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NEW DEVELOPMENTS IN GEOMETRIC MECHANICS

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Abstract. We review the concept of a graded bundle, which is a generalisation of a vector bundle, its linearisation, and a double structure of this kind. We then present applications of these structures in geometric mechanics including systems with higher order Lagrangian and the Plateau problem.

1. Introduction. In this paper we report on the key results of our collective works [2, 3, 8]. The broad idea of these works can be stated as *applying the notion of graded bundles to the setting of geometric mechanics in the spirit of W. M. Tulczyjew.*

Recall that Grabowski and Rotkiewicz [12] established the one-to-one correspondence between manifolds that admit non-negatively graded local coordinates and manifolds equipped with an action of the monoid of multiplicative reals, or *homogeneity structures* in the language developed in that paper. Such manifolds are referred to as *graded bundles* for reasons that we will shortly explain. The cardinal examples of graded bundles are the higher order tangent bundles, which of course play a central rôle in (standard) higher order mechanics.

We describe a quite general geometric set-up of higher order mechanics for which the (higher order) velocities get replaced with elements of a graded bundle. To realise this we employ the notion of a *weighted Lie algebroid* [3]. Such manifolds have simultaneously the

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structure of a graded bundle and a Lie algebroid that are compatible in a precise sense. The approach we develop makes use of first order mechanics on Lie algebroids subject to affine vakonomic constraints and, as a result, we are lead to consider relations as opposed to genuine maps. The higher order flavour is due to the fact that underlying a weighted Lie algebroid is a graded bundle: there is an associated series of affine fibrations that mimic higher order tangent bundles. The standard description of higher order Lagrangian mechanics can naturally be accommodated within this framework, as can higher order mechanics on a Lie algebroid. For the case of higher order mechanics on a Lie algebroid, the Euler–Lagrange equations we obtain agree with the ones obtained by Colombo & Martín de Diego [5], Jóźwikowski & Rotkiewicz [15, 16] and Martínez [21]. One should note that our approach is completely geometric and we do not employ any standard tools from the calculus of variations (like variations etc.). The Hamiltonian formalism can also be described within this framework, however we will not discuss this in this paper and refer the reader to [2].

The challenge of describing mechanical systems configured on Lie groupoids and their reduction to Lie algebroids was first posted by Weinstein [32]. This challenge was taken up by many authors and various approaches developed, a rather incomplete list is given in [2]. The notion of the *Tulczyjew triple* for a Lie algebroid, as we shall understand it, was first given in [9]. It was based on a framework for Lagrangian and Hamiltonian formalisms developed by Tulczyjew [27, 28, 29] and a corresponding description of Lie algebroids [13, 14]. The motivation for extending the geometric tools of the Lagrangian formalism on tangent bundles to Lie algebroids comes from the fact that reductions usually push one out of the environment of tangent bundles and into the world of Lie algebroids. In a similar way, reductions of higher order tangent bundles, which is where higher order mechanical Lagrangians "live", will push one into the environment of "higher Lie algebroids". Weighted Lie algebroids turn out to give a clear geometric framework for reductions of higher order tangent bundles.

Graded bundles play also an important rôle in our approach to the geometric mechanics of (classical bosonic) *strings* [8]. The constructions cover the case of higher dimensional *branes*, though for simplicity we only discuss strings in any detail. The basic tools for point-particles are not adequate for the description of strings in geometric mechanics. The geometric approach to classical strings as presented here is based on morphisms of double graded bundles and the (canonical) multisymplectic structure on $\wedge^2\mathsf{T}^*M$. Again, we have a generalisation of the Tulczyjew triple, now adapted for the setting of classical strings. The phase space, the phase equations and Legendre transformation relating the Lagrangian and Hamiltonian pictures are obtained in a purely geometric way. The Euler–Lagrange equations for strings in this geometric set-up are of course the equations of motion for the Nambu–Goto string.

2. Graded and double graded bundles. A *vector bundle* is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibres

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n,$$

$A(x) \in \mathrm{GL}(n, \mathbb{R})$.

The latter property can also be expressed in the terms of the gradation in which base coordinates x have degree 0 and 'linear coordinates' y have degree 1. Linearity in y's is now equivalent to the fact that changes of coordinates respect the degrees. Morphisms in the category of vector bundles are represented by commutative diagrams of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

being linear in fibres. As canonical examples and constructions one can take e.g. $\mathsf{T}M$, T^*M, $E \otimes_M F$, $\wedge^k E$, etc.

A natural generalisation of the vector bundle is the concept of a *graded bundle* $\tau : F \to M$ (cf. [12]). We have a local trivialisation by $U \times \mathbb{R}^n$ as before, but with the difference that the local coordinates $(y^1, \dots, y^n)$ in the fibres have now associated positive integer *weights* (or *degrees*) $w_1, \dots, w_n$, that are preserved by changes of local trivialisations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n\,.$$

One can show that in this case $A(x, y)$ must be polynomial in fibre coordinates, i.e. any graded bundle is a *polynomial bundle* [12]. As these polynomials need not be linear, graded bundles do not have, in general, vector space structure in fibres. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.

If all $w_i \leq r$, we say that the graded bundle is *of degree r*. In the above terminology, vector bundles are just graded bundles of degree 1. Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by the bundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M.$$

Example 2.1. *Higher tangent bundles* $\mathsf{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3$, etc. are canonical examples of graded bundles.

Example 2.2. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathsf{T}E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathsf{T}\tau : \wedge^r \mathsf{T}E \to \wedge^r \mathsf{T}M.$$

Remark 2.3. Note that similar objects has been used in supergeometry by Voronov [31] under the name of *non-negatively graded (super)manifolds*. If the Grassmann parity of the coordinates is given by the weight of the coordinates (mod 2) then the resulting supermanifolds are known as *N-manifolds* following Ševera [26] and Roytenberg [24]. However, we will work with classical purely even manifolds only.

With the use of coordinates (x^α, y^a) with degree 0 for basic coordinates x^α, and degree $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined *weight vector field* (*Euler vector field*)

$$\nabla_F = \sum_a w_a y^a \partial_{y^a}\,.$$

The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, will be called a *homogeneity structure* [12].

Definition 2.4. A function $f : F \to \mathbb{R}$ is called *homogeneous of degree (weight) k* if $(h_t^* f)(x) := f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields. *Morphisms* of two homogeneity structures (F_i, h^i), $i = 1, 2$, are smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h_t^1 = h_t^2 \circ \Phi$. Consequently, a *homogeneity substructure* is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

2.1. Double graded bundles. The fundamental fact (cf. [12]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem 2.5. *For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathsf{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathsf{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.*

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following [12].

Definition 2.6. A *double graded bundle* is a manifold equipped with two homogeneity structures h^1, h^2 which are *compatible* in the sense that

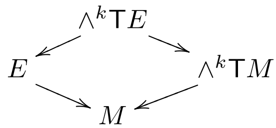
$$h_t^1 \circ h_s^2 = h_s^2 \circ h_t^1 \quad \text{for all } s, t \in \mathbb{R}\,.$$

This covers the concept of a *double vector bundle* of Pradines [23] and extends to *n-tuple* graded bundles in the obvious way.

2.2. Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear will be called a *$\mathcal{GL}$-bundle*. There are also lifts of graded structures on F to $\mathsf{T}^r F$.

In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*. As a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$ (see [13, 14]).

Example 2.7. If $\tau : E \to M$ is a vector bundle, then $\wedge^k \mathsf{T}E$ is canonically a $\mathcal{GL}$-bundle:



3. Tulczyjew triples. The canonical symplectic form ω_M on T^*M induces an isomorphism

$$\beta_M : \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}^*M.$$

Composing it with $\mathcal{R}_{\mathsf{T}M}$, where

$$\mathcal{R}_E : \mathsf{T}^*E^* \to \mathsf{T}^*E$$

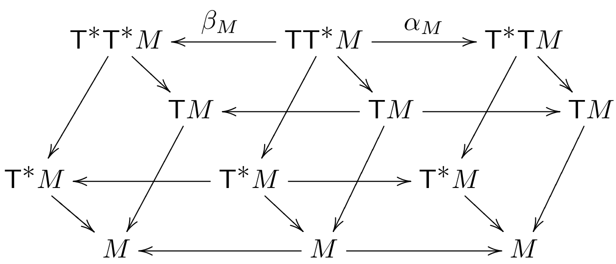
is the well-known canonical isomorphism (see e.g. [17, 18, 30]), we get the map

$$\alpha_M : \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M\,.$$

Using the standard coordinates $(x^\mu, \dot{x}^\nu)$ and (x^μ, p_ν) on $\mathsf{T}M$ and T^*M, respectively, and the adapted coordinates on $\mathsf{T}^*\mathsf{T}M$ and $\mathsf{T}\mathsf{T}^*M$, we can write

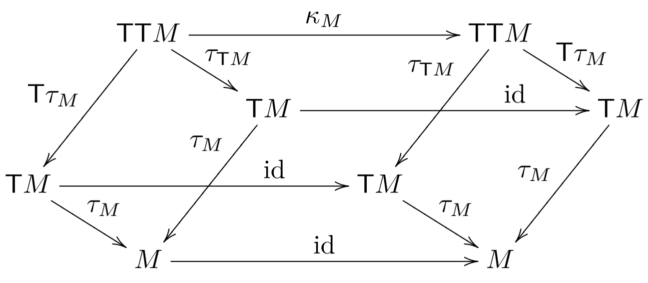
$$\alpha(x, p, \dot{x}, \dot{p}) = (x, \dot{x}, \dot{p}, p). \tag{1}$$

This gives rise to the commutative diagram of *double vector bundle (iso)morphisms* (Tulczyjew triple)



(2)

Note that the mapping α_M can be obtained directly as the dual to the 'canonical flip' $\kappa_M : \mathsf{T}\mathsf{T}M \to \mathsf{T}\mathsf{T}M$ which is an isomorphism of two vector bundle structures on $\mathsf{T}\mathsf{T}M$:



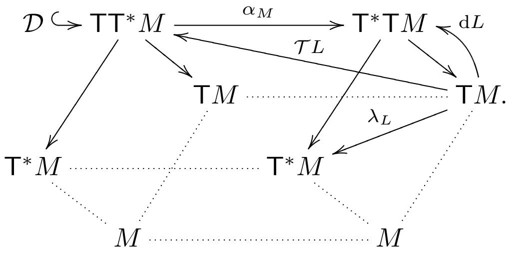
(3)

Indeed, the duals of these two vector bundle structures on $\mathsf{T}\mathsf{T}M$ are $\mathsf{T}^*\mathsf{T}M$ and $\mathsf{T}\mathsf{T}^*M$, and α_M can be understood as the dual map of κ_M.

The map κ_M, together with α_M and β_M, encodes the Lie algebroid structure of $\mathsf{T}M$ and note that no brackets are needed (cf. [13, 14]).

The Lagrangian and Hamiltonian formalisms have a simple description in terms of the Tulczyjew triple. The true physical dynamics, the *phase dynamics*, will be described as an implicit first order differential equation on the *phase space* T^*M, given by a submanifold $\mathcal{D} \subset \mathsf{T}\mathsf{T}^*M$. Note that a solution of an implicit differential equation $\mathcal{D} \subset \mathsf{T}N$ is a curve $\gamma : \mathbb{R} \to N$ such that its *tangent prolongation* $\mathsf{t}\gamma : \mathbb{R} \to \mathsf{T}N$ takes values in $\mathcal{D}$.

3.1. The Tulczyjew triple — Lagrangian side. Denote with M positions of our system, with $\mathsf{T}M$ (kinematic) configurations, and let $L : \mathsf{T}M \to \mathbb{R}$ be a Lagrangian function. We have the diagram



The dynamics

$$\mathcal{D} = \alpha_M^{-1}(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M),$$

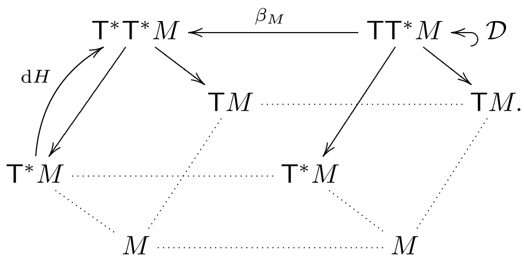
is given as the range of the *Tulczyjew differential* $\mathcal{T}L = \alpha_M^{-1} \circ \mathrm{d}L$. In local coordinates,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \ p = \frac{\partial L}{\partial \dot{x}} \ , \ \dot{p} = \frac{\partial L}{\partial x}\right\}.$$

Notice in the diagram the *Legendre map*,

$$\lambda_L : \mathsf{T}M \to \mathsf{T}^*M, \quad \lambda_L(x, \dot{x}) = \left(x, \frac{\partial L}{\partial \dot{x}}\right).$$

3.2. The Tulczyjew triple — Hamiltonian side. The Hamiltonian formalism looks analogously. If $H : \mathsf{T}^*M \to \mathbb{R}$ is a Hamiltonian function, from the Hamiltonian side of the triple



we derive the phase dynamics in the form

$$\mathcal{D} = \beta_M^{-1}(\mathrm{d}H(\mathsf{T}^*M)).$$

It is automatically explicit, i.e. generated by the corresponding Hamiltonian vector field, so it corresponds to a phase dynamics induced by a Lagrangian function only in regular cases. In local coordinates,

$$\mathcal{D} = \left\{(x, p, \dot{x}, \dot{p}) : \dot{p} = -\frac{\partial H}{\partial x} \ , \ \dot{x} = \frac{\partial H}{\partial p}\right\},$$

so we obtain the standard Hamilton equations.

3.3. Euler–Lagrange equations. Let now $\gamma : \mathbb{R} \to M$ be a curve in M (of course, $\mathbb{R}$ can be replaced by an open interval), and $\mathsf{t}\gamma : \mathbb{R} \to \mathsf{T}M$ be its tangent prolongation. It is easy to see that both curves, $\mathrm{d}L \circ \mathsf{t}\gamma$ and $\alpha_M \circ \mathsf{t}(\lambda_L \circ \mathsf{t}\gamma)$ are curves in $\mathsf{T}^*\mathsf{T}M$ covering $\mathsf{t}\gamma$. Therefore, their difference makes sense and, as easily seen, takes values in the annihilator

$V^0\mathsf{T}M$ of the vertical subbundle $V\mathsf{T}M \subset \mathsf{T}\mathsf{T}M$. Since $V^0\mathsf{T}M \simeq \mathsf{T}M \times_M \mathsf{T}^*M$, we obtain a map $\delta L_\gamma : \mathbb{R} \to \mathsf{T}^*M$. The above map is interpreted as the external force along the trajectory. Its value at $t \in \mathbb{R}$ depends on the second jet $\mathsf{t}^2\gamma(t)$ of γ only, so defines the variation of the Lagrangian, understood as a map

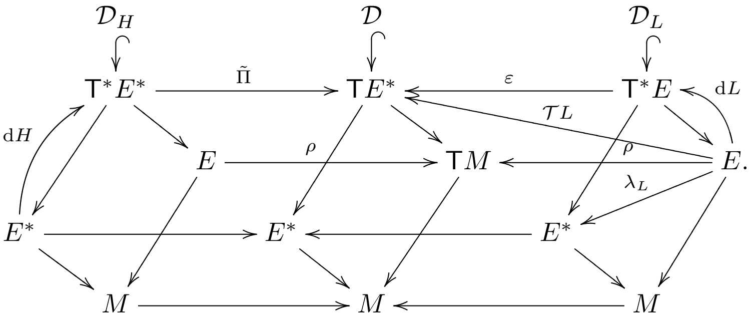
$$\delta L : \mathsf{T}^2M \to \mathsf{T}^*M, \tag{4}$$

where T^2M, the second tangent bundle, is the bundle of all second jets of curves $\mathbb{R} \to M$ at $0 \in \mathbb{R}$. The equation

$$\delta L_\gamma = \delta L \circ \mathsf{t}^2\gamma = 0 \tag{5}$$

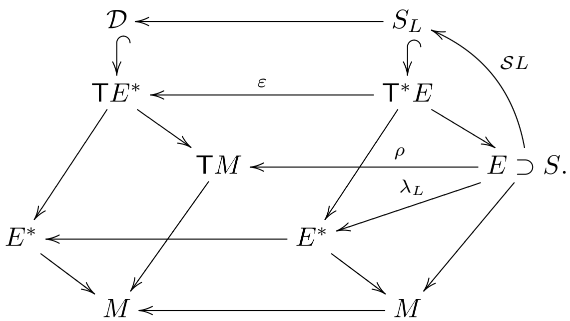
is known as the *Euler–Lagrange equation* and tells us that the curve $\mathrm{d}L \circ \mathsf{t}\gamma$ corresponds *via* α_M to an *admissible curve* in $\mathsf{T}\mathsf{T}^*M$, i.e. the tangent prolongation of a curve in T^*M. Here, of course, $\mathsf{t}^2\gamma$ is the second tangent prolongation of γ to T^2M.

4. Mechanics on algebroids with vakonomic constraints. The whole model of inducing dynamics out of a Lagrangian or a Hamiltonian can be repeated practically without changes when we replace $\mathsf{T}M$ with a vector bundle $\tau : E \to M$ and β_M with a map $\tilde{\Pi} : \mathsf{T}^*E^* \to \mathsf{T}E^*$, associated with a linear bivector field Π on E^*. The fact that it reverses the direction is not really changing the picture, as we used α_M^{-1} and β_M^{-1} to obtain the dynamics. Note that $\varepsilon = \tilde{\Pi} \circ \mathcal{R}_E^{-1}$. Thus, we get the diagram



Here, $\mathcal{D}_H = \mathrm{d}H(E^*) \subset \mathsf{T}^*E^*$ is the Lagrangian submanifold associated with a Hamiltonian $H : E^* \longrightarrow \mathbb{R}$, the Lagrangian submanifold $\mathcal{D}_L \subset \mathsf{T}^*E$ is associated with a Lagrangian, and $\mathcal{D} = \varepsilon(\mathcal{D}_\mathcal{L}) = \mathcal{TL}(\mathcal{E})$ or $\mathcal{D} = \tilde{\Pi}(\mathcal{D}_\mathcal{H})$, depending on the formalism used.

Starting with a Lagrangian defined on a constraint manifold $S \subset E$, we can slightly modify the above picture and get the diagram

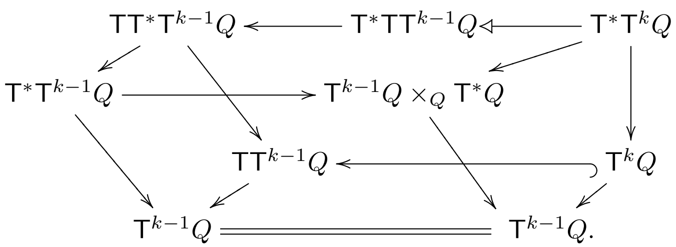


Here, S_L is the Lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\},$$

and $\mathcal{S}L : S \to \mathsf{T}^*E$ is the corresponding relation. The *vakonomically constrained phase dynamics* is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$. We stress that, due to the fact that we are dealing with a vakonomically constrained system, relations and not just genuine smooth maps naturally appear in the formalism.

5. Higher order Lagrangians. The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1}Q$ as an affine subbundle of *holonomic vectors*. Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1}Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1}Q$:



The Lagrangian function $L = L(q, \dot{q}, \dots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \dot{v}_{i-1} = v_i,\ \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}\,,\ \dot{p}_0 = \frac{\partial L}{\partial q}\,,\ p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the *higher Euler–Lagrange equations* in the traditional form:

$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}\,, \quad i = 1, \dots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}}\Big) + \dots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as ordinary differential equation of order $2k$ on Q.

6. Linearisation of graded bundles. The possibility of constructing mechanics on graded bundles is based on the following generalisation of the embedding $\mathsf{T}^k Q \hookrightarrow \mathsf{T}\mathsf{T}^{k-1}Q$ (see [3]).

THEOREM 6.1 (Bruce–Grabowska–Grabowski). *There is a canonical functor from the category of graded bundles into the category of $\mathcal{GL}$-bundles which canonically assigns, for an arbitrary graded bundle F_k of degree k, a $\mathcal{GL}$-bundle $D(F_k)$ which is linear over F_{k-1}, called the* linearisation of F_k, *together with a* graded *embedding $\iota : F_k \hookrightarrow D(F_k)$ of F_k as an affine subbundle of the vector bundle $D(F_k)$.*

Elements of $F_k \subset D(F_k)$ may be viewed as *holonomic vectors* in the linear-graded bundle $D(F_k)$. Another geometric part we need is a (Lie) algebroid structure on the vector bundle $D(F_k) \to F_{k-1}$, compatible with the second graded structure (homogeneity). We will call such $\mathcal{GL}$-bundles D *weighted (Lie) algebroids* and view them as abstract generalisations of the Lie algebroid $\mathsf{T}\mathsf{T}^{k-1}M$. Such D is called a *$\mathcal{VB}$-algebroid* if it is a double vector bundle.

Remark 6.2. In [3] a slightly more general notion of a weighted (Lie) algebroid is given for which the underlying $\mathcal{GL}$-bundle is not necessarily associated with the linearisation of a graded bundle. Furthermore, in [1] we discuss the corresponding groupoid objects which are named *weighted Lie groupoids*. Without details, a weighted Lie groupoid is both a graded bundle and a Lie groupoid at the same time. In a more categorical language, we have a Lie groupoid in the category of graded bundles or indeed vice versa. The basic Lie theory relating weighted Lie groupoids and weighted Lie algebroids is discussed in [1].

Example 6.3 (Weighted Lie algebroids out of reductions). Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\big|_M,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

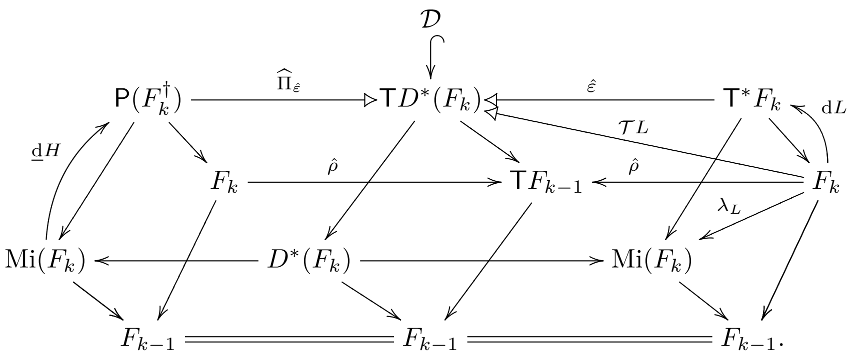
Theorem 6.4. *The linearisation of $A^k(\mathcal{G})$ is given as*

$$D(A^k(\mathcal{G})) \simeq \big\{(Y,Z) \in A(\mathcal{G}) \times_M \mathsf{T}A^{k-1}(\mathcal{G}) \mid \rho(Y) = \mathsf{T}\tau(Z)\big\},$$

viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid.

The above weighted algebroid is an example of a *Lie algebroid prolongation* in the sense of Cariñena, Martínez and Popescu [4, 19, 22]. For $k = 2$ it appeared also in [25] as the Lie algebroid of prolonged Lie groupoid.

7. Lagrangian framework for graded bundles. A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple



Here, the diagram consists of relations, $\hat{\varepsilon} : \mathsf{T}^* F_k \longrightarrow\!\!\triangleright \mathsf{T}^* D(F_k) \to \mathsf{T} D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called *Mironian* of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^k M) = \mathsf{T}^{k-1} M \times_M \mathsf{T}^* M$. We will not discuss the Hamiltonian side of the triple and direct the reader to [2]. The map $\mathcal{T}L = \hat{\varepsilon} \circ \mathrm{d}L$ we call the *Tulczyjew differential*, and λ_L the *Legendre relation*.

The fact that we obtain the Euler–Lagrange equations of higher order comes from the fact that we deal with a vakonomic constraint and the additional gradation.

EXAMPLE 7.1. Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$. The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$. The Lie algebroid structure $\varepsilon : \mathsf{T}^* D(g_2) \to \mathsf{T} D^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y \beta, \alpha).$$

Given a Lagrangian $L : g_2 \to \mathbb{R}$, the *Tulczyjew differential relation* $\mathcal{T}L : g_2 \to \mathsf{T} D^*(g_2)$ is

$$\mathcal{T}L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x \beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\}.$$

Hence, for the phase dynamics,

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, z) \right).$$

This leads to the *Euler–Lagrange equations* on g_2:

$$\dot{x} = z, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, z) \right) \right) = \mathrm{ad}^*_x \left(\frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, z) \right) \right).$$

These equations are second order and induce the *Euler–Lagrange equations* on g which are of order 3:

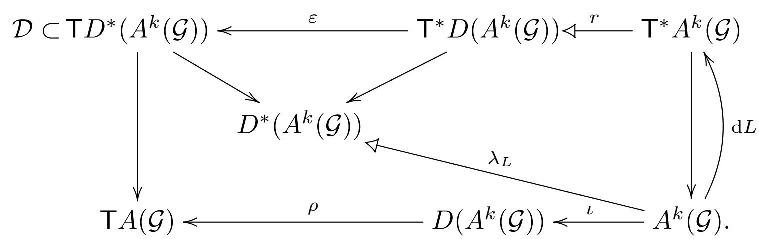
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, \dot{x}) \right) \right) = \mathrm{ad}^*_x \left(\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, \dot{x}) \right) \right).$$

For instance, the 'free' Lagrangian $L(x, z) = \frac{1}{2} \sum_i I_i (z^i)^2$ induces the equations on g (here, c^k_{ij} are structure constants, no summation convention):

$$I_j \dddot{x}^j = \sum_{i,k} c^k_{ij} I_k x^i \ddot{x}^k .$$

The latter can be viewed as *'higher Euler equations'*.

8. Higher order Lagrangian mechanics on Lie algebroids. Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a *k-th order Lagrangian system on the Lie algebroid* $A(\mathcal{G})$. The relevant diagram here is



Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the *Legendre relation*. Note that we actually deal with reductions: if $\mathcal{G}$ is a Lie group, then

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad D(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}.$$

For instance, using x^A as base coordinates, and y_i^a as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π_b^j of degree $j = 1, \ldots, k$ in $D^*(A^k)$, we get the equations for the Legendre relation in the form *(no Lie algebroid structure is relevant)*:

$$\begin{aligned}
k\pi_a^1 &= \frac{\partial L}{\partial y_k^a}\,, \\
(k-1)\pi_b^2 &= \frac{\partial L}{\partial y_{k-1}^b} - \frac{1}{k}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial y_k^b}\Big), \\
&\vdots \\
\pi_d^k &= \frac{\partial L}{\partial y_1^d} - \frac{1}{2!}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial y_2^d}\Big) + \frac{1}{3!}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(\frac{\partial L}{\partial y_3^d}\Big) - \cdots \\
&\quad + (-1)^k\,\frac{1}{(k-1)!}\,\frac{\mathrm{d}^{k-2}}{\mathrm{d}t^{k-2}}\Big(\frac{\partial L}{\partial y_{k-1}^d}\Big) - (-1)^k\,\frac{1}{k!}\,\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\Big(\frac{\partial L}{\partial y_k^d}\Big),
\end{aligned}$$

which we recognise as the *Jacobi–Ostrogradski momenta*. The remaining equation for the dynamics makes use of the Lie algebroid structure and reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_a^k = \rho_a^A(x)\,\frac{\partial L}{\partial x^A} + y_1^b C_{ba}^c(x)\pi_c^k\,,$$

where ρ_a^A and C_{ba}^c are structure functions of the Lie algebroid $A = A(\mathcal{G})$.

The above equation can then be rewritten as

$$\rho_a^A(x)\,\frac{\partial L}{\partial x^A} = \Big(\delta_a^c\,\frac{\mathrm{d}}{\mathrm{d}t} - y_1^b C_{ba}^c(x)\Big)\Big(\frac{\partial L}{\partial y_1^c} - \frac{1}{2!}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial y_2^c}\Big) - \ldots - (-1)^k\,\frac{1}{k!}\,\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}\Big(\frac{\partial L}{\partial y_k^c}\Big)\Big),$$

which we define to be the *k-th order Euler–Lagrange equations* on $A(\mathcal{G})$.

The above higher order algebroid E–L equations are in complete agreement with the ones obtained by Colombo & Martín de Diego [5], Jóźwikowski & Rotkiewicz [15, 16], as well as Martínez [21]. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example as well as the k-th order invariant system on a Lie group discussed in [6].

Example 8.1 (The tip of a javelin). For instance, let L be the Lagrangian governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x, y, z) = \frac{1}{2}\Big(\sum_{i=1}^{3}(y^i)^2 - (z^i)^2\Big).$$

We can understand $G = \mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1] \times \mathbb{R}^3[2]$. The Euler–Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial y^i} - \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial z^i}\Big)\Big) = 0,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}^2 z^i}{\mathrm{d}t^2}\,,$$

so the Euler–Lagrange equation on $\mathbb{R}^3$ reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}^4 x^i}{\mathrm{d}t^4}\,.$$

9. Geometric mechanics of strings. Another example of a Tulczyjew triple constructed from double graded bundles and morphisms is the one in which the dynamics lives in $\wedge^n\mathsf{T}\wedge^n\mathsf{T}^*M$ (see [8]). The justification is the following. We want to build a framework for higher dimensional objects, being motivated by the study of dynamics of one-dimensional non-parametrised objects (strings).

The *motion* of a system will be given by an n-dimensional submanifold in the manifold M (“space-time”). Therefore, an infinitesimal piece of the motion is the first jet of the submanifold. However, this model leads to essential complications even in one-dimensional case (relativistic particle). For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a ‘dual’ of the bundle of “first jets with volumes”. Therefore we will take the compromise: use for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume. It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n\mathsf{T}M$ of n-vectors on M. In this way we get the following principles:

- A *Lagrangian* L is a function on infinitesimal motions, $L : \wedge^n\mathsf{T}M \to \mathbb{R}$. If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^n\mathsf{T}^*M$ (the phase space).
- The *dynamics* should be an equation (in general, implicit) for n-dimensional submanifolds in the phase space, i.e.

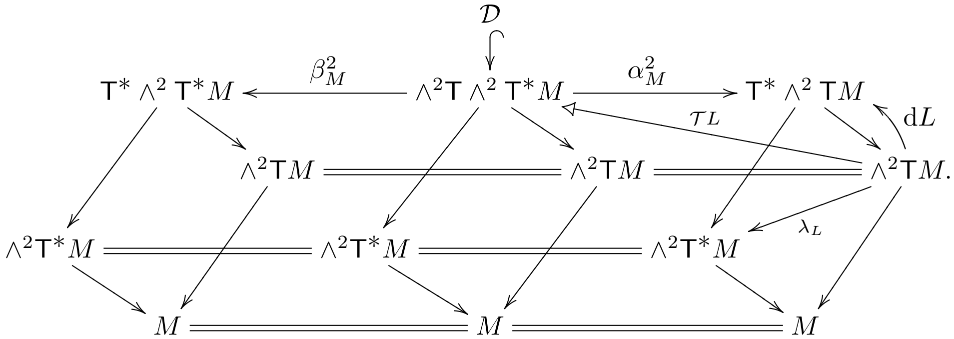
$$\mathcal{D} \subset \wedge^n\mathsf{T}\wedge^n\mathsf{T}^*M\,.$$

- A submanifold N in the phase space $\wedge^n\mathsf{T}^*M$ is a *solution* of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha N$ at $\alpha \in \wedge^n\mathsf{T}^*M$ is represented by a n-vector from $\mathcal{D}_\alpha$. If we use a parametrization, then the tangent n-vectors associated with this parametrization must belong to $\mathcal{D}$.

For simplicity, in what follows we will consider the ‘string case’ $n = 2$, but the constructions remain valid for arbitrary n. We will use canonical coordinates $(x^\rho, \dot{x}^{\mu\nu})$ and $(x^\rho, p_{\mu\nu})$ on $\wedge^2\mathsf{T}M$ and $\wedge^2\mathsf{T}^*M$ (with the convention $\dot{x}^{\mu\nu} = -\dot{x}^{\nu\mu}$, $p_{\mu\nu} = -p_{\nu\mu}$), respectively, representing the decomposition of bivectors:

$$\dot{x}^{\mu\nu}\,\partial_{x^\mu}\wedge\partial_{x^\nu} \in \wedge^2\mathsf{T}M\,,\quad p_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \in \wedge^2\mathsf{T}^*M.$$

Using the canonical multisymplectic structure on $\wedge^2\mathsf{T}^*M$, we get the following *Tulczyjew* *triple* for multivector bundles, consisting of *double graded bundle morphisms*:

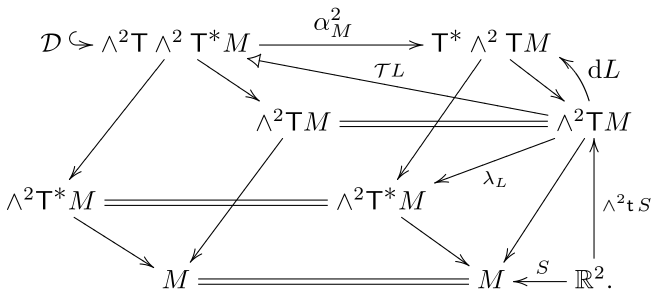


The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, from a Lagrangian $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ (or from a Hamiltonian $H:\wedge^2\mathsf{T}^*M\to\mathbb{R}$) is now standard: $\mathcal{D}=\mathcal{T}L(\wedge^2\mathsf{T}M)$.

9.1. The Euler–Lagrange equations. To define *Euler–Lagrange equations* for strings, consider a surface $S:\mathbb{R}^2\ni(t,s)\mapsto(x^\sigma(t,s))$ in M and its bi-tangent prolongation

$$\wedge^2\mathsf{t}\,S:\mathbb{R}^2\to\wedge^2\mathsf{T}M\,,\quad \wedge^2\mathsf{t}\,S=\mathsf{t}_tS\wedge\mathsf{t}_sS.$$

On the diagram:



It is easy to see that both parametrized surfaces, $\mathrm{d}L\circ\wedge^2\mathsf{t}\,S$ and $\alpha^2_M\circ\wedge^2\mathsf{t}(\lambda_L\circ\wedge^2\mathsf{t}\,S)$ in $\mathsf{T}^*\wedge^2\mathsf{T}M$ cover $\wedge^2\mathsf{t}\,S$. Therefore, their difference makes sense and, as easily seen, takes values in the annihilator $V^0\wedge^2\mathsf{T}M$ of the vertical subbundle $V\wedge^2\mathsf{T}M\subset\mathsf{T}\wedge^2\mathsf{T}M$. Since $V^0\wedge^2\mathsf{T}M\simeq\wedge^2\mathsf{T}M\times_M\mathsf{T}^*M$, we obtain a map $\delta L_S:\mathbb{R}^2\to\mathsf{T}^*M$. The above map is interpreted as external forces along the string trajectory S. Its value at (t,s) depends on the second jet $\mathsf{j}^2S(t,s)$ of S only, so defines the variation of the Lagrangian understood as a map

$$\delta L:\mathsf{J}^2_0(\mathbb{R}^2,M)\to\mathsf{T}^*M\,, \tag{6}$$

where $\mathsf{J}^2_0(\mathbb{R}^2,M)$ is the bundle of all second jets of maps $\mathbb{R}^2\to M$ at $0\in\mathbb{R}^2$. The equation

$$\delta L_S=0 \tag{7}$$

will be called the *Euler–Lagrange equation*. It tells that the surface $\mathrm{d}L\circ\wedge^2\mathsf{t}\,S$ corresponds *via* α^2_M to an *admissible surface* in $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, i.e. the bi-tangent prolongation of a parametrised surface in $\wedge^2\mathsf{T}^*M$.

A surface $S:(t,s)\mapsto(x^\sigma(t,s))$ in M satisfies the Euler–Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2\mathsf{T}M$,

$$(t,s)\mapsto\Big(x^\sigma(t,s),\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\Big),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface, living in the phase space $\wedge^2\mathsf{T}^*M$, to $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$.

In coordinates, the Euler–Lagrange equations read

$$\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\,\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\,\frac{\partial x^\nu}{\partial t}\,,$$
$$\frac{\partial L}{\partial x^\sigma}=\frac{\partial x^\mu}{\partial t}\,\frac{\partial}{\partial s}\Big(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\Big)-\frac{\partial x^\mu}{\partial s}\,\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\Big).$$

Example 9.1 (Plateau problem). In particular, if $M=\mathbb{R}^3=\{(x^1=x,x^2=y,x^3=z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2\mathsf{T}M$ reads

$$L(x^\mu,\dot{x}^{\kappa\lambda})=\sqrt{\sum_{\kappa,\lambda}(\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler–Lagrange equation for surfaces being graphs $(x,y)\mapsto(x,y,z(x,y))$ provides the well-known equation for *minimal surfaces*, found already by Lagrange:

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=0.$$

In another form,

$$(1+z_x^2)z_{yy}-2z_xz_yz_{xy}+(1+z_y^2)z_{xx}=0.$$

Starting with a Lorentz metric, we can obtain analogously the Euler–Lagrange equations for the *Nambu–Goto Lagrangian.*

10. Concluding remarks. We hope that the reader now appreciates that graded bundles and multi-graded bundles are interesting objects from a geometric perspective, but moreover that they are potentially very important in geometric mechanics. We have only 'scratched the surface' here with the applications of graded bundles in geometric mechanics and expect further results to follow.

The name of this volume suggest one line of further investigation: that is to develop jets and field theories within the general framework presented here. A little more specifically, the notion of a *Lie algebroid valued jet* was developed by Martínez [20] and it is natural to wonder what further structure come available when passing to weighted Lie algebroids. From there one may be able to develop a framework for higher order field theory using the Tulczyjew triple approach, modifying the constructions of [7, 10] as needed. This is work to be done in the near future.

In conclusion, the formalism of (n-tuple) graded bundles offers a clear and powerful way to view many constructions in differential geometry, most notably iterations of (higher) tangent bundles and cotangent bundles, as well as multivector bundles. Phrasing non-negatively graded geometry in terms of homogeneity structures can offer solutions to questions that would not otherwise be so readily obtainable. As an example we offer the problem of integrating weighted Lie algebroids as presented in [2].

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