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NONABELIAN OMNI-LIE ALGEBRAS

HONGLEI LANG

Department of Mathematics and LMAM, Peking University
Beijing 100871, China
E-mail: hllang@pku.edu.cn

YUNHE SHENG

Department of Mathematics, Jilin University
Changchun 130012, China
and
Kavli Institute for Theoretical Physics China, CAS
Beijing 100190, China
E-mail: shengyh@jlu.edu.cn

XIAOMENG XU

Section of Mathematics, University of Geneva
2-4 Rue de Lièvre, c.p. 64, 1211-Genève 4, Switzerland
E-mail: Xiaomeng.Xu@unige.ch

Abstract. We introduce the notion of a nonabelian omni-Lie algebra associated to a Lie algebra. From a geometric point of view, it is the linearization of the Courant algebroid associated to a Poisson manifold. As an algebraic application, we show that its Dirac structures describe deformations of the Lie algebra.

1. Introduction. Courant algebroids were introduced in [LWX] (see also [K2, R] for more details), and have many applications. Roughly speaking, a Courant algebroid is a vector bundle together with a nondegenerate symmetric pairing, a bracket operation, and an anchor map satisfying some compatibility conditions. The standard Courant algebroid associated to a manifold M is the quadruple $(TM \oplus T^*M, (\cdot,\cdot)_+, \{\cdot,\cdot\}, \rho)$, where ρ is the projection to the first summand, $(\cdot,\cdot)_+$ is the nondegenerate symmetric pairing given by

$$(X+\alpha, Y+\beta)_+ = i_X\beta + i_Y\alpha, \quad \forall\ X+\alpha, Y+\beta \in \mathfrak{X}^1(M)\oplus\Omega^1(M), \tag{1}$$

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and $\{\cdot,\cdot\}$ is the standard Dorfman bracket given by

$$\{X+\alpha, Y+\beta\} = [X,Y] + L_X\beta - i_Y d\alpha.$$

The notion of omni-Lie algebras was proposed by Weinstein in [W] to study the linearization of the standard Courant algebroid. Then it was studied from several aspects [KW, SZ, U]. An *omni-Lie algebra* associated to a vector space V is a triple $(\mathfrak{gl}(V)\oplus V, (\cdot,\cdot)_+, \{\cdot,\cdot\})$, where $(\cdot,\cdot)_+$ is the V-valued pairing given by

$$(A+u, B+v)_+ = Av + Bu, \quad \forall\ A+u, B+v \in \mathfrak{gl}(V)\oplus V, \tag{2}$$

and $\{\cdot,\cdot\}$ is the bilinear bracket operation given by

$$\{A+u, B+v\} = [A,B] + Av. \tag{3}$$

They are compatible in the sense that

$$(\{A+u,B+v\}, C+w)_+ + (B+v, \{A+u, C+w\})_+ = A(B+v, C+w)_+.$$

An omni-Lie algebra has several important properties:

- It is the linearization of the standard Courant algebroid; see [W] for more details.
- $(\mathfrak{gl}(V)\oplus V, \{\cdot,\cdot\})$ is a Leibniz algebra, i.e. the following equality holds:

  $$\{e_1,\{e_2,e_3\}\} = \{\{e_1,e_2\},e_3\} + \{e_2,\{e_1,e_3\}\}, \quad \forall\ e_1,e_2,e_3 \in \mathfrak{gl}(V)\oplus V.$$

  Thus, an omni-Lie algebra provides a fundamental example of Leibniz algebras, which could be used to study representations of Leibniz algebras; see [SL] for more details.
- Even though $(\mathfrak{gl}(V)\oplus V, \{\cdot,\cdot\})$ is not a Lie algebra, its Dirac structures given by the graphs of operators from V to $\mathfrak{gl}(V)$ characterize all Lie algebra structures on V. Here, by a Dirac structure, we mean a maximal isotropic subspace of $\mathfrak{gl}(V)\oplus V$ which is closed under the bracket operation.
- We can construct a Lie 2-algebra from an omni-Lie algebra; see [SZ] for more details.

In this paper, we introduce the notion of a nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_{\mathfrak{g}})$ associated to a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. We first study its algebraic properties. We show that it is a trivial deformation of the omni-Lie algebra, as well as the double of a matched pair of Leibniz algebras $\mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{g}$. This is the content of Section 2. In Section 3, we give a geometric explanation of the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_{\mathfrak{g}})$ and show that it is the linearization of the Courant algebroid $T\mathfrak{g}^*\oplus T^*_{\pi_{\mathfrak{g}}}\mathfrak{g}^*$, where $\pi_{\mathfrak{g}}$ is the Lie–Poisson structure on $\mathfrak{g}^*$. In Section 4, we show that Dirac structures of the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_{\mathfrak{g}})$ give rise to deformations of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. In Section 5, we construct several examples of Leibniz algebras using 2-term complexes of vector spaces and Lie algebra crossed modules.

2. Nonabelian omni-Lie algebras. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra and $\mathrm{ad}: \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ the induced adjoint map.

Definition 2.1. A nonabelian omni-Lie algebra associated to the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a triple $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_{\mathfrak{g}})$, where $(\cdot,\cdot)_+$ is the symmetric V-valued pairing given by (2) and $\{\cdot,\cdot\}_{\mathfrak{g}}$ is the bilinear bracket operation given by

$$\{A+u,B+v\}_{\mathfrak{g}}=[A,B]+[A,\mathrm{ad}_v]+[\mathrm{ad}_u,B]-\mathrm{ad}_{Av}+Av+[u,v]_{\mathfrak{g}}. \tag{4}$$

PROPOSITION 2.2. *With the above notations, $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},\{\cdot,\cdot\}_{\mathfrak{g}})$ is a Leibniz algebra. Furthermore, the pairing $(\cdot,\cdot)_+$ and the bracket $\{\cdot,\cdot\}_{\mathfrak{g}}$ are compatible in the sense that*

$$(\{e_1,e_2\}_{\mathfrak{g}},e_3)_++(e_2,\{e_1,e_3\}_{\mathfrak{g}})_+=\rho_{\mathfrak{g}}(e_1)(e_2,e_3)_+, \tag{5}$$

where $\rho_{\mathfrak{g}}:\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ is given by

$$\rho_{\mathfrak{g}}(A+u)=A+\mathrm{ad}_u. \tag{6}$$

Proof. One can prove that $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},\{\cdot,\cdot\}_{\mathfrak{g}})$ is a Leibniz algebra directly by a tedious computation. We will not do it here since in the next subsection, we will show that $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},\{\cdot,\cdot\}_{\mathfrak{g}})$ is a trivial deformation of the Leibniz algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},\{\cdot,\cdot\})$ via a Nijenhuis operator, and thus it is a Leibniz algebra.

For all $A+u,B+v,C+w\in\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}$, we have

$$\begin{aligned}&(\{A+u,B+v\}_{\mathfrak{g}},C+w)_++(B+v,\{A+u,C+w\}_{\mathfrak{g}})_+\\ =&[A,B]w+[A,\mathrm{ad}_v]w+[\mathrm{ad}_u,B]w-\mathrm{ad}_{Av}w+C(Av+[u,v]_{\mathfrak{g}})\\ &+[A,C]v+[A,\mathrm{ad}_w]v+[\mathrm{ad}_u,C]v-\mathrm{ad}_{Aw}v+B(Aw+[u,w]_{\mathfrak{g}})\\ =&ABw+[u,Bw]_{\mathfrak{g}}+ACv+[u,Cv]_{\mathfrak{g}}\\ =&\rho_{\mathfrak{g}}(A+u)(B+v,C+w)_+,\end{aligned}$$

which finishes the proof. ∎

REMARK 2.3. The quadruple $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},(\cdot,\cdot)_+,\{\cdot,\cdot\}_{\mathfrak{g}},\rho_{\mathfrak{g}})$ is an E-Courant algebroid in the sense of [CLS].

REMARK 2.4. Now we consider the skew-symmetric bracket $[\![\cdot,\cdot]\!]$ given by

$$[\![e_1,e_2]\!]=\frac{1}{2}(\{e_1,e_2\}_{\mathfrak{g}}-\{e_2,e_1\}_{\mathfrak{g}}).$$

So that we have

$$[\![A+u,B+v]\!]=[A,B]+[A,\mathrm{ad}_v]+[\mathrm{ad}_u,B]-\frac{1}{2}(\mathrm{ad}_{Av}-\mathrm{ad}_{Bu})+\frac{1}{2}(Av-Bu)+[u,v]_{\mathfrak{g}}. \tag{7}$$

This bracket $[\![\cdot,\cdot]\!]$ fails to be a Lie bracket, but it can be completed to a Lie 2-algebra. This Lie 2-algebra was found in [LSX] by the authors in the study of homotopy Poisson structures; see [LSX, Example 4.10] for more details.

2.1. Trivial deformations of an omni-Lie algebra. In this subsection, we show that a nonabelian omni-Lie algebra can be viewed as a trivial deformation of an omni-Lie algebra. For details of deformations of Leibniz algebras, see [CGM, K1].

Let $(\mathfrak{L},[\cdot,\cdot]_{\mathfrak{L}})$ be a Leibniz algebra. For an endomorphism N of $\mathfrak{L}$, define

$$[e_1,e_2]_N=[Ne_1,e_2]_{\mathfrak{L}}+[e_1,Ne_2]_{\mathfrak{L}}-N[e_1,e_2]_{\mathfrak{L}},$$

and set

$$TN(e_1,e_2)=[Ne_1,Ne_2]_{\mathfrak{L}}-N[e_1,e_2]_N.$$

The endomorphism N is called a *Nijenhuis operator* if $TN=0$.

Let $\omega : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}$ be a bilinear map. Consider a λ-parametrized family of bilinear operations

$$[e_1, e_2]_\lambda = [e_1, e_2]_{\mathfrak{L}} + \lambda\omega(e_1, e_2).$$

We say that ω generates a *deformation* of the Leibniz algebra $\mathfrak{L}$ if all the brackets $[\cdot,\cdot]_\lambda$ endow $\mathfrak{L}$ with Leibniz algebras. A deformation is said to be *trivial* if there exists a linear operator $N : \mathfrak{L} \to \mathfrak{L}$ such that for $T_\lambda = id + \lambda N$ we have

$$T_\lambda[e_1, e_2]_\lambda = [T_\lambda e_1, T_\lambda e_2]_{\mathfrak{L}}.$$

A Nijenhuis operator gives a trivial deformation of the Leibniz algebra $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$ [CGM].

PROPOSITION 2.5. *Let N be a Nijenhuis operator on the Leibniz algebra $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$. Then*

(1) *$(\mathfrak{L}, [\cdot,\cdot]_N)$ is a Leibniz algebra;*
(2) *N is a morphism of Leibniz algebras from $(\mathfrak{L}, [\cdot,\cdot]_N)$ to $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$;*
(3) *$(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}} + [\cdot,\cdot]_N)$ is a Leibniz algebra.*

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. Then we define a linear map $N : \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}$ by

$$N(A + u) = \mathrm{ad}_u. \tag{8}$$

LEMMA 2.6. *The linear map N given by* (8) *is a Nijenhuis operator on the Leibniz algebra $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \{\cdot,\cdot\})$, where the Leibniz bracket $\{\cdot,\cdot\}$ is given by* (3).

Proof. First by definition, we have

$$\begin{aligned}\{A+u, B+v\}_N &= \{N(A+u), B+v\} + \{A+u, N(B+v)\} - N\{A+u, B+v\}\\ &= [\mathrm{ad}_u, B] + [u,v]_{\mathfrak{g}} + [A, \mathrm{ad}_v] - \mathrm{ad}_{Av}.\end{aligned}$$

Hence it is clear that

$$N\{A+u, B+v\}_N = \mathrm{ad}_{[u,v]_{\mathfrak{g}}} = [\mathrm{ad}_u, \mathrm{ad}_v] = \{N(A+u), N(B+v)\},$$

which says that N is a Nijenhuis operator. ∎

It is easy to see that

$$\{A+u, B+v\}_{\mathfrak{g}} = \{A+u, B+v\} + \{A+u, B+v\}_N.$$

Therefore, by Proposition 2.5 and Lemma 2.6, we have

THEOREM 2.7. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. Then the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_{\mathfrak{g}})$ is a trivial deformation of the omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\})$. In particular, $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \{\cdot,\cdot\}_{\mathfrak{g}})$ is a Leibniz algebra.*

2.2. A matched pair of Leibniz algebras. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. We view both $\mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{g}$ as Leibniz algebras. In this subsection, we explain that the nonabelian omni-Lie algebra is actually the double of a matched pair of Leibniz algebras $\mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{g}$. First we recall that a *representation* of a Leibniz algebra $(\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}})$ [LP] is a triple (V, ρ^L, ρ^R), where V is a vector space equipped with two linear maps $\rho^L, \rho^R : \mathfrak{L} \to \mathfrak{gl}(V)$ such that the following equalities hold for $g, h \in \mathfrak{L}$:

$$\rho^L([g,h]_{\mathfrak{L}}) = [\rho^L(g), \rho^L(h)],\ \rho^R([g,h]_{\mathfrak{L}}) = [\rho^L(g), \rho^R(h)],\ \rho^R(h)\circ\rho^L(g) = -\rho^R(h)\circ\rho^R(g).$$

A pair $(\mathfrak{L}, \mathfrak{M})$ of two Leibniz algebras is called a *matched pair* [A] if there exists a representation (ρ_1^L, ρ_1^R) of $\mathfrak{L}$ on $\mathfrak{M}$ and a representation of (ρ_2^L, ρ_2^R) of $\mathfrak{M}$ on $\mathfrak{L}$ such that the identities

(1) $\rho_1^R(g)[x,y]_{\mathfrak{M}} = [x, \rho_1^R(g)y]_{\mathfrak{M}} - [y, \rho_1^R(g)x]_{\mathfrak{M}} + \rho_1^R(\rho_2^L(y)g)x - \rho_1^R(\rho_2^L(x)g)y;$
(2) $\rho_1^L(g)[x,y]_{\mathfrak{M}} = [\rho_1^L(g)x, y]_{\mathfrak{M}} + [x, \rho_1^L(g)y]_{\mathfrak{M}} + \rho_1^L(\rho_2^R(x)g)y + \rho_1^R(\rho_2^R(y)g)x;$
(3) $[\rho_1^L(g)x, y]_{\mathfrak{M}} + \rho_1^L(\rho_2^R(x)g)y + [\rho_1^R(g)x, y]_{\mathfrak{M}} + \rho_1^L(\rho_2^L(x)g)y = 0;$
(4) $\rho_2^R(x)[g,h]_{\mathfrak{L}} = [g, \rho_2^R(x)h]_{\mathfrak{L}} - [h, \rho_2^R(x)g]_{\mathfrak{L}} + \rho_2^R(\rho_1^L(h)x)g - \rho_2^R(\rho_1^L(g)x)h;$
(5) $\rho_2^L(x)[g,h]_{\mathfrak{L}} = [\rho_2^L(x)g, h]_{\mathfrak{L}} + [g, \rho_2^L(x)h]_{\mathfrak{L}} + \rho_2^L(\rho_1^R(g)x)h + \rho_2^R(\rho_1^R(h)x)g;$
(6) $[\rho_2^L(x)g, h]_{\mathfrak{L}} + \rho_2^L(\rho_1^R(g)x)h + [\rho_2^R(x)g, h]_{\mathfrak{L}} + \rho_2^L(\rho_1^L(g)x)h = 0$

hold for all $g, h \in \mathfrak{L}$ and $x, y \in \mathfrak{M}$.

LEMMA 2.8 ([A]). *Given a matched pair $(\mathfrak{L}, \mathfrak{M})$ of Leibniz algebras, there is a Leibniz algebra structure $\mathfrak{L} \bowtie \mathfrak{M}$ on the direct sum vector space $\mathfrak{L} \oplus \mathfrak{M}$ with bracket*

$$[g+x, h+y]_{\mathfrak{L}\bowtie\mathfrak{M}} = [g,h]_{\mathfrak{L}} + \rho_2^R(y)g + \rho_2^L(x)h + [x,y]_{\mathfrak{M}} + \rho_1^L(g)y + \rho_1^R(h)x.$$

Conversely, if $\mathfrak{L} \oplus \mathfrak{M}$ has a Leibniz algebra structure for which $\mathfrak{L}$ and $\mathfrak{M}$ are Leibniz subalgebras, then the representations defined by

$$[g,x]_{\mathfrak{L}\oplus\mathfrak{M}} = \rho_2^R(x)g + \rho_1^L(g)x, \qquad [x,g]_{\mathfrak{L}\oplus\mathfrak{M}} = \rho_2^L(x)g + \rho_1^R(g)x,$$

endow the couple $(\mathfrak{L}, \mathfrak{M})$ with the structure of a matched pair.

The following conclusion can be obtained by straightforward computation.

LEMMA 2.9. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. Then $(\mathfrak{g}, \rho_1^L = \mathrm{id}, \rho_1^{\mathrm{R}} = 0)$ is a representation of the Leibniz algebra $\mathfrak{gl}(\mathfrak{g})$ on $\mathfrak{g}$, and $(\mathfrak{gl}(\mathfrak{g}), \rho_2^L, \rho_2^R)$ is a representation of the Leibniz algebra $\mathfrak{g}$ on $\mathfrak{gl}(\mathfrak{g})$, where ρ_2^L, ρ_2^R are given by*

$$\rho_2^L(u)B = [\mathrm{ad}_u, B], \qquad \rho_2^R(u)B = [B, \mathrm{ad}_u] - \mathrm{ad}_{Bu}.$$

It is straightforward to prove that

$$\{A+u, B+v\}_{\mathfrak{g}} = [A,B] + \rho_2^R(v)A + \rho_2^L(u)B + \rho_1^L(A)v + \rho_1^R(B)u + [u,v]_{\mathfrak{g}}.$$

By Lemma 2.9 and the fact that $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \{\cdot,\cdot\}_{\mathfrak{g}})$ is a Leibniz algebra, we have

PROPOSITION 2.10. $(\mathfrak{g}, \rho_1^L = \mathrm{id}, \rho_1^{\mathrm{R}} = 0)$ *and* $(\mathfrak{gl}(\mathfrak{g}), \rho_2^L, \rho_2^R)$ *form a matched pair of Leibniz algebras with the Leibniz algebra* $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \{\cdot,\cdot\}_{\mathfrak{g}})$ *as the double.*

3. A linearization of the Courant algebroid $TM \oplus T^*_\pi M$. It is well-known that the omni-Lie algebra is a linearization of the standard Courant algebroid $TM \oplus T^*M$ [W]. In this section, we show that the nonabelian omni-Lie algebra is a linearization of the Courant algebroid $TM \oplus T^*_\pi M$ associated to a Poisson manifold (M, π).

Let (M, π) be a Poisson manifold, and denote by $\pi^\sharp$ the induced bundle map from T^*M to TM. It is well-known that $(T^*_\pi M, [\cdot,\cdot]_\pi, a_\pi)$ is a Lie algebroid, where the Lie bracket $[\cdot,\cdot]_\pi$ and the anchor a_π are given by

$$[\xi, \eta]_\pi = L_{\pi^\sharp \xi}\eta - L_{\pi^\sharp \eta}\xi - d\pi(\xi,\eta), \qquad a_\pi(\xi) = \pi^\sharp(\xi).$$

Then $(TM \oplus T^*_\pi M, (\cdot,\cdot)_+, \{\cdot,\cdot\}_\pi, \rho_\pi)$ is a Courant algebroid, where the pairing $(\cdot,\cdot)_+$ is given by (1), ρ_π is given by $\rho_\pi(X+\xi) = X + \pi^\sharp(\xi)$, and the Dorfman bracket $\{\cdot,\cdot\}_\pi$ is given by

$$\{X+\xi, Y+\eta\}_\pi = [X,Y] + L_X\eta - i_Y d\xi + L^\pi_\xi Y - i_\eta d_\pi X + [\xi,\eta]_\pi, \tag{9}$$

where L^π and d_π are the Lie derivative and the differential associated to the Lie algebroid $T^*_\pi M$.

The Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ yields a Lie–Poisson manifold $(\mathfrak{g}^*, \pi_\mathfrak{g})$. Then we obtain a Courant algebroid structure on $T\mathfrak{g}^* \oplus T^*_{\pi_\mathfrak{g}}\mathfrak{g}^*$. Denote the vector spaces of linear vector fields and constant 1-forms on $\mathfrak{g}^*$ by $\mathfrak{X}_{lin}(\mathfrak{g}^*)$ and $\Omega^1_{con}(\mathfrak{g}^*)$ respectively. It is quite obvious that $\mathfrak{X}_{lin}(\mathfrak{g}^*) \oplus \Omega^1_{con}(\mathfrak{g}^*) \cong \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}$. To make this explicit, for any $x \in \mathfrak{g}$, denote by l_x the corresponding linear function on $\mathfrak{g}^*$. Let $\{x^i\}$ be a basis of the vector space underlying $\mathfrak{g}$. Then $\{l_{x^i}\}$ forms a coordinate chart for $\mathfrak{g}^*$. So $\{\frac{\partial}{\partial l_{x^i}}\}$ constitutes a basis of vector fields on $\mathfrak{g}^*$ and $\{dl_{x^i}\}$ constitutes a basis of 1-forms on $\mathfrak{g}^*$. For $A \in \mathfrak{gl}(\mathfrak{g})$, we get a linear vector field $\hat{A} = \sum_j l_{A(x^j)} \frac{\partial}{\partial l_{x^j}}$ on $\mathfrak{g}^*$. Also $u = \sum_i u_i x^i \in \mathfrak{g}$ defines a constant 1-form $\hat{u} = dl_u = \sum_i u_i dl_{x^i}$ on $\mathfrak{g}^*$. Moreover, we have

$$\pi_\mathfrak{g} = \frac{1}{2}\sum_{i,j} l_{[x^i,x^j]_\mathfrak{g}} \frac{\partial}{\partial l_{x^i}} \wedge \frac{\partial}{\partial l_{x^j}}.$$

Define $\Phi : \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g} \to \mathfrak{X}_{lin}(\mathfrak{g}^*) \oplus \Omega^1_{con}(\mathfrak{g}^*)$ by

$$\Phi(A+u) = \hat{A} + \hat{u}.$$

Obviously, Φ is an isomorphism between vector spaces.

THEOREM 3.1. *The nonabelian omni-Lie algebra* $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, (\cdot,\cdot)_+, \{\cdot,\cdot\}_\mathfrak{g})$ *is induced from the Courant algebroid* $(T\mathfrak{g}^* \oplus T^*_{\pi_\mathfrak{g}}\mathfrak{g}^*, (\cdot,\cdot)_+, \{\cdot,\cdot\}_{\pi_\mathfrak{g}}, \rho_{\pi_\mathfrak{g}})$ *via the restriction to* $\mathfrak{X}_{lin}(\mathfrak{g}^*) \oplus \Omega^1_{con}(\mathfrak{g}^*)$. *More precisely, we have*

$$\begin{aligned}
(\Phi(A+u), \Phi(B+v))_+ &= l_{(A+u,B+v)_+},\\
\{\Phi(A+u), \Phi(B+v)\}_{\pi_\mathfrak{g}} &= \Phi\{A+u, B+v\}_\mathfrak{g},\\
\rho_{\pi_\mathfrak{g}}(\Phi(A+u)) &= \Phi(\rho_\mathfrak{g}(A+u)).
\end{aligned}$$

We first prove a lemma.

LEMMA 3.2. *With the above notations, for all* $A, B \in \mathfrak{gl}(\mathfrak{g})$ *and* $u, v \in \mathfrak{g}$, *we have*

$$(\hat{A}, \hat{u})_+ = l_{Au}, \tag{10}$$

$$\pi^\sharp_\mathfrak{g}(\hat{u}) = \widehat{\mathrm{ad}_u}, \tag{11}$$

$$L_{\hat{A}}\hat{u} = \widehat{Au}, \tag{12}$$

$$[\hat{u}, \hat{v}]_{\pi_\mathfrak{g}} = \widehat{[u,v]_\mathfrak{g}}, \tag{13}$$

$$i_{\hat{v}} d_{\pi_\mathfrak{g}} \hat{A} = \widehat{[\mathrm{ad}_v, A]} + \widehat{\mathrm{ad}_{Av}}, \tag{14}$$

$$L^{\pi_\mathfrak{g}}_{\hat{u}} \hat{B} = \widehat{[\mathrm{ad}_u, B]}. \tag{15}$$

Proof. (10) follows from

$$(\hat{A}, \hat{u})_+ = \Big(\sum_j l_{A(x^j)} \frac{\partial}{\partial l_{x^j}}, \sum_i u_i dl_{x^i}\Big)_+ = \sum_i u_i l_{A(x^i)} = l_{Au}.$$

(11) follows from

$$\pi_{\mathfrak{g}}^{\sharp}(\hat{u})=\sum_{i,j}u_i l_{[x^i,x^j]_{\mathfrak{g}}}\frac{\partial}{\partial l_{x^j}}=\sum_j l_{\mathrm{ad}_u(x^j)}\frac{\partial}{\partial l_{x^j}}=\widehat{\mathrm{ad}_u}.$$

By (10) and the fact that $d\hat{u}=0$, we have

$$L_{\hat{A}}\hat{u}=i_{\hat{A}}d\hat{u}+di_{\hat{A}}\hat{u}=dl_{Au}=\widehat{Au},$$

which says that (12) holds. By (11) and (12), we have

$$[\hat{u},\hat{v}]_{\pi_{\mathfrak{g}}}=L_{\pi_{\mathfrak{g}}^{\sharp}(\hat{u})}\hat{v}-L_{\pi_{\mathfrak{g}}^{\sharp}(\hat{v})}\hat{u}-d\pi_{\mathfrak{g}}(\hat{u},\hat{v})=\widehat{[u,v]_{\mathfrak{g}}}-\widehat{[v,u]_{\mathfrak{g}}}-\widehat{[u,v]_{\mathfrak{g}}}=\widehat{[u,v]_{\mathfrak{g}}},$$

so (13) holds. Then we have

$$\begin{aligned}(i_{\hat{v}}d_{\pi_{\mathfrak{g}}}\hat{A},\hat{u})_+&=d_{\pi_{\mathfrak{g}}}\hat{A}(\hat{v},\hat{u})=\pi_{\mathfrak{g}}^{\sharp}(\hat{v})(\hat{A},\hat{u})_+-\pi_{\mathfrak{g}}^{\sharp}(\hat{u})(\hat{A},\hat{v})_+-(\hat{A},[\hat{v},\hat{u}]_{\pi_{\mathfrak{g}}})_+\\&=l_{[v,Au]_{\mathfrak{g}}}-l_{[u,Av]_{\mathfrak{g}}}-l_{A[v,u]_{\mathfrak{g}}},\end{aligned}$$

which implies that $i_{\hat{v}}d_{\pi_{\mathfrak{g}}}\hat{A}=\widehat{[\mathrm{ad}_v,A]}+\widehat{\mathrm{ad}_{Av}}$, i.e. (14) holds. Moreover, we have

$$(L_{\hat{u}}^{\pi_{\mathfrak{g}}}\hat{B},\hat{v})_+=\pi_{\mathfrak{g}}^{\sharp}(\hat{u})(\hat{B},\hat{v})_+-(\hat{B},[\hat{u},\hat{v}]_{\pi_{\mathfrak{g}}})_+=l_{[\mathrm{ad}_u,B]v},$$

which implies that $L_{\hat{u}}^{\pi_{\mathfrak{g}}}\hat{B}=\widehat{[\mathrm{ad}_u,B]}$, i.e. (15) holds. This ends the proof. ■

Proof of Theorem 3.1. By (10), we have

$$(\Phi(A+u),\Phi(B+v))_+=(\hat{A},\hat{v})_++(\hat{u},\hat{B})_+=l_{Av}+l_{Bu}=l_{(A+u,B+v)_+}.$$

By (11)–(15), we deduce that

$$\begin{aligned}\{\hat{A}+\hat{u},\hat{B}+\hat{v}\}_{\pi_{\mathfrak{g}}}&=[\hat{A},\hat{B}]+L_{\hat{u}}^{\pi_{\mathfrak{g}}}\hat{B}-i_{\hat{v}}d_{\pi_{\mathfrak{g}}}\hat{A}+[\hat{u},\hat{v}]_{\pi_{\mathfrak{g}}}+L_{\hat{A}}\hat{v}-i_{\hat{B}}d\hat{u}\\&=\widehat{[A,B]}+\widehat{[\mathrm{ad}_u,B]}+\widehat{[A,\mathrm{ad}_v]}-\widehat{\mathrm{ad}_{Av}}+\widehat{[u,v]_{\mathfrak{g}}}+\widehat{Av}\\&=\Phi\{A+u,B+v\}_{\mathfrak{g}}.\end{aligned}$$

Finally, by (11), we have

$$\rho_{\pi_{\mathfrak{g}}}(\Phi(A+u))=\hat{A}+\pi_{\mathfrak{g}}^{\sharp}(\hat{u})=\hat{A}+\widehat{\mathrm{ad}_u}=\Phi(\rho_{\mathfrak{g}}(A+u)).$$

The proof is completed.

4. Dirac structures. In this section we study Dirac structures of the nonabelian omni-Lie algebra and give our algebraic application.

Recall that for a vector space V, there is a graded Lie bracket on the vector space $\oplus_k\mathrm{Hom}(\wedge^k V,V)$, which is known as the Nijenhuis–Richardson bracket [NR]. In particular, for $\omega\in\mathrm{Hom}(\wedge^2V,V)$, we have

$$[\omega,\omega](e_1,e_2,e_3)=2\big(\omega(e_1,\omega(e_2,e_3))+\omega(e_3,\omega(e_1,e_2))+\omega(e_2,\omega(e_3,e_1))\big),\tag{16}$$

for all $e_1,e_2,e_3\in V$. Thus, ω defines a Lie algebra structure on V if and only if $[\omega,\omega]=0$.

DEFINITION 4.1. A *Dirac structure* of the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},(\cdot,\cdot)_+,\{\cdot,\cdot\}_{\mathfrak{g}})$ is a maximal isotropic subspace $\mathbb{L}$, i.e. $\mathbb{L}=\mathbb{L}^{\perp}$, which is closed under the bracket $\{\cdot,\cdot\}_{\mathfrak{g}}$.

For any bilinear map $F:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, define $\mathrm{ad}^F:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ by $\mathrm{ad}^F(u)=F(u,\cdot)$. The following theorem says that Dirac structures of the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},(\cdot,\cdot)_+,\{\cdot,\cdot\}_\mathfrak{g})$ could give deformations of the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$.

THEOREM 4.2. *Let $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra and $F:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ a bilinear map. The graph of ad^F, which we denote by $\mathcal{G}_F$, is a Dirac structure of the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},(\cdot,\cdot)_+,\{\cdot,\cdot\}_\mathfrak{g})$ if and only if F is skewsymmetric and satisfies the Maurer–Cartan equation*

$$dF+\frac{1}{2}[F,F]=0,$$

where d is the coboundary operator on Lie algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ with the coefficients in the adjoint representation and $[\cdot,\cdot]$ is given by (16). *In other words, $[\cdot,\cdot]_\mathfrak{g}+F$ defines another Lie algebra structure on $\mathfrak{g}$.*

Proof. Obviously, F is skew-symmetric if and only if $\mathcal{G}_F$ is maximal isotropic. Moreover, for any skew-symmetric map F, we have

$$\{\mathrm{ad}^F_u+u,\mathrm{ad}^F_v+v\}_\mathfrak{g}=[\mathrm{ad}^F_u,\mathrm{ad}^F_v]+[\mathrm{ad}^F_u,\mathrm{ad}_v]+[\mathrm{ad}_u,\mathrm{ad}^F_v]-\mathrm{ad}_{F(u,v)}+F(u,v)+[u,v]_\mathfrak{g}.$$

Thus, $\mathcal{G}_F$ is a Dirac structure if and only if F is skew-symmetric and the following equality is satisfied:

$$[\mathrm{ad}^F_u,\mathrm{ad}^F_v]+[\mathrm{ad}^F_u,\mathrm{ad}_v]+[\mathrm{ad}_u,\mathrm{ad}^F_v]-\mathrm{ad}_{F(u,v)}=\mathrm{ad}^F_{F(u,v)+[u,v]_\mathfrak{g}}.$$

Applying this to an element $w\in\mathfrak{g}$, we obtain

$$\begin{aligned}&F(u,F(v,w))-F(v,F(u,w))+F(u,[v,w]_\mathfrak{g})-[v,F(u,w)]_\mathfrak{g}+[u,F(v,w)]_\mathfrak{g}\\&-F(v,[u,w]_\mathfrak{g})-[F(u,v),w]_\mathfrak{g}=F(F(u,v),w)+F([u,v]_\mathfrak{g},w),\end{aligned}$$

which implies that $dF+\frac{1}{2}[F,F]=0$. ∎

5. Examples of Leibniz algebras. Recently, people pay a lot of attention to Leibniz algebras; see [FMM, HPL] for more information. Motivated by the definition of (non-abelian) omni-Lie algebras and the work in [LSX], we give several interesting examples of Leibniz algebras in this section.

EXAMPLE 5.1. Let $\phi:V\to W$ be a linear map of vector spaces. Define a bilinear operation $\{\cdot,\cdot\}_\phi$ on $\mathrm{Hom}(W,V)\oplus W$ by

$$\{A+u,B+v\}_\phi=A\circ\phi\circ B-B\circ\phi\circ A+\phi(Av),\qquad\forall\ A,B\in\mathrm{Hom}(W,V),\ u,v\in W.$$

It is straightforward to check that $(\mathrm{Hom}(W,V)\oplus W,\{\cdot,\cdot\}_\phi)$ is a Leibniz algebra.

Let $\mathfrak{m},\mathfrak{g}$ be Lie algebras. We call $(\mathfrak{m}\xrightarrow{\phi}\mathfrak{g},\triangleright)$ a *Lie algebra crossed module* [G] if $\phi:\mathfrak{m}\to\mathfrak{g}$ is a Lie algebra homomorphism and $\triangleright:\mathfrak{g}\times\mathfrak{m}\to\mathfrak{m}$ is an action of $\mathfrak{g}$ on $\mathfrak{m}$ by derivations such that

$$[a,b]_\mathfrak{m}=\phi(a)\triangleright b,\qquad\phi(u\triangleright a)=[u,\phi(a)]_\mathfrak{g},\qquad\forall\ a,b\in\mathfrak{m},u\in\mathfrak{g}.$$

EXAMPLE 5.2. Let $(\mathfrak{m}\xrightarrow{\phi}\mathfrak{g},\triangleright)$ be a Lie algebra crossed module. We get a Leibniz algebra structure on $\mathrm{Hom}(\mathfrak{g},\mathfrak{m})\oplus\mathfrak{g}$ with the bracket given by

$$\{A+u,B+v\}_\phi = A\circ\phi\circ B - B\circ\phi\circ A + A\circ \mathrm{ad}^0_v - \mathrm{ad}^1_v\circ A + \cdot\triangleright Av \\ - B\circ\mathrm{ad}^0_u + \mathrm{ad}^1_u\circ B + \phi(Av) + [u,v]_\mathfrak{g},$$

for all $A,B\in\mathrm{Hom}(\mathfrak{g},\mathfrak{m}), u,v\in\mathfrak{g}$, where $\mathrm{ad}^0_v\in\mathrm{End}(\mathfrak{g})$ and $\mathrm{ad}^1_v\in\mathrm{End}(\mathfrak{m})$ are given by $\mathrm{ad}^0_v(u)=[v,u]_\mathfrak{g}$ and $\mathrm{ad}^1_v(a)=v\triangleright a$ respectively and $\cdot\triangleright Av\in\mathrm{Hom}(\mathfrak{g},\mathfrak{m})$ is defined by $(\cdot\triangleright Av)(u)=u\triangleright Av$.

Example 5.3. The subalgebras of the (nonabelian) omni-Lie algebra give examples of Leibniz algebras. For example, we have Leibniz algebras $\mathfrak{sl}(V)\oplus V$, $\mathfrak{so}(V)\oplus V$ for a vector space V as Leibniz subalgebras of $\mathfrak{gl}(V)\oplus V$, and $\mathfrak{sl}(\mathfrak{g})\oplus\mathfrak{g}$, $\mathfrak{so}(\mathfrak{g})\oplus\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$ as Leibniz subalgebras of $\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g}$.

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