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SYMPLECTIC SPACE FORMS AND SUBMANIFOLDS

SIMONE GUTT

Membre de l'Académie Royale de Belgique
Université Libre de Bruxelles, Campus Plaine, C. P. 218, B-1050 Brussels, Belgium
and
Université de Lorraine, Ile du Saulcy, F-57045 Metz Cedex 01, France
E-mail: sgutt@ulb.ac.be

It is a pleasure to dedicate this paper to Janusz Grabowski
on the occasion of his 60-th birthday

Abstract. This is a report on some ongoing work with Michel Cahen and Thibaut Grouy: the aim of our project is to define Radon-type transforms in symplectic geometry. The chosen framework is that of symplectic symmetric spaces whose canonical connection is of Ricci-type. These can be considered as symplectic analogues of the space forms, i.e. the spaces of constant sectional curvature, in Riemannian geometry. I shall focus here on their submanifold theory and I shall recall constructions of models of such spaces.

Introduction. Radon-type transforms were constructed from the beginning of the 20th century. Funk first observed in 1913 that a symmetric function on the sphere S^2 can be described from its great circle integrals. Radon showed in 1917 that a smooth function f on the Euclidean space $\mathbb{R}^3$ can be determined by its integrals over the planes in $\mathbb{R}^3$; if $J(\omega, p)$ is the integral of f over the plane defined by $x \cdot \omega = p$ for ω a fixed unit vector and p a fixed constant in $\mathbb{R}$, then

$$f(x) = -\frac{1}{8\pi^2} L_x\Big(\int_{S^2} J(\omega, \omega\cdot x)\, d\omega\Big)$$

where L is the Laplacian.

This yields a correspondence between the space $\mathbb{R}^3$ and the space of planes in $\mathbb{R}^3$, denoted $P(2,\mathbb{R}^3)$, in the following way:

— the Radon transform associates to a "nice" function f on $\mathbb{R}^3$, a function $\hat f$ on $P(2,\mathbb{R}^3)$:

$$\hat f(\xi) = \int_{x\in\xi} f(x)\, dm(x) \qquad \text{for any 2-plane } \xi;$$

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— the dual Radon transform associates to a "nice" function φ on $P(2,\mathbb{R}^3)$, a function $\check\varphi$ on $\mathbb{R}^3$:

$$\check\varphi(x) = \int_{\xi\,|\,x\in\xi} \varphi(\xi)\, d\mu(\xi).$$

This idea was widely generalized, constructing correspondences between a class of objects on a space X and a class of objects on a space Y. For instance, in a complex context, the Penrose transform builds a correspondence between the complex projective space $X = \mathbb{P}_3(\mathbb{C})$ and the Grassmannian space $Y = Gr(2,\mathbb{C}^4)$.

Radon-type transforms between spaces of functions, built by integrating functions, have mostly been developed in the framework of homogeneous (Riemannian) spaces, using a "double fibration", and a natural incidence relation (we refer for details to Helgason's nice book [6]). The problems of finding volume forms to integrate the functions, relating functions spaces, inverting the transforms, finding a corresponding map between invariant differential operators, studying the support of f when $\hat f$ has compact support… have been studied in various examples. It has been done in particular for spaces of constant curvature, which provide a rich supply of totally geodesic submanifolds. Recall that a connected submanifold S of a Riemannian manifold X is said to be *totally geodesic* if each geodesic in X which is tangent to S at a point lies entirely in S. The "abstract Radon transform" generalizes the original transform, associating to a function f on a space form X the function $\hat f$ on a space Y_p of totally geodesic submanifolds of dimension p, whose value at a point $S \in Y_p$ is given by the integral of f on the corresponding submanifold S

$$f \to \hat f(S) = \int_{x\in S} f(x)\, dm(x);$$

and with "dual abstract Radon transform":

$$\varphi \to \check\varphi(x) = \int_{S\supset\{x\}} \varphi(S)\, d\mu(S).$$

As a first step towards building correspondences in a symplectic framework, we introduce the notion of symplectic space forms and we study the space of totally geodesic symplectic submanifolds of those spaces. Section 1 is devoted to the definition of symplectic space forms; we recall constructions of models of such spaces. We show that there is a nice theory of symplectic submanifolds in symplectic space forms in Section 2. Section 3 gives a short description of the space of totally geodesic symplectic submanifolds in a symplectic space form and of the associated Radon transforms.

1. Symplectic space forms. Let us first recall the notions of pseudo-Riemannian and Kähler space forms.

1.1. Space forms in pseudo-Riemannian geometry. Let $(N, \mathring{g})$ be a pseudo-Riemannian manifold. It is endowed with the Levi-Civita connection which is the unique torsion free connection $\mathring{\nabla}$ such that $\mathring{\nabla}\mathring{g} = 0$.

Definition 1.1. *Pseudo-Riemannian space forms* are connected pseudo-Riemannian manifolds $(N, \mathring{g})$ of dimension $n \geq 4$, which are geodesically complete (for the Levi-Civita connection) and with constant sectional curvature, i.e.

$$\mathring{g}_x(\mathring{R}_x(X,Y)X,Y) = k_x\big(\mathring{g}_x(X,X)\mathring{g}_x(Y,Y) - \mathring{g}_x(X,Y)^2\big), \qquad k_x \in \mathbb{R},$$

for all $x \in N$ and for all $X, Y \in T_xN$ (where $\mathring{R}$ is the curvature of the Levi-Civita connection). Remark that it is equivalent to ask the above equality only for pairs of vectors $X, Y \in T_xN$ which span a non-degenerate 2-dimensional subspace of $(T_xN, \mathring{g}_x)$. Then

$$\mathring{g}_x(\mathring{R}_x(X,Y)Z,T) = k_x\big(\mathring{g}_x(X,Z)\mathring{g}_x(Y,T) - \mathring{g}_x(X,T)\mathring{g}_x(Y,Z)\big) \qquad \forall X,Y,Z,T \in T_xN. \quad (1.1)$$

Bianchi's second identity implies $k_x = k$. The space $(N, \mathring{g})$ is thus locally symmetric ($\mathring{\nabla}\mathring{R} = 0$).

The curvature tensor $\mathring{R}_x$ of a Levi-Civita connection at a point x of a pseudo-Riemannian manifold $(N, \mathring{g})$ is an algebraic tensor on T_xN with the following symmetry properties:

$$\mathring{g}_x(\mathring{R}_x(X,Y)Z,T) = -\mathring{g}_x(\mathring{R}_x(Y,X)Z,T) = -\mathring{g}_x(\mathring{R}_x(X,Y)T,Z) = \mathring{g}_x(\mathring{R}_x(Z,T)X,Y)$$

and

$$\underset{XYZ}{\circlearrowleft\!\!\!\!\!\!\bigoplus} \mathring{R}_x(X,Y)Z = 0$$

where $\underset{XYZ}{\circlearrowleft\!\!\!\!\!\!\bigoplus}$ denotes the sum over cyclic permutations of X, Y and Z. When $\dim M = 2n \geq 4$, the space of tensors having those symmetries splits, under the action of the pseudo-orthogonal group, into three irreducible components. Only one of them is one-dimensional. One has a decomposition of the curvature into three components and the condition to be a space form implies that only the one-dimensional component of the curvature may not vanish.

Equivalent definitions are thus given as follows:

— Pseudo-Riemannian space forms are connected, pseudo-Riemannian, geodesically complete, manifolds $(N, \mathring{g})$ of dimension $n \geq 4$ such that the curvature is a polynomial in the tensor algebra only involving the metric tensor $\mathring{g}$;
— Pseudo-Riemannian space forms are connected, pseudo-Riemannian, geodesically complete, manifolds $(N, \mathring{g})$ of dimension $n \geq 4$ such that the curvature at each point x is invariant under the whole pseudo-orthogonal group $O(T_xN, \mathring{g}_x)$.

In the Riemannian setting, it is well known that space forms in dimension n are all quotients of the Euclidean space $\mathbb{R}^n$, the sphere S^n or the hyperbolic space H^n.

Remark that if a symplectic manifold (M, ω) is endowed with a symplectic connection (i.e. a torsion free connection ∇ such that $\nabla\omega = 0$), whose curvature is a polynomial in the tensor algebra only involving the symplectic tensor ω, then the curvature is identically zero. To define a good notion of symplectic space forms, we obviously want to go beyond the flat case, so we consider what happens in the Kähler case, Kähler manifolds being both Riemannian and symplectic.

1.2. Kähler space forms

Definition 1.2. *Kähler space forms* are connected, Kähler, geodesically complete, manifolds $(N, \mathring{g}, \mathring{J})$ of dimension $n \geq 4$ with constant holomorphic sectional curvature, i.e.

$$\mathring{g}_x(\mathring{R}_x(X, JX)X, JX) = k_x\big(\mathring{g}_x(X, X)^2\big) \qquad k_x \in \mathbb{R},$$

for all $x \in N$ and for all $X \in T_xN$. Then

$$\begin{aligned}\mathring{g}_x(\mathring{R}_x(X, Y)Z, T) = \frac{k_x}{4}\big(&\mathring{g}_x(X, Z)\mathring{g}_x(Y, T) - \mathring{g}_x(X, T)\mathring{g}_x(Y, Z) + \mathring{g}_x(X, JZ)\mathring{g}_x(Y, JT) \\ &- \mathring{g}_x(X, JT)\mathring{g}_x(Y, JZ) + 2\mathring{g}_x(X, JY)\mathring{g}_x(Z, JT)\big) \quad \forall X, Y, Z, T \in T_xN \quad (1.2)\end{aligned}$$

and by Bianchi's second identity $k_x = k$, so the space is locally symmetric.

The curvature tensor at a point $\mathring{R}_x$ is an algebraic tensor on T_xN with the symmetry properties of a Riemann curvature tensor and furthermore:

$$\mathring{g}_x\big(\mathring{R}_x(X, Y)JZ, JT\big) = \mathring{g}_x\big(\mathring{R}_x(X, Y)Z, T\big) = \mathring{g}_x\big(\mathring{R}_x(JX, JY)Z, T\big).$$

When $\dim M = 2n \geq 4$, the space of tensors on T_xN having those symmetries splits under the action of the unitary group into three irreducible components with only one of them which is one-dimensional, and the condition to be a space form implies that only this component may not vanish in the curvature.

Equivalently:

— Kähler space forms are connected, Kähler, geodesically complete, manifolds $(N, \mathring{g}, \mathring{J})$ of dimension $n \geq 4$ such that the curvature is a polynomial in the tensor algebra (with catenation) involving only $\mathring{g}$ and J, or

— Kähler space forms are connected, Kähler, geodesically complete, manifolds $(N, \mathring{g}, \mathring{J})$ of dimension $n \geq 4$ such that the curvature at each x is invariant under the unitary group $U(T_xN, \mathring{g}_x, \mathring{J}_x)$.

It is again well known that Kähler space forms in complex dimension n are all quotients of $\mathbb{C}^n$, the complex projective space $\mathbb{CP}^n$, or the complex hyperbolic space $\mathbb{CH}^n$.

1.3. Symplectic connections. A linear connection ∇ on a symplectic manifold (M, ω) is called *symplectic* if the symplectic form ω is parallel and if its torsion T^∇ vanishes. Let me emphazise here that it is also interesting to consider connection $\tilde{\nabla}$ for which $\tilde{\nabla}\omega = 0$ but which can have torsion. In particular, given a compatible almost complex structure J on (M, ω), there is a unique connection $\tilde{\nabla}$ such that $\tilde{\nabla}\omega = 0$, $\tilde{\nabla}J = 0$ and such that its torsion is given—up to factor $\frac{1}{4}$—by the Nijenhuis tensor of J.

Let us recall that a symplectic connection exists on any symplectic manifold. Indeed take any torsion free connection ∇^0, define N by $\nabla^0_X\omega(Y, Z) =: \omega(N(X, Y), Z)$ and set

$$\nabla_X Y := \nabla^0_X Y + \frac{1}{3}N(X, Y) + \frac{1}{3}N(Y, X).$$

Then ∇ is a symplectic connection. On the other hand such a connection is far from being unique: given ∇ symplectic, the connection $\nabla'_X Y := \nabla_X Y + S(X, Y)$ is symplectic if and only if $\omega(S(X, Y), Z)$ is totally symmetric.

1.3.1. *Symplectic symmetric spaces.* Without further data, there is no preferred choice of a specific symplectic connection. Recall from the above that simply connected space forms in the Riemannian or Kähler context are locally symmetric spaces. Now, in a symmetric context there is always a unique "symmetric" connection [7]. In the symplectic context, symmetric spaces were studied in [2]. A *symplectic symmetric space* is a symplectic manifold (M,ω) endowed with "symmetries", i.e. with a smooth map

$$S : M \times M \to M : (x,y) \mapsto s_x y,$$

so that, for each $x \in M$, the symmetry $s_x : M \to M$ is an involutive symplectomorphism (i.e. $s_x^*\omega = \omega$ and $s_x^2 = \mathrm{Id}$), with x an isolated fixed point ($s_x x = x$), and such that $s_x s_y s_x = s_{s_x y}$, for any $x, y \in M$.

On a symplectic symmetric space there is a unique connection for which each s_x is an affinity; it is given by

$$(\nabla_X Y)_x = \tfrac{1}{2}[X - s_{x*}X, Y]_x.$$

This connection is automatically symplectic.

1.3.2. *Decomposition of the curvature.* We have seen that Riemannian and Kähler space forms have a curvature tensor whose decomposition into irreducible factors only contains the smallest dimensional irreducible part. The curvature tensor of a symplectic connection at a point x has the following symmetry properties:

$$\omega_x\big(R^\nabla_x(X,Y)Z,T\big) = -\omega_x\big(R^\nabla_x(Y,X)Z,T\big) = \omega_x\big(R^\nabla_x(X,Y)T,Z\big)$$

and

$$\underset{XYZ}{\circlearrowright\!\!\!\!\!\!\bigoplus} R^\nabla_x(X,Y)Z = 0.$$

When $\dim M = 2n \geq 4$, Izu Vaisman [10] has shown that the space of tensors having those symmetries splits under the action of the symplectic group into two irreducible components so that one has a decomposition of the curvature into

$$R^\nabla = W^\nabla + E^\nabla$$

where W^∇ has no trace and

$$\begin{aligned} E^\nabla(X,Y)Z = \frac{1}{2n+2}\big(&2\omega(X,Y)\rho^\nabla Z + \omega(X,Z)\rho^\nabla Y - \omega(Y,Z)\rho^\nabla X \\ &+ \omega(X,\rho^\nabla Z)Y - \omega(Y,\rho^\nabla Z)X\big) \end{aligned} \tag{1.3}$$

with the Ricci tensor r^∇ (defined by $r^\nabla(X,Y) := \mathrm{Tr}[Z \to R^\nabla(X,Z)Y]$), which is automatically symmetric, converted into a so called Ricci endomorphism ρ^∇ by

$$\omega(X,\rho^\nabla Y) = r^\nabla(X,Y). \tag{1.4}$$

A symplectic connection for which $W^\nabla = 0$ is said to be of *Ricci-type*. One can show [3] that a Ricci-type symplectic connection is determined by the second jet of its curvature at a point. Precisely, if ∇ is a symplectic connection of Ricci-type on the connected symplectic manifold (M,ω), there exist a vector field U^∇ on M and a constant k^∇ such that

$$\begin{aligned}(\nabla_X\rho^\nabla)Y &= X\omega(U^\nabla,Y)+U^\nabla\omega(X,Y),\\ \nabla_X U^\nabla &= \frac{1}{2n+1}(\rho^\nabla)^2X+f^\nabla X,\\ f^\nabla &= \frac{-1}{4n+1}\operatorname{Tr}(\rho^\nabla)^2+k^\nabla,\end{aligned}$$

so that all iterated derivatives of the curvature at the point x are determined by $\rho^\nabla_x, U^\nabla_x$ and k^∇.

1.4. Symplectic space forms. Observe that the Levi-Civita connection on a Kähler manifold is obviously symplectic and equation (1.2) shows that the Levi-Civita connection on a Kähler space form is of Ricci-type.

Definition 1.3. A *symplectic space form* is a connected symplectic manifold endowed with a symplectic connection ∇ which is complete, locally symmetric, and such that its curvature is of Ricci-type.

The curvature is thus given by

$$\begin{aligned}R^\nabla(X,Y)Z = \frac{1}{2n+2}\big(&2\omega(X,Y)\rho^\nabla Z+\omega(X,Z)\rho^\nabla Y-\omega(Y,Z)\rho^\nabla X\\ &+\omega(X,\rho^\nabla Z)Y-\omega(Y,\rho^\nabla Z)X\big)\end{aligned}\tag{1.5}$$

with ρ^∇ the Ricci endomorphism. A Ricci-type connection is locally symmetric if and only if $(\rho^\nabla)^2=k\,\mathrm{Id}$.

Symplectic spaces with Ricci-type connections have been studied in [3]. Nicolas Richard studied the analogue of the notion of constant holomorphic sectional curvature in a symplectic context in his thesis [8].

1.4.1. *Construction of Ricci-type connections by reduction.* Let $(\mathbb{R}^{2n+2},\Omega)$ be the standard symplectic vector space and let A be an element in the symplectic Lie algebra $\mathrm{sp}(\mathbb{R}^{2n+2},\Omega)=\{B\in\mathrm{Mat}((2n+2)\times(2n+2),\mathbb{R})\mid {}^{\mathrm{tr}}B\Omega+\Omega B=0\}$.

Consider the embedded hypersurface $\Sigma_A=\{x\in\mathbb{R}^{2n+2}\mid \Omega(x,Ax)=1\}$ and suppose it is not empty. The 1-parameter subgroup $\{\exp tA\}$ acts on Σ_A and we consider the quotient

$$M^{\mathrm{red}}:=\Sigma_A/\exp tA \quad \text{(it exists at least locally since } Ax\neq 0!)$$

with the canonical projection

$$\pi:\Sigma_A\to M^{\mathrm{red}}.$$

For any $x\in\Sigma_A$, we define a $2n$-dimensional "horizontal" subspace H_x of the tangent space $T_x\Sigma_A$ given by the Ω-orthogonal to the subspace spanned by x and Ax:

$$H_x:=\langle x,Ax\rangle^{\perp_\Omega}\subset T_x\Sigma_A\simeq\langle Ax\rangle^{\perp_\Omega}\subset\mathbb{R}^{2n+2};$$

the differential of the projection π induces an isomorphism

$$\pi_*:H_x\sim T_{\pi(x)}M^{\mathrm{red}}.$$

Given a tangent vector $X \in T_y M^{\mathrm{red}}$, we denote by $\overline{X}$ its horizontal lift:

$$\overline{X}_x \in H_x, \quad X_{y=\pi(x)} = \pi_{*x}\overline{X}_x.$$

The reduced 2-form ω^{red} on M^{red} is defined in the standard way:

$$\omega^{\mathrm{red}}_{y=\pi(x)}(X,Y) := \Omega_x(\overline{X}_x, \overline{Y}_x);$$

and $(M^{\mathrm{red}}, \omega^{\mathrm{red}})$ is a symplectic manifold. The flat connection on $\mathbb{R}^{2n+2}$ induces a connection on Σ_A

$$(\nabla^A_U V)_x := \nabla_U V - \Omega(\nabla_U V, Ax)x = \nabla_U V - \Omega(AU, V)x$$

and a connection ∇^{red} on M^{red} given by

$$(\nabla^{\mathrm{red}}_X Y)_y := \pi_{*_x}\big(\nabla_{\overline{X}}\overline{Y} - \Omega(A\overline{X}, \overline{Y})x + \Omega(\overline{X}, \overline{Y})Ax\big).$$

The reduced connection ∇^{red} on $(M^{\mathrm{red}}, \omega^{\mathrm{red}})$ constructed in [1] is symplectic and of Ricci-type. With Cahen and Schwachhöfer [4], we have proven that any symplectic manifold endowed with a symplectic connection of Ricci-type is locally of this form. In the above construction the Ricci endomorphism is $-2(n+1)\overline{A}$, i.e.

$$-\frac{1}{2n+2}\rho^{\nabla^{\mathrm{red}}}_{\pi(x)}(X) = \pi_{*x}\big(A\overline{X} - \Omega(A\overline{X}, Ax)x\big).$$

The space is locally symmetric if and only if $A^2 = \lambda\,\mathrm{Id}$.

1.5. Models of symplectic space forms. We can construct models of symplectic space forms by the reduction procedure explained in Section 1.4.1 using an element $A \in \mathrm{sp}(\mathbb{R}^{2n+2}, \Omega)$ so that $A^2 = \lambda\,\mathrm{Id}$; in these cases the quotient of the hypersurface Σ_A by the action of $\{\exp tA\}$ is globally defined and we have

$$\pi : \Sigma_A = \{x \in \mathbb{R}^{2n+2} \mid \Omega(x, Ax) = 1\} \to M_A := M^{\mathrm{red}}_{\mathrm{cc}} = (\Sigma_A/_{\{\exp tA\}})_{\mathrm{cc}}$$

where $_{\mathrm{cc}}$ indicates that we take a connected component.

Any $B \in G_A := \{B \in \mathrm{Sp}(\mathbb{R}^{2n+2}, \Omega) \mid BA = AB\}$ induces an automorphism of M_A in the obvious way:

$$B \cdot \pi(x) := \pi(Bx).$$

The space M_A is a symplectic symmetric space, with the symmetry at $y = \pi(x)$ induced by

$$S_x(v) = -v - 2\Omega(v, x)Ax + 2\Omega(v, Ax)x.$$

With Cahen and Schwachhöfer [4], we proved that any symplectic space form is diffeomorphic to a quotient of the universal cover of such a model M_A and those models are explicitly given as follows:

THEOREM 1.4 ([4]). *Let $A \in \mathrm{sp}(\mathbb{R}^{2n+2}, \Omega)$ be so that $A^2 = \lambda\,\mathrm{Id}$ and consider the reduced manifold $M_A = (\{x \in \mathbb{R}^{2n+2} \mid \Omega(x, Ax) = 1\}/\{\exp tA\})_{\mathrm{cc}}$.*

- *If $\lambda > 0$ then $M_A = T^*S^n = Sl(n+1, \mathbb{R})/Gl^+(n, \mathbb{R})$ with the usual cotangent bundle symplectic structure.*
- *If $\lambda < 0$, a multiple of the matrix A defines a complex structure on $\mathbb{R}^{2n+2}$. Denoting by p the integer such that the signature of the pseudo-Hermitian structure defined by $\Omega(x, Ax)$ on $\mathbb{C}^{n+1}$ is $(p+1, n-p)$ (thus $0 \le p \le n$), then $M_A = SU(p+1, n-p)/U(p, n-p)$ with its natural pseudo-Kähler structure and*

— $M_A = \mathbb{C}H^n = SU(1,n)/U(n)$ *is the complex hyperbolic space for* $p=0$;
— $M_A = E$ *is a holomorphic vector bundle of rank* $n-p$ *over* $\mathbb{C}P^p$ *for* $1 \leq p \leq n-1$;
— $M_A = \mathbb{C}P^n = SU(n+1)/U(n)$ *is the complex projective space for* $p=n$.

• *If* $\lambda = 0$ *there are two integers* r *and* q *attached to the space form,* r *being the rank of* A *(so that* $1 \leq r \leq n+1$*) and* $(q, r-q)$ *being the signature of* $\Omega(x, Ax)$*; then*

$$M_A = T(S^{q-1} \times \mathbb{R}^{r-q}) \times \mathbb{R}^{2(n+1-r)}$$

with a transitive automorphism group given by the semidirect product

$$\big(O(q, r-q).\mathrm{Sp}(\mathbb{R}^{2(n+1-r)})\big) \cdot R$$

where $R = \mathrm{Mat}((r-1) \times 2(n+1-r)) \cdot \mathrm{MatSym}(r \times r)$ *is a solvable group.*

2. Submanifold theory in space forms. Pseudo-Riemannian space forms form a very nice framework for the theory of submanifolds. As is well known, when $j : (M, g) \to (N, \mathring{g})$ is a pseudo-Riemannian embedding, in a space form, the curvature of M and the curvature of the normal bundle νM are determined (Gauss' and Ricci's equations) by the constant k of scalar curvature in N, the metric $\mathring{g}$ and the second fundamental form α. Furthermore, this α obeys a system of differential equations (Codazzi's equations). The fundamental theorem of submanifolds in space forms gives a reciprocal of the above: given an m-dimensional Riemannian manifold (M, g), a Riemannian vector bundle $(E, \tilde{g})$ of rank p on M with a metric connection $\tilde{\nabla}$, and given α a symmetric 2-form on M with values in E, if the curvature of M, the curvature of E, and α satisfy Gauss, Codazzi, and Ricci equations, then, for any point x in M, there is a local isometric immersion of a neighborhood U_x of x in M in a space of constant curvature, $j : (U_x, g) \to (N, \mathring{g})$, such that α is the second fundamental form and E is diffeomorphic to the normal bundle. We now state how those results generalize in the framework of symplectic space forms.

2.1. The second fundamental form and the shape operator. Consider a manifold $(N, \mathring{\mathrm{b}}, \mathring{\nabla})$ with $\mathring{\mathrm{b}}$ a non-degenerate symmetric or skewsymmetric 2-form and with $\mathring{\nabla}$ a torsionfree connection such that $\mathring{\nabla}\mathring{\mathrm{b}} = 0$. Observe that in the symmetric case, $(N, \mathring{\mathrm{b}})$ is a pseudo-Riemannian manifold and $\mathring{\nabla}$ is its Levi-Civita connection; in the skewsymmetric case, $\mathring{\nabla}\mathring{\mathrm{b}} = 0$ implies $d\mathring{\mathrm{b}} = 0$, so $(N, \mathring{\mathrm{b}})$ is a symplectic manifold and $\mathring{\nabla}$ is a symplectic connection. Assume that we have an embedded submanifold $j : M \to N$ so that $j^*\mathring{\mathrm{b}} = \mathrm{b}$ is non-degenerate (i.e. $(M, j^*\mathring{\mathrm{b}})$ is pseudo-Riemannian or symplectic). We consider the splitting

$$T_{j(x)}N = j_{*_x} T_x M \oplus T_x M^{\perp}$$

where $^{\perp}$ is the orthogonal relative to $\mathring{\mathrm{b}}$. The normal bundle to M, $\nu M = \bigcup_{x \in M} T_x M^{\perp}$, is endowed with $\tilde{\mathrm{b}}_x$ which is the restriction of $\mathring{\mathrm{b}}_{j(x)}$ to $T_x M^{\perp}$. The pullback bundle of the tangent bundle splits as $j^*TN = TM \oplus \nu M$ and one can decompose the covariant derivatives of its sections accordingly. If X, Y denote vector fields on M (which can be seen as sections of j^*TN) and if ξ, η denote sections of νM, one defines a torsionfree *induced connection* ∇ *on* M preserving b via:

$$(\nabla_X Y)_x := p_x^{TM}(\mathring{\nabla}_X Y),$$

the *second fundamental form* α, which is a 2-form on M with values in νM:

$$\alpha_x(X,Y) := p_x^{\nu M}(\mathring{\nabla}_X Y),$$

the *shape operator* A:

$$A_{\xi,x}(X) := -p_x^{TM}(\mathring{\nabla}_X \xi)$$

and an *induced connection on the normal bundle* preserving $\tilde{\mathrm{b}}$:

$$(\tilde{\nabla}_X \xi)_x = p_x^{\nu M}(\mathring{\nabla}_X \xi).$$

The shape operator and the second fundamental form are equivalent data through the relation

$$\tilde{\mathrm{b}}(\xi, \alpha(X,Y)) = \mathrm{b}(A_\xi(X), Y).$$

2.2. Gauss, Codazzi and Ricci equations. Projecting similarly the curvature tensor $R^{\mathring{\nabla}}$ yields:

$$\begin{aligned}
&p^{TM}(R^{\mathring{\nabla}}(X,Y)Z) = R^{\nabla}(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X \quad \text{(Gauss's formula)}\\
&p^{\nu M}(R^{\mathring{\nabla}}(X,Y)Z) = (D_X^{\nabla\tilde{\nabla}}\alpha)(Y,Z) - (D_Y^{\nabla\tilde{\nabla}}\alpha)(X,Z) \qquad \text{(Codazzi's formula)}
\end{aligned} \tag{2.1}$$

with $(D_X^{\nabla\tilde{\nabla}}\alpha)(Y,Z) = \tilde{\nabla}_X \alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$,

$$\begin{aligned}
&p^{\nu M}(R^{\mathring{\nabla}}(X,Y)\xi) = R^{\tilde{\nabla}}(X,Y)\xi + \alpha(Y, A_\xi X) - \alpha(X, A_\xi Y) \quad \text{(Ricci's formulas)}\\
&p^{TM}(R^{\mathring{\nabla}}(X,Y)\xi) = (D_Y^{\nabla\tilde{\nabla}}A)_\xi X - (D_X^{\nabla\tilde{\nabla}}A)_\xi Y
\end{aligned} \tag{2.2}$$

with $(D_X^{\nabla\tilde{\nabla}}A)_\xi Y = \nabla_X(A_\xi Y) - A_{\tilde{\nabla}_X \xi} Y - A_\xi \nabla_X Y$.

Definition 2.1. A *good submanifold* of dimension m in the sympletic space form $(N, \mathring{\omega}, \mathring{\rho})$ (where $\mathring{\rho}$ denotes the Ricci endomorphism of the symmetric connection) is a manifold M with $j : M \to N$ an embedding such that $j^*\mathring{\omega} =: \omega$ is non-degenerate and $\mathring{\rho}_{j(x)}$ stabilizes $j_* T_x M$.

Theorem 2.2. *Let $(N, \mathring{\omega}, \mathring{\rho})$ be a symplectic space form and (M, j) be a good submanifold. Then, on the pullback bundle $j^*TN = TM \oplus \nu M$, the Ricci endomorphism splits $\mathring{\rho} = \rho_1 \oplus \rho_2$ into two parallel fields of endomorphisms $\nabla\rho_1 = 0$, $\tilde{\nabla}\rho_2 = 0$, so that $\rho_1^2 = \mu\,\mathrm{Id}$ and $\rho_2^2 = \mu\,\mathrm{Id}$ for a real constant μ, and one has the following intertwining involving the second fundamental form*:

$$\rho_2 \alpha(X,Y) = \alpha(X, \rho_1 Y).$$

The curvature on M, on νM, and the form α are solutions of the equations:

$$\begin{aligned}
R^{\nabla}(X,Y)Z = \frac{-1}{2(n+1)}\big(&-2\omega(X,Y)\rho_1 Z - \omega(X,Z)\rho_1 Y + \omega(Y,\rho_1 Z)X\\
&+ \omega(Y,Z)\rho_1 X - \omega(X,\rho_1 Z)Y\big) + A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y,
\end{aligned} \tag{2.3}$$

$$(D_X^{\nabla\tilde{\nabla}}\alpha)(Y,Z) = (D_Y^{\nabla\tilde{\nabla}}\alpha)(X,Z), \tag{2.4}$$

$$R^{\tilde{\nabla}}(X,Y)\xi = \frac{1}{n+1}\omega(X,Y)\rho_2\xi + \alpha(X, A_\xi Y) - \alpha(Y, A_\xi X). \tag{2.5}$$

2.3. Fundamental theorem for submanifolds in symplectic space forms

Theorem 2.3.

— *Given a symplectic manifold* (M,ω) *of* $\dim 2m$ *with a symplectic connection* ∇, *endowed with a field* ρ_1 *of endomorphisms of* TM, *with values in the symplectic Lie algebra* $\mathrm{sp}(T_xM,\omega_x)$, *which is parallel* ($\nabla\rho_1 = 0$) *and squares to a multiple of the identity* $\rho_1^2 = \mu\,\mathrm{Id}$;
— *given a symplectic vector bundle* $(E,\tilde{\omega})$ *of rank* $2p$ *on* M, *with a connection* $\tilde{\nabla}$ *so that* $\tilde{\nabla}\tilde{\omega} = 0$, *endowed with a field* ρ_2 *of parallel endomorphisms* ($\nabla\rho_2 = 0$), *with values in* $\mathrm{sp}(E_x,\tilde{\omega}_x)$, *so that* $\rho_2^2 = \mu\,\mathrm{Id}$ (*with the same constant* μ);
— *given* α *a symmetric* 2*-form on* M *with values in* E, *such that* $\rho_2\alpha(X,Y) = \alpha(X,\rho_1 Y)$;
and assuming these data satisfy the symplectic Gauss (2.3), *Codazzi* (2.4) *and Ricci* (2.5) *equations, then, for any point* $x \in M$, *there is a local symplectic immersion of a neighborhood of* x *in* M *into a* $2(m+p)$*-dimensional symplectic space form, such that* α *is the second fundamental form and* E *is isomorphic to the normal bundle* νM. *Furthermore the immersion is unique up to an affine symplectic diffeomorphism of* N.

The proof generalizes the one given in Spivak [9], Chapter 7, Theorem 20, for submanifolds in Riemannian space forms: one shows that the equations are the integrability conditions for a distribution built on a principal bundle of adapted frames for $(TM\oplus E)\times TN$. Details were presented in [5].

3. Totally geodesic symplectic submanifolds and Radon transforms

3.1. Totally geodesic symplectic submanifolds in the model M_A of a symplectic space form

Theorem 3.1. *Let* $M_A := (\Sigma_A/\{\exp tA\})_{\mathrm{cc}}$ *be a symplectic space form of dimension* n *as constructed above. Let* N *be a totally geodesic symplectic submanifold of* M_A, *of dimension* $2q$, *passing through* y *and let* $V = T_yN \subset T_yM_A$. *Then* V *is a symplectic subspace of* $T_y(M_A)$ *stable by* ρ_y.

Conversely, let V *be a* $2q$*-dimensional symplectic subspace of* $T_y(M_A)$ *stable by* ρ_y. *There exists a unique maximal totally geodesic submanifold* N *of* M_A, *of dimension* $2q$, *passing through* y *and tangent to* V. *It is given by*

$$N = (\Sigma_A \cap W/\{\exp tA\})_{\mathrm{cc}} \quad \text{with} \quad W = \overline{V} \oplus \mathbb{R}x \oplus \mathbb{R}Ax$$

where x *is a point in* Σ_A *so that* $\pi(x) = y$ *and where* $\overline{V}$ *the* $2q$*-subspace of* $\mathbb{R}^{2n+2}$ *which is the horizontal lift of* V *in* $T_x(\Sigma_A) = \langle Ax\rangle^{\perp_\Omega} \subset \mathbb{R}^{2n+2}$, *i.e. the subspace defined by* $\Omega(\overline{V},x) = 0$, $\Omega(\overline{V},Ax) = 0$ *and* $\pi_{*x}\overline{V} = V$.

Such a totally geodesic submanifold is automatically symplectic; *it is a symplectic space form and carries an invariant measure.*

3.2. The space of totally geodesic submanifolds in M_A. The group $G_A := \{B \in \mathrm{Sp}(\mathbb{R}^{2n+2},\Omega) \mid BA = AB\}$ acts by symplectic affine transformations on $(M_A,\omega^{\mathrm{red}},\nabla^{\mathrm{red}})$; it maps a symplectic totally geodesic submanifold of dimension $2q$ on a symplectic totally geodesic submanifold of dimension $2q$; this action corresponds to the action of G_A on the set of $(2q+2)$-dimensional symplectic subspaces W of $\mathbb{R}^{2n+2}$ which are stable by A and intersect $\Sigma_A = \{x \in \mathbb{R}^{2n+2} \,|\, \Omega(x, Ax) = 1\}$.

THEOREM 3.2. *There exist a finite number of orbits of G_A in the set of symplectic maximal totally geodesic submanifolds of M_A. The orbit is determined by the dimension $2q$ of the submanifold, the rank of $A_{|W}$ and the signature of the symmetric 2-form on W defined by $\Omega(\cdot, A\cdot)$. Each of these G_A-orbits in the space of symplectic totally geodesic submanifolds is a symmetric space. If $A^2 \neq 0$, those orbits are symplectic symmetric spaces.*

3.3. The Radon transform. Let $M_A := \Sigma_A/\{\exp tA\}$ be a symplectic space form of dimension n with $A^2 = \lambda \operatorname{Id} \neq 0$. For a fixed k (with $0 < k < \dim M$), let N_k be one orbit of the automorphism group G_A in the set of symplectic maximal totally geodesic submanifolds of dimension k.

The Radon transform associates to a continuous function f with compact support on M_A, the function $\hat{f}$ on N_k defined by

$$\hat{f}(\xi) = \int_{x\in\xi} f(x)\, d\mu(x)$$

with $d\mu$ an invariant measure on the totally geodesic submanifold ξ (which exists since it is a symplectic space form).

The dual Radon transform associates to a continuous function F with compact support on N_k, the function $\tilde{F}$ on M_A defined in a similar way

$$\hat{F}(x) = \int_{\xi\,|\,x\in\xi} F(\xi)\, d\nu(\xi)$$

with $d\nu$ an invariant measure on N_k (which we know exist if $A^2 \neq 0$ since N_k is then a symplectic symmetric space).

3.4. Spaces in duality

• If $A^2 = -\operatorname{Id}$, we view A as a complex structure and identify $\mathbb{R}^{2n+2}$ to $\mathbb{C}^{n+1}$. The Hermitian structure $\langle u, v\rangle = \Omega(u, Jv) - i\Omega(u, v)$ has signature $(p+1, n-p)$ and we get:

— When $p = n$, the group G_A identifies with $SU(n+1)$ and the space form M is the complex projective space

$$M = P_n(\mathbb{C}) = SU(n+1)/U(n).$$

Every symplectic maximal totally geodesic submanifold of dimension $2q$ is diffeomorphic to $P_q(\mathbb{C})$. There is only one orbit of symplectic maximal totally geodesic symplectic submanifolds for a given q; it is given by

$$N = SU(n+1)/S(U(q+1) \times U(n-q)).$$

The Radon transform in the case where $q = n-1$ corresponds to the transform defined by antipodal submanifolds in $P_n(\mathbb{C})$ (see [6]).

— When $1 < p < n$, the group G_A identifies with $SU(p+1, n-p)$ and the space form is

$$M = SU(p+1, n-p)/S(U(p) \times U(n-p)).$$

Any symplectic maximal totally geodesic submanifold of dimension $2q$ is of the form

$$SU(p'+1, q-p')/S(U(p') \times U(q-p'))$$

for $p' < \min(p,q)$. Each value of p' corresponds again to just one orbit. Each of these orbits is a symmetric symplectic space.

— When $p = 0$, the group G_A is $SU(1,n)$ and the space form is the complex hyperbolic space

$$M = SU(1,n)/U(n) = H_n(\mathbb{C}).$$

Every symplectic totally geodesic submanifold of dimension $2q$ is diffeomorphic to $H_q(\mathbb{C})$. For any dimension $2q$, there is one G_A orbit of such submanifolds; it is given by

$$N = SU(1,n)/S(U(1,q) \times U(n-q)).$$

The Radon transform in the case where $q = n-1$ corresponds to the transform defined by antipodal submanifolds in $H_n(\mathbb{C})$ (see again [6]).

• When $A^2 = \mathrm{Id}$, we view $\mathbb{R}^{2n+2}$ as a sum of two Lagrangian subspaces $\mathbb{R}^{2n+2} = L_+ \oplus L_-$ corresponding to the ± 1 eigenspaces for A. The group G_A identifies with $Sl(n+1,\mathbb{R})$ and the space form is the cotangent bundle to the sphere with its canonical symplectic structure.

$$M = T^*S^n = Sl(n+1,\mathbb{R})/Gl_+(n,\mathbb{R}).$$

Any symplectic totally geodesic symplectic submanifold of dimension $2q$ is diffeomorphic to T^*S^q. All such submanifolds are in the same orbit of $Sl(n+1,\mathbb{R})$; the space of such manifolds is

$$N = Sl(n+1,\mathbb{R})/S(Gl(q+1,\mathbb{R}) \times Gl(n-q,\mathbb{R})),$$

i.e. the space of pairs of supplementary spaces (one of dimension $q+1$) in $\mathbb{R}^{n+1}$. It is a symmetric symplectic space.

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