that equation (19), with $U = Z_I$, implies

$$g(Z_I, M^{kj} \nabla_{X_k} X_j) = 0,$$

that is, the immersion $pr_2 \circ \nu^0 \circ s_0 : X \to Q$ is minimal.

On the other hand, when $U = X_i$, using (7), (32), (34) and Lemma 5.3, it follows that this equation does not give rise to new relations. ∎

5.3. Hamilton–Poincaré field equations for minimal immersions. Now, suppose that G is a Lie subgroup of the isometry Lie group of the Riemannian manifold (Q, g) which acts freely and properly on Q. Then, it is clear that the Hamiltonian section is G-equivariant and we can apply the results in Section 4. In particular, we will use Theorem 4.1 in order to find the Hamilton–Poincaré field equations for minimal immersions. As before, we need a result analogous to Lemma 5.2. We will use the same notation as in Section 4.

LEMMA 5.5. *If a section* $\widehat{s_0} : X \to \widehat{\mathcal{M}^0\pi}$ *locally given by*

$$\widehat{s_0}(x^i) = (x^i, u^a(x), p_b^k(x), \overleftarrow{p}_\alpha^l(x))$$

is a solution of the equations of motion given by Theorem 4.1, *then*

$$\begin{aligned} \frac{\partial u^a}{\partial x^i} &= \frac{\widehat{H}}{m-1}\, g^{ab} N_{ik} p_b^k \\ \frac{\partial \overleftarrow{p}_\alpha^i}{\partial x^i} &= -\frac{\widehat{H}}{m-1}\, c_{\alpha\beta}^\gamma N_{ki} \overleftarrow{p}_\gamma^i (g^{a\beta} p_a^k + g^{\sigma\beta} \overleftarrow{p}_\sigma^k). \end{aligned} \tag{35}$$

Proof. The π-vertical (local) vector fields

$$U_\alpha := \overleftarrow{\xi}_\alpha, \qquad U_c^i := x^i \frac{\partial}{\partial u^c}$$

are G-invariant. Then, the result follows if we apply Theorem 4.1 to U_α and U_c^i. ■

We can choose coordinates (u^i, u^p, g^α) on $U \subset Q$ such that:

- the restriction to U of the canonical projection $\pi_Q : Q \to Q/G$ is given by $\pi_Q(u^i, u^p, g^\alpha) = (u^i, u^p)$, and
- the image set of the sections $pr_2 \circ \nu^0 \circ s_0 : X \to Q$ and $pr_2 \circ \nu^0/G \circ \widehat{s_0} : X \to Q/G$ restricted to U become

$$\begin{aligned} i_U(X) &= U \cap (pr_2 \circ \nu^0 \circ s_0)(X) = \{(u^i, u^p, g^\alpha) : u^p = 0,\ (g^\alpha) = e\} \\ \widehat{i_U(X)} &:= U \cap (pr_2 \circ \nu^0/G \circ \widehat{s_0})(X) = \{(u^i, u^p) : u^p = 0\}. \end{aligned}$$

For every $p \in i_U(X)$ we can define the basis adapted to the immersion

$$\left\{X_i := \frac{\partial}{\partial u^i},\ Z_p := \frac{\partial}{\partial u^p} - g_{pi} M^{ki} \frac{\partial}{\partial u^k},\ Z_\alpha := \overleftarrow{\xi}_\alpha - g_{\alpha i} M^{ik} \frac{\partial}{\partial u^k}\right\}.$$

On $i_U(X)$, the remaining equations of motion become

$$\frac{\partial p_q^i}{\partial u^i} = \frac{\widehat{H}}{2(m-1)}\, g_{ik} \frac{\partial M^{ik}}{\partial u^q}. \tag{36}$$

Since g is G-invariant, we deduce that g induces a Riemannian metric $\hat{g}$ on the reduced space Q/G in such a way that the canonical projection $\pi_Q : Q \to Q/G$ is a Riemannian submersion (see [1, 29]).

The geometrical meaning of Hamilton–Poincaré field equations (35), (36) and the relation between the solutions of these equations and the minimal immersions $i : X \to (Q/G, \hat{g})$ from X on the Riemannian manifold $(Q/G, \hat{g})$ will be discussed in a forthcoming paper.

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