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GEOMETRY OF VARIATIONAL PARTIAL DIFFERENTIAL EQUATIONS AND HAMILTONIAN SYSTEMS

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Abstract. This is a survey of Hamiltonian field theory in jet bundles with a particular stress on geometric structures associated with Euler–Lagrange and Hamilton equations. Our approach is based on the concept of *Lepage manifold*, a fibred manifold endowed with a closed Lepage $(n+1)$-form where n is the dimension of the base manifold, which serves as a background for formulation of a covariant Hamilton field theory related to an Euler–Lagrange form (representing variational equations), hence to the class of equivalent Lagrangians. Compared with conventional approaches, dependent upon choice of a particular Lagrangian, this is an important distinction which enables us to enlarge substantially the family of field Lagrangians which possess a canonical multisymplectic Hamiltonian formulation on the affine dual of the jet bundle, and can thus be treated without using the Dirac constraint formalism. Within the Hamiltonian theory on Lepage manifolds, the concepts of regularity and Legendre transformation are revisited and extended, and new formulas for the Hamiltonian and momenta are obtained. In this paper we focus on De Donder–Hamilton equations which arise from "short" (at most 2-contact) Lepage $(n+1)$-forms. To illustrate the results we present regular Lepage manifolds (and the corresponding Hamiltonian formulation) for the Einstein and Maxwell equations.

1. Introduction. The present paper is based on a lecture delivered at the international conference "Geometry of Jets and Fields" at Będlewo in May 2015. The aim is to survey the current state of the covariant Hamiltonian field theory. We shall leave aside the evolution Hamiltonian field theory which is another important direction in generalizing classical Hamiltonian mechanics to several independent variables, and well-suited to study

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the initial value problem (the interested reader is referred to the books [20] and [5] and references therein), as well as we shall not discuss the relationship between the evolution and the covariant approach and we refer to Gotay [16].

Foundations of the covariant Hamiltonian formalism in the calculus of variations have been built since late 1970's inspired mainly by works of De Donder and Weyl. There is a vast literature of miscellaneous approaches to these questions which cannot be discussed in more detail here, but can roughly be characterized as representing two overlapping streams: those based on multisymplectic forms, and those based on Lepage forms and their generalizations (to see this diversity see, for instance, [2], [3], [11], [13], [15], [18], [22], [29] and references therein). A multivariable generalization of the symplectic setting for Hamiltonian mechanics has been successfully achieved in terms of affine duals of jet bundles. In contrast to mechanics on tangent bundles, in jet bundles one has two dually related manifolds, the affine dual equipped with the canonical multisymplectic form, and the reduced dual (see e.g. [4], [3], [8], [40], [39]). Within the conventional approach, Hamilton equations are associated *with a Lagrangian* via Legendre map where momenta and the Hamiltonian are defined by means of the Lagrangian. This approach works well in classical mechanics, however, if one considers multiple variational integrals, or even higher-order variational problems (with Ostrogradskii and De Donder higher-order generalization of the formula for the Legengre map), there appear unexpected complications: The Hamilton formulation is non-unique—the higher-order Poincaré–Cartan form is generically not global, and, if 'globalized', it is non-unique, moreover, a Lagrangian has many Lepage equivalents which differ by order of contactness (the Poincaré–Cartan form is the 'shortest' among them). Equivalent Lagrangians (with the same Euler–Lagrange expressions) may essentially differ in regularity/degeneracy properties: one may be regular and thus admit a multisymplectic formulation and Hamilton equations equivalent to the Euler–Lagrange equations, while some other one may be degenerate, hence the corresponding Hamilton equations are not equivalent to the (same!) Euler–Lagrange equations and, moreover, have to be treated with the help of the Dirac constraint formalism. These 'strange' properties can be understood and disappear if Hamiltonian formulation is related to *Euler–Lagrange form*, representing variational equations, hence to *the class of equivalent Lagrangians*, rather than to a particular Lagrangian. The possibility to achieve such a formulation is based on a fundamental property of variational equations, namely, that they can be globally extended via adding contact forms of higher degrees to *closed* $(n+1)$-forms, called *Lepage* $(n+1)$*-forms*. Formalizing the procedure one is led to the concept of *Lepage manifold*, a fibred manifold endowed with a closed Lepage $(n+1)$-form where n is the dimension of the base manifold, which is a well-designed background for variational problems in full generality. Foundations of this new framework were laid in [25] and [27], [31] (see also [26]) for higher–order mechanics, and [28], [29], [40], [39], [38] for field theory. Compared with conventional approaches, dependent upon a choice of a particular Lagrangian, this makes an important distinction which enables one to enlarge substantially the family of field Lagrangians which possess a canonical multisymplectic Hamiltonian formulation on the affine dual of the jet bundle, and can thus be treated without using the Dirac constraint formalism.

In this paper we first introduce the canonical Hamiltonian field theory on affine duals. We also present recent results on the structure of solutions of De Donder–Hamilton equations [39] which can be understood in terms of Ehresmann connections. Section 3 is a brief reminder on conventional De Donder–Hamilton field theory based on the concept of an individual Lagrangian. In the last section we then present Lepage manifolds, Euler–Langrange and Hamilton equations modelled on Lepage manifolds, and the revisited concepts of regularity and Legendre transformation, including explicit new formulas for Hamiltonian and momenta on a Lepage manifold [38]. To illustrate the results we construct regular Lepage manifolds and the corresponding Hamiltonian formulation avoiding the use of the Dirac constraint formalism for the Einstein and Maxwell equations.

2. Hamiltonian systems on affine duals. Throughout the paper we shall assume that all manifolds and mappings are smooth, and use the standard summation convention whenever appropriate.

2.1. Jets and contact forms. We shall consider a fibred manifold $\pi : Y \to X$, $\dim X = n > 1$ and $\dim Y = m + n$, and its jet prolongations, $\pi_r : J^r Y \to X$. We assume that X is orientable. Local fibred coordinates on Y are denoted by (x^i, y^σ) where $1 \le i \le n$ and $1 \le \sigma \le m$, and we put

$$\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\,, \quad \omega_j = i_{\partial/\partial x^j}\omega_0\,. \tag{1}$$

We shall often work just with the first and second prolongation of π. The associated coordinates on J^1Y and J^2Y are denoted by $(x^i, y^\sigma, y^\sigma_i)$ and $(x^i, y^\sigma, y^\sigma_i, y^\sigma_{ij})$, $i \le j$, respectively. If appropriate, we shall use all the functions y^σ_{ij} where $1 \le i, j \le n$, defined with the help of symmetry of the second derivatives. For convenience we shall often denote $J^0Y = Y$. The natural projections $J^rY \to J^sY$ are denoted by $\pi_{r,s}$.

Typically, (x^i) are local coordinates on a space-time manifold, (y^σ) are components of fields and $y^\sigma_i, y^\sigma_{ij}$ are their first and second derivatives. Often π is a vector bundle, but generally it is a surjective submersion, like, for example, in general relativity, where π is a bundle of metrics (regular symmetric covariant 2-tensors) over a space-time X.

Jet bundles inherit a canonical contact structure. A differential form η on J^rY, $r \ge 1$, is called *contact* if it vanishes on prolongations of sections of π. The ideal of contact forms on J^rY is generated by contact 1-forms of order r and their exterior derivatives. In particular, contact 1-forms on J^rY annihilate a horizontal distribution $\mathcal{C}_r$, called *Cartan distribution.* We shall need $\mathcal{C}_1$ and $\mathcal{C}_2$, locally annihilated by contact 1-forms ω^σ and $\omega^\sigma, \omega^\sigma_i$, respectively, where

$$\omega^\sigma = dy^\sigma - y^\sigma_j dx^j, \quad \omega^\sigma_i = dy^\sigma_i - y^\sigma_{ij} dx^j\,. \tag{2}$$

A contact q-form η is called k-contact if for every vertical vector field ξ the contraction $i_\xi\eta$ is $(k-1)$-contact, where $k = 1, 2, \dots q$. Here a 0-contact form, also called *horizontal*, is such that $i_\xi\eta = 0$ whenever ξ is a vertical vector field. Every q-form η on J^rY, if lifted to $J^{r+1}Y$, has a unique global decomposition

$$\pi^*_{r+1,r}\eta = h\eta + p_1\eta + \cdots + p_q\eta \tag{3}$$

where $h\eta$ is horizontal and $p_k\eta$ is a k-contact q-form ($1 \leq k \leq q$) on $J^{r+1}Y$, respectively. Note that for $q \geq n$ we have $h\eta = p_1\eta = \cdots = p_{q-n-1}\eta = 0$.

Horizontal n-forms on J^rY (where $n = \dim X$, and $r \geq 1$) play a fundamental role in the calculus of variations, and are called *Lagrangians* of order r.

2.2. Affine duals of jet bundles. The manifold J^1Y is also the total space of an affine bundle $\pi_{1,0} : J^1Y \to Y$ modelled on the vector bundle $V_XY \otimes \pi^*T^*X \to Y$, where $V_XY \to Y$ is the bundle of vectors tangent to Y and vertical over X. We denote the fibre over $y \in Y$ by J^1_yY (so that $J^1_yY = \pi^{-1}_{1,0}(y)$) and let $x = \pi(y)$.

As an affine space of dimension m, every fibre J^1_yY is associated with a vector space W_y of dimension $m+1$, its *extended dual*, whose elements are the affine functions $J^1_yY \to \mathbb{R}$. Applying this construction to all the fibres of the affine bundle $\pi_{1,0} : J^1Y \to Y$ gives the manifold $\bigcup_y W_y = J^\dagger Y$, called *the extended dual of* J^1Y. If $\phi \in J^\dagger Y$ and $\phi_y : J^1_yY \to \mathbb{R}$, we put $\pi^\dagger_{1,0}(\phi) = y$, and then $\pi^\dagger_{1,0} : J^\dagger Y \to Y$ becomes a vector bundle. With our notations as above, we shall write $J^\dagger_yY = (\pi^\dagger_{1,0})^{-1}(y) = W_y$. The manifold $J^\dagger Y$ is equipped with local coordinates related to fibred coordinates on Y. Let (x^i, y^σ) be a chart around y, and $y^\sigma_{i(y)}$ denote the restrictions of the induced coordinate functions y^σ_i to the fibre J^1_yY. The $y^\sigma_{i(y)}$ are affine functions, hence elements of $J^\dagger_yY$. Together with the constant function $1_y : J^1_yY \to \mathbb{R}$, they form a basis $(1_y, y^\sigma_{i(y)})$ of the vector space $J^\dagger_yY$. Denote the dual basis of $(J^\dagger_yY)^*$ by $(P_{(y)}, P^i_{\sigma(y)})$. The functions $P_{(y)}, P^i_{\sigma(y)}$ on $J^\dagger_yY$ then give rise to smooth local functions P, P^i_σ on $J^\dagger Y$ which together with x^i, y^σ form a local coordinate system $(x^i, y^\sigma, P, P^i_\sigma)$ on $J^\dagger Y$. We can see that if $\phi \in J^\dagger Y$ and $\phi_y = f_0 + f^i_\sigma y^\sigma_{i(y)}$ then (f_0, f^i_σ) are the components of the vector ϕ_y in the basis $(1_y, y^\sigma_{i(y)})$, hence $f_0 = P_{(y)}(\phi_y)$, and $f^i_\sigma = P^i_{\sigma(y)}(\phi_y)$.

With a choice of a fixed volume element on X, the manifold $J^\dagger Y$ is equipped with a canonical multisymplectic $(n+1)$-form Ω which makes $(J^\dagger Y, \Omega)$ a multi-variable generalization of the canonical symplectic structure on the cotangent bundle $T^*M \to M$, used in mechanics (where M serves as a configuration manifold). To remind how this arises ([39], [40]), let us consider a chart $(\bar{U}, (x^i, y^\sigma, P, P^i_\sigma))$ on $J^\dagger Y$ and the associated chart $(\bar{U}_0, (x^i))$ on X, and suppose there is given a volume form vol on X such that $\mathrm{vol}|_{\bar{U}_0} = \omega_0 = dx^1 \wedge \cdots \wedge dx^n$. For every n-form η on Y belonging to the subbundle Λ^n_1Y of Λ^nY such that $i_\xi i_\zeta \eta = 0$ whenever $\xi, \zeta \in V_XY$, we have $\eta = \eta_0\omega_0 + \eta^i_\sigma dy^\sigma \wedge \omega_i$, and for any section γ of $\pi : Y \to X$ such that $y = \gamma(x) \in \pi^\dagger_{1,0}(\bar{U})$,

$$\gamma^*\eta(x) = \Big((\eta_0 \circ \gamma)(x) + (\eta^i_\sigma \circ \gamma)(x)\Big(\frac{\partial \gamma^\sigma}{\partial x^i}\Big)_x\Big)\omega_0(x) = (\eta_0 + \eta^i_\sigma y^\sigma_i)(j^1_x\gamma)\,\omega_0(x)\,, \tag{4}$$

yielding a local fiberwise affine function $\phi = \eta_0 + \eta^i_\sigma y^\sigma_i$ on J^1Y, hence an element of $J^\dagger Y$. With the volume form fixed, the above correspondence between elements of Λ^n_1Y and $J^\dagger Y$ is a linear isomorphism of the fibres, under which the basis $(1_y, y^\sigma_{i(y)})$ of $J^\dagger_yY$ corresponds to the basis $(\omega_0(y), dy^\sigma \wedge \omega_i(y))$ of $(\Lambda^n_1)_yY$, and gives rise to a diffeomorphism $F : J^\dagger Y \to \Lambda^n_1Y$ [1]. Now, for every $\phi \in J^\dagger Y$ and every system of $\pi^\dagger_{1,0}$-projectable vector

[1]Note that without a fixed volume element on X we do not have a mapping assigning to every affine function $\phi \in J^\dagger Y$, $\phi_y = f_0 + f^i_\sigma y^\sigma_{i(y)}$, an element of Λ^n_1Y.

fields $\xi_1, \dots, \xi_n$ on $J^\dagger Y$ we can put

$$\Theta(\phi)(\xi_1, \dots, \xi_n) = F(\phi)(T\pi^\dagger_{1,0}.\xi_1, \dots, T\pi^\dagger_{1,0}.\xi_n)\,, \tag{5}$$

and thus obtain a canonical n-form Θ on $J^\dagger Y$. Then $\Omega = d\Theta$ is the desired canonical multisymplectic $(n+1)$-form.

In the given coordinates, if we denote $\phi = \eta_0 + \eta^i_\sigma y^\sigma_i$ we get $F(\phi) = \eta_0\omega_0 + \eta^i_\sigma dy^\sigma \wedge \omega_i$, and as we have seen above, $\eta_0 = P(\phi)$ and $\eta^i_\sigma = P^i_\sigma(\phi)$. Without loss of generality we can restrict ourselves to the subbundle of $\Lambda^n(J^\dagger Y)$ spanned by $(\omega_0, dy^\sigma \wedge \omega_i)$, hence Θ must take a coordinate expression $\Theta = a\omega_0 + a^i_\sigma dy^\sigma \wedge \omega_i$ for some functions a, a^i_σ on U, and we finally obtain $a(\phi) = P(\phi)$, $a^i_\sigma(\phi) = P^i_\sigma(\phi)$. This means that the canonical forms on $J^\dagger Y$ become

$$\Theta = P\omega_0 + P^i_\sigma dy^\sigma \wedge \omega_i\,, \qquad \Omega = dP \wedge \omega_0 + dP^i_\sigma \wedge dy^\sigma \wedge \omega_i\,. \tag{6}$$

We observe that the dimension of $J^\dagger Y$ is greater than the dimension of $J^1 Y$, hence $J^\dagger Y$ canot serve as a 'true dual space' for $J^1 Y$. This difficulty is avoided by taking the quotient of $J^\dagger Y$ by functions constant on fibres of $J^1 Y$. More precisely, for every y, the vector space $J^\dagger_y Y$ contains a distinguished subspace of constant maps, and we may consider the quotient space, denoted $J^\star_y Y$. Setting $J^\star Y = \bigcup_y J^\star_y Y$ gives another vector bundle called the *reduced dual.* We shall let $\rho : J^\dagger Y \to J^\star Y$ denote the quotient map, $\pi^\star_{1,0} : J^\star Y \to Y$ the induced map, and τ the composition $\pi \circ \pi^\star_{1,0}$. We also observe that local coordinates $(x^i, y^\sigma, P, P^i_\sigma)$ on $J^\dagger Y$ induce local coordinates $(x^i, y^\sigma, P^i_\sigma)$ on $J^\star Y$, and every local section h of ρ gives rise to a *local 'Hamiltonian' form* $\Omega_h = h^*\Omega$ on $J^\star Y$. In coordinates, if $P \circ h = -\mathcal{H}$, we get

$$\Omega_h = -d\mathcal{H} \wedge \omega_0 + dP^i_\sigma \wedge dy^\sigma \wedge \omega_i\,. \tag{7}$$

We call h a *Hamiltonian section*, and its component $\mathcal{H}$ a *Hamiltonian.*

Given a Hamiltonian section on $U \subset J^\star Y$, the bundle $(\tau|_U, \Omega_h)$ represents a framework for a regular Hamiltonian field theory[2]. Instead of a Hamiltonian vector field (or, better of the Pfaffian system annihilating a Hamiltonian vector field), describing dynamics of a mechanical system, we now obtain rather an exterior differential system generated by n-forms,

$$\mathcal{D}_h = \{i_\xi \Omega_h\}, \quad \xi \text{ runs over vertical vector fields on } U \subset J^\star Y\,. \tag{8}$$

Integral sections of $\mathcal{D}_h$ (being local sections $\psi : X \to J^\star Y$) satisfy the following system of first order PDEs, called *De Donder–Hamilton equations*:

$$\frac{\partial(y^\sigma \circ \psi)}{\partial x^i} = \frac{\partial \mathcal{H}}{\partial P^i_\sigma} \circ \psi, \qquad \frac{\partial(P^i_\sigma \circ \psi)}{\partial x^i} = -\frac{\partial \mathcal{H}}{\partial y^\sigma} \circ \psi\,. \tag{9}$$

The name refers to De Donder [7] who obtained these equations as a multivariable generalization of Hamilton equations of classical mechanics.

[2]We notice that this is *not* a straightforward generalization of the symplectic setting for time-independent mechanics. On the other hand, it is a straightforward generalization of (generally) time-dependent regular Hamiltonian mechanics, where in place of the canonical $(n+1)$-form Ω on $J^\dagger Y$, and a related local form Ω_h one has the canonical symplectic form $\Omega = dP \wedge dt + dP_\sigma \wedge dq^\sigma$ on $J^\dagger(\mathbb{R} \times M) \sim T^*(\mathbb{R} \times M)$ over $\mathbb{R}$, and a local cosymplectic form $\Omega_h = -d\mathcal{H} \wedge dt + dP_\sigma \wedge dq^\sigma$ on $J^\star(\mathbb{R} \times M) \to \mathbb{R}$ (for more details see e.g. [35], [31]).

It is worth noticing that equations (9) satisfy the Helmholtz conditions (for first order PDEs), meaning that they are variational; the corresponding variational principle concerns the fibred manifold τ, so that the Lagrangian κ is a local n-form on $J^1(J^\star Y)$ (affine in the first derivatives). We can see that $\kappa = \tilde{h}\Theta_h$, where $\Theta_h = h^*\Theta$, and $\tilde{h}$ denotes the horizontalization with respect to the projection τ. Due to the fact that solutions of De Donder–Hamilton equations are extremals of a Lagrangian, we speak about *De Donder–Hamilton extremals.*

2.3. De Donder–Hamilton equations and connections. The use of Ehresmann connections in classical field theory has been considered in several papers, as e.g. [9]. In [39] it was proved that the exterior differential system (8) is *equivalent* with a *certain family* of Ehresmann connections on the fibred manifold $\tau : J^\star Y \to X$, and the family was completely characterized, thus shedding light on the structure of solutions of De Donder–Hamilton equations.

Denote by $\hat{\Gamma}$ an Ehresmann connection (jet field) on $\tau : J^\star Y \to X$. By definition, $\hat{\Gamma}$ is a local section of $\tau_{1,0}$, and it is represented by its horizontal projector

$$\Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \Gamma^\sigma_j \frac{\partial}{\partial y^\sigma} + \Gamma^i_{\sigma j} \frac{\partial}{\partial P^i_\sigma} \right), \tag{10}$$

where Γ^σ_j, $\Gamma^i_{\sigma j}$ are components of $\hat{\Gamma}$. A local section ψ of τ is called an *integral section* of $\hat{\Gamma}$ if it satisfies $\hat{\Gamma} \circ \psi = J^1\psi$. In coordinates this is a system of overdetermined first order PDEs

$$\frac{\partial \psi^\sigma}{\partial x^j} = \Gamma^\sigma_j \, , \qquad \frac{\partial \psi^i_\sigma}{\partial x^j} = \Gamma^i_{\sigma j} \, . \tag{11}$$

By a direct calculation we obtain:

THEOREM 2.1. *If an Ehresmann connection $\hat{\Gamma}$ on $\tau : J^\star Y \to X$ satisfies the compatibility condition $i_\Gamma \Omega_h = (n-1)\Omega_h$ then any integral section of $\hat{\Gamma}$ is a solution of (9).*

In view of the above theorem we call Ehresmann connections compatible with Ω_h *fields of De Donder–Hamilton extremals.*

Apparently, a compatible connection is non-unique; however, we are able to characterize *all* of them:

THEOREM 2.2 ([39]). *The family of Ehresmann connections $\hat{\Gamma}$ on $\tau : J^\star Y \to X$ compatible with Ω_h is locally described by the horizontal projectors*

$$\Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \frac{\partial \mathcal{H}}{\partial P^j_\sigma} \frac{\partial}{\partial y^\sigma} - \left(\frac{1}{n} \delta^i_j \frac{\partial \mathcal{H}}{\partial y^\sigma} + F^i_{\sigma j} \right) \frac{\partial}{\partial P^i_\sigma} \right), \tag{12}$$

where for every σ, the $(F^i_{\sigma j})$ is an arbitrary $(n \times n)$-matrix on U, traceless at each point of U.

Formula (12) gives a family of local Ehresmann connections such that every local section of any of these connections is a De Donder–Hamilton extremal. In particular, for every *completely integrable* connection $\hat{\Gamma}$ (in the sense of Frobenius integrability) the maximal De Donder–Hamilton extremals of $\hat{\Gamma}$ form an n-dimensional foliation of $\operatorname{Dom} \hat{\Gamma} \subset J^\star Y$.

The question arises if all solutions of Ω_h are (at least locally) included, i.e., if every solution of De Donder–Hamilton equations can be locally embedded in a field of De Donder–Hamilton extremals—and the answer is affirmative: *Every solution of De Donder–Hamilton equations is locally an integral section of some Ω_h-compatible connection $\hat{\Gamma}$, which, moreover, is* maximal *in the sense that it is defined on the domain of the section h.* To be more precise, one can prove the following:

THEOREM 2.3 ([39]). *Let h be a section of ρ defined on $U \subset J^\star Y$, and let W be a nonempty open subset of $\tau(U) \subset X$. If ψ is a local section of $\tau : J^\star Y \to X$ defined on W and satisfying*

$$\psi^*(i_\xi \Omega_h) = 0 \quad \textit{for every vertical vector field } \xi \textit{ on } U \tag{13}$$

then for each $x \in W$ there is a connection $\hat{\Gamma}$ defined on $U \subset J^\star Y$ and satisfying the compatibility condition $i_\Gamma \Omega_h = (n-1)\Omega_h$ such that for some neighbourhood $\mathcal{N}$ of $\psi(x)$ the restriction $\psi|_\mathcal{N}$ is an integral section of $\hat{\Gamma}$.

DEFINITION 2.4. A De Donder–Hamilton system is called *Cauchy integrable* if the Cauchy problem for the given De Donder–Hamilton equations has, for every initial condition, at least one maximal solution.

COROLLARY 2.5. *Existence of a flat Ehresmann connection compatible with Ω_h is a sufficient condition for Cauchy integrability.*

Indeed, assume that there exists a flat (completely integrable) Ω_h-compatible connection $\hat{\Gamma}$ on $U = \operatorname{Dom} h$. Then $\hat{\Gamma}$ induces an n-dimensional foliation of U such that the leaves are solutions of the De Donder–Hamilton equations of Ω_h. It follows that the Cauchy problem for the system of PDEs (9) has for any given initial condition (at each point in U) at least one maximal solution, corresponding to the unique maximal integral manifold of $\hat{\Gamma}$ passing through that point.

3. Field theory in jet bundles: Lagrangian-based approach

3.1. Goldschmidt–Sternberg's setting for first order Hamiltonian field theory. Let λ be a Lagrangian on J^1Y, in fibred coordinates $\lambda = L\omega_0$ where L may depend on $(x^i, y^\sigma, y^\sigma_j)$. An *action* over Ω (where $\Omega \subset X$ is a compact connected manifold of dimension n with boundary) is then a real function on the set $\mathrm{Sec}(\pi)$ of local sections of π

$$\mathrm{Sec}(\pi) \ni \gamma \to \int_\Omega J^1\gamma^*\lambda \in \mathbb{R}. \tag{14}$$

Variation of the action gives rise to the splitting of the action into the 'Euler–Lagrange term' and boundary term; this splitting can be intrinsically given in the form

$$\int_\Omega J^1\gamma^* \mathcal{L}_{J^1\xi}\lambda = \int_\Omega J^1\gamma^* i_{J^1\xi} d\theta_\lambda + \int_{\partial\Omega} J^1\gamma^* i_{J^1\xi}\theta_\lambda \,, \tag{15}$$

where ξ is a projectable vector field on Y, $J^1\xi$ denotes the first jet prolongation of ξ, and θ_λ is the *Poincaré–Cartan form*

$$\theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_i}\omega^\sigma \wedge \omega_i = -H\omega_0 + p^i_\sigma dy^\sigma \wedge \omega_i \,. \tag{16}$$

The local functions H and p^i_σ, components of θ_λ with respect to the canonical basis $(dx^i, dy^\sigma, dy^\sigma_j)$ of $\Lambda^1(J^1Y)$, are called the *Hamiltonian* and *momenta* of the Lagrangian λ; obviously,

$$H = -L + p^i_\sigma y^\sigma_i\,, \quad p^i_\sigma = \frac{\partial L}{\partial y^\sigma_i}\,. \tag{17}$$

A Lagrangian λ is called *regular* if at each point of J^1Y

$$\det\left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_k}\right) \neq 0. \tag{18}$$

We observe that the above Hessian matrix of L is equal to the matrix $(\partial p^j_\sigma/\partial y^\nu_k)$, so that if λ is regular then $(x^i, y^\sigma, p^j_\sigma)$ are local coordinates on J^1Y; they are called *Legendre coordinates.*

With the help of the first variation formula (15) it is proved that a section $\gamma \in \mathrm{Sec}(\pi)$ is an extremal of λ if and only if it satisfies the *Euler–Lagrange equations* which can be written with the help of the Poincaré–Cartan form as follows:

$$J^1\gamma^* i_\Xi d\theta_\lambda = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \Xi \text{ on } J^1Y. \tag{19}$$

In terms of the exterior differential systems theory, the above equations can be viewed as equations for *holonomic sections* in $\mathrm{Sec}(\pi_1)$ (i.e. local sections of π_1 having the form of prolongations of sections of π) which are *solutions of the exterior differential system $\mathcal{D}_\lambda$ generated by n-foms $i_\Xi d\theta_\lambda$ where Ξ runs over all vertical vector fields on J^1Y*. However, as proposed by Goldschmidt and Sternberg [14] one can consider *all integral sections* $\delta \in \mathrm{Sec}(\pi_1)$ of $\mathcal{D}_\lambda$. This yields *first order* PDEs

$$\delta^* i_\Xi d\theta_\lambda = 0 \quad \text{for every vertical vector field } \Xi \text{ on } J^1Y. \tag{20}$$

If λ is regular then one can express generators of $\mathcal{D}_\lambda$ and equations (20) in the Legendre coordinates, and obtain

$$\frac{\partial y^\sigma}{\partial x^i} = \frac{\partial H}{\partial p^i_\sigma}, \qquad \frac{\partial p^i_\sigma}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma}\,. \tag{21}$$

Equations (21) have a form of De Donder–Hamilton equations (cf. (9)). Since they appear as a coordinate form of (20), the global equations (20) are also called *De Donder–Hamilton equations.* It should be stressed, however, that unlike (21), equations (20) make sense and hold for *any Lagrangian* (not only a regular one, which has to be assumed to obtain (21)).

The meaning of the regularity condition (18) is easily seen to be the following:

Proposition 3.1. *If λ is regular then the De Donder–Hamilton equations (20) are equivalent to the Euler–Lagrange equations in the sense that the solutions are in bijective correspondence.*

In general, every extremal, if prolonged to J^1Y, is a solution of the De Donder–Hamilton equations, however, the De Donder–Hamilton equations may possess solutions that do not correspond to extremals. To deal with these equations then requires a quite complicated Dirac constraint algorithm (see e.g. [12], [17], [10]).

3.2. Lagrangian–Hamiltonian duality. Equations (20) are defined on J^1Y and concern sections of the bundle $\pi_1 : J^1Y \to X$. However, equipped with the concept of affine duals, we may transfer the Hamilton theory from J^1Y to its extended and reduced dual. This enables one to consider at least some Lagrangian systems within a canonical Hamiltonian setting on a dual multisymplectic bundle in a similar spirit as one considers the symplectic setting for time-independent mechanics on cotangent bundles. The tool to move from J^1Y to the dual side is, naturally, the *Legendre map*.

As above, let $(J^\dagger Y, \Omega)$ be the extended dual of J^1Y, $J^\star Y$ be the reduced dual, and $\rho : J^\dagger Y \to J^\star Y$ be the quotient map. Given a Lagrangian λ on J^1Y, $\lambda = L\omega_0$, one has the following maps, called *Legendre map*, and *reduced Legendre map*, respectively:

$$\mathrm{Leg} : J^1Y \to J^\dagger Y\,, \qquad p = -L + \frac{\partial L}{\partial y^\nu_k} y^\nu_k\,, \quad p^j_\sigma = \frac{\partial L}{\partial y^\sigma_j}\,, \tag{22}$$

$$\mathrm{leg} = \rho \circ \mathrm{Leg} : J^1Y \to J^\star Y\,, \qquad p^j_\sigma = \frac{\partial L}{\partial y^\sigma_j}\,. \tag{23}$$

In this context we say that the Lagrangian λ is *hyperregular* if there is an extended Legendre map defined globally, and such that the corresponding reduced Legendre map is a diffeomorphism.

If λ is regular, there arises locally a Hamiltonian section $h = \mathrm{Leg} \circ \mathrm{leg}^{-1}$ of ρ, giving a dual Hamiltonian system related to λ, $\Omega_h = h^*\Omega$. By construction,

$$\mathrm{leg}^*\Omega_h = d\theta_\lambda\,. \tag{24}$$

For regular (resp. hyperregular) Lagrangians the duality equation (24) gives locally (resp. globally) the identification of the Lagrangian and the De Donder–Hamiltonian system on J^1Y with a Hamiltonian system on $J^\star Y$.

4. Field theory in jet bundles: A non-Lagrangian viewpoint

4.1. An example. The above geometric setting for Lagrangian field theories on J^1Y and its affine duals looks to be, at a first sight, a natural extension of symplectic mechanics. However, a deeper insight reveals some unpleasant problems and questions from both the mathematical and physical point of view.

First of all, unfortunately, many interesting Lagrangians for physical field theories do not satisfy the regularity condition (18), hence the advantage of dealing with canonical dual systems cannot be fully explored. Instead, one is forced to consider the Dirac formalism for singular Lagrangians. This, of course, makes the whole dualization procedure for PDEs much more complicated and less useful than in the case of ODEs. Moreover, compared to mechanics, the field theories carry another strange property which does not exist in mechanics, namely that *equivalent Lagrangians* may possess *essentially different regularity/degeneracy properties*, and hence *essentially different Hamiltonian de-*

scriptions[3]. To illustrate the situation, let us consider the following easy example [41]: The Lagrangians

$$L_1 = u_x^2\,, \qquad L_2 = u_x^2 + u_x v_y - u_y v_x \tag{25}$$

are equivalent, giving the same Euler–Lagrange expressions. L_2 is regular, however L_1 is not. This means that the De Donder–Hamilton equations $\delta^* i_\xi d\theta_{\lambda_2} = 0$ are equivalent to the Euler–Lagrange equations, and we have the *equivalent* dual representation on $J^\star Y$, giving us *one-to-one correspondence between extremals and Hamilton extremals.* On the other hand, the De Donder–Hamilton equations $\delta^* i_\xi d\theta_{\lambda_1} = 0$ are not equivalent to the Euler–Lagrange equations, which leads to the conclusion that there is no duality between the Hamiltonian and Lagrangian description of the extremals, and the field equations are constrained in the sense of Dirac.

This apparent contradiction suggests the idea first pointed out by Dedecker [6] that the conventional understanding of regularity and Legendre transformation is not fully appropriate and has to be revisited. An escape, leading to quite surprising issues, was presented first for mechanics in [25] (see also [27], [26]), and later generalized to field theory in [29]. The main idea was to *associate the Hamiltonian theory with the Euler–Lagrange form* in order to obtain *Hamiltonian equations related to the class of equivalent Lagrangians* rather than to a particular Lagrangian as is the 'standard' case (Sec. 3).

As a result of this new philosophy, *the class of regular variational problems having an equivalent dual Hamiltonian description is enlarged*, moreover, the extension guarantees that *'regularity'* (or, more generally, 'the extent of degeneracy') *is a geometric property of equations (dynamics, solutions), rather than a Lagrangian* which is not a physically observable quantity. As a 'surprise' it turns out that some first order Lagrangians, or even some second order Lagrangians, which are traditionally considered degenerate, have a regular Hamiltonian counterpart on $J^\star Y$ (coming from the canonical multisymplectic form); this concerns e.g. the Dirac Lagrangian, electromagnetic-type fields, or the scalar curvature Lagrangian ([32], [33], [24], [19], [37]).

4.2. Lepage manifolds. The idea is to extend the Euler–Lagrange form to a *(proper) closed* $(n+1)$*-form.* This form then serves as a basic object for representing the Euler–Lagrange equations, carries information about the Hamiltonian and momenta and gives Hamiltonian equations, independently of the choice of a particular Lagrangian (within this setting, a global Lagrangian even need not exist).

In what follows we shall consider a dynamical form E on J^2Y (which by definition is a 1-contact and ω^σ-generated $(n+1)$-form); in coordinates

$$E = E_\sigma\, \omega^\sigma \wedge \omega_0 \qquad E_\sigma = E_\sigma(x^i, y^\nu, y^\nu_j, y^\nu_{jk})\,.$$

Then local sections γ of π such that E vanishes along $J^2\gamma$ are solutions of a system of m

[3]More precisely, this does not exist in mechanics if one considers equivalent first order Lagrangians for second order ODEs (or, in higher-order mechanics, equivalent minimal-order Lagrangians for even order equations). Allowing equivalent Lagrangians of different orders, or considering odd-order variational equations, we come exactly to the same discrepancy. We refer to [26] for more details, and for an approach solving this problem.

second order partial differential equations of the form

$$E_\sigma\left(x^i, f^\nu, \frac{\partial f^\nu}{\partial x^i}, \frac{\partial^2 f^\nu}{\partial x^i \partial x^j}\right) = 0$$

where f^ν are components of a section, $\gamma = (x^i, f^\nu)$.

Taking into account the decomposition of differential forms into contact components (3), we have the following definition:[4]

Definition 4.1 ([29], [40]). By a *Lepage* $(n+1)$*-form* we shall mean a closed $(n+1)$-form α on J^1Y such that $p_1\alpha$ is a dynamical form. A *Lepage manifold* will then be a fibered manifold $\pi : Y \to X$ where $\dim X = n \geq 1$, equipped with a Lepage $(n+1)$-form defined on J^1Y.

The importance of Lepage $(n+1)$-forms follows from the fact that they are closed counterparts of Euler–Lagrange forms (variational equations):

Theorem 4.2. *If α is a Lepage $(n+1)$-form then the dynamical form $E = p_1\alpha$ is locally variational: in a neighbourhood of every point $x \in \operatorname{Dom}\alpha$ there exists a Lagrangian λ such that E is the Euler–Lagrange form of λ* [21].

Conversely, to every locally variational dynamical form E there exists a closed Lepage $(n+1)$-form α such that $E = p_1\alpha$ [25], [36].

The form α above is not unique (unless $n = 1$); it is called a *Lepage equivalent of E*.

Equations for the dynamical form arising from a Lepage $(n+1)$-form are Euler–Lagrange equations. The closedness conditions are then the Helmholtz variationality conditions [1], [21].

The question about the structure of first order Lepage $(n+1)$-forms is answered by the following theorem:

Theorem 4.3 ([34]). *Any Lepage $(n+1)$-form may be locally written as $\alpha = \alpha_E + \eta$ where α_E is closed and completely determined by E, and η is a closed and at least 2-contact form. The restriction of α to a suitably small open set U satisfies*

$$\alpha|_U = d\theta_\lambda + d\mu, \tag{26}$$

where θ_λ is the Poincaré–Cartan form of a local Lagrangian, and μ is an at least 2-contact n-form.

Note that we can regard a Lepage manifold as a fibred manifold, equipped (covered) with a family of equivalent local Lagrangians; in general there is no global Lagrangian (even of higher order): obstructions come from the topology of Y [1].

On a Lepage manifold (π_1, α) we have the *exterior differential system $\mathcal{D}_\alpha$*, generated by the n-forms $i_\xi\alpha$ where ξ runs over all π_1-vertical vector fields. It is easy to see that the global Euler–Lagrange equations take the form of equations for *holonomic integral sections* of $\mathcal{D}_\alpha$,

$$J^1\gamma^* i_\xi\alpha = 0, \quad \forall\ \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y\,. \tag{27}$$

[4] Here we deal only with *closed* Lepage $(n+1)$-forms *of order* 1. However, one can extend the definitions and results to higher orders, and even to forms that are *not closed*, including in this way also *non-Lagrangian PDEs*. For details we refer e.g. to [26], [30], [23], [36] and [38].

The first order PDEs

$$\delta^* i_\xi \alpha = 0, \quad \forall\ \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y \tag{28}$$

for all integral sections of $\mathcal{D}_\alpha$ (i.e., local sections of the fibred manifold $\pi_1 : J^1Y \to X$), are called *Hamilton equations*; the solutions δ are then called *Hamilton extremals.*

It is worth noticing that:

• Hamilton and Euler–Lagrange equations are not equivalent, as there might exist Hamilton extremals that are not prolongations of extremals.

• On a Lepage manifold both the Euler–Lagrange equations and the Hamilton equations are independent of the choice of a concrete Lagrangian for E.

• Due to non-uniqueness of the Lepage equivalent of E, for given Euler–Lagrange equations, one can have different (and distinct) Hamiltonian systems represented by different α's. This opens the possibility to consider different Hamilton theories associated with given variational equations, and the question on 'regularity' can be posed in a completely different way: instead of asking whether 'a given Lagrangian is regular' (in the sense of the previous section), better to ask whether a family of equivalent Lagrangians *admits* Hamilton equations equivalent to the Euler–Lagrange equations. Of course, this is then a question about the *choice of a proper* α for E.

4.3. De Donder–Hamilton systems, regularity. In view of the above remark, let us consider the *simplest case* of a Lepage manifold, when α is at most 2-contact and $\{\omega^\sigma\}$-generated. As we shall see, in this case the Hamilton equations (28) become of De Donder type. We also note that the assumption that α is defined on J^1Y implies that the corresponding Euler–Lagrange expressions are affine in the second derivatives, and variationality gives (among other things) an additional symmetry of the coefficients, so that

$$E_\sigma = A_\sigma + B^{ij}_{\sigma\nu} y^\nu_{ij}\,, \quad B^{ij}_{\sigma\nu} = B^{ji}_{\sigma\nu} = B^{ij}_{\nu\sigma}\,. \tag{29}$$

The form α is closed by definition. However, $\operatorname{rank}\alpha$ need not be maximal, even need not be constant. We say that α is *regular* if $\operatorname{corank}\alpha = \dim X$, or, equivalently, $\operatorname{rank}\alpha = \operatorname{rank}\mathcal{D}_\alpha = m + nm$.

With the help of the Poincaré Lemma we can prove:

THEOREM 4.4 (Canonical form of Lepage $(n+1)$-form [38]). *Let (π_1, α) be a Lepage manifold and assume that α is at most 2-contact and $\{\omega^\sigma\}$-generated. Then around every point in J^1Y there is a neighbourhood U and functions H and p^j_σ defined on U such that*

$$\alpha|_U = -dH \wedge \omega_0 + dp^j_\sigma \wedge dy^\sigma \wedge \omega_j. \tag{30}$$

If, moreover, α is regular then the functions p^j_σ are independent:

$$\operatorname{rank}\left(\frac{\partial p^j_\sigma}{\partial y^\nu_k}\right) = \max = mn\,. \tag{31}$$

In fibred coordinates we have

$$\pi^*_{2,1}\alpha = E_\sigma \omega^\sigma \wedge \omega_0 + \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial y^\nu_j} - d_k f^{j,k}_{\sigma\nu}\right)\omega^\sigma \wedge \omega^\nu \wedge \omega_j + \left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - f^{i,j}_{\sigma\nu}\right)\omega^\sigma \wedge \omega^\nu_i \wedge \omega_j\,, \tag{32}$$

where $f^{i,j}_{\sigma\nu} = -f^{j,i}_{\sigma\nu} = f^{j,i}_{\nu\sigma}$ are some first order functions such that $d\alpha = 0$. In terms of α the regularity condition (31) reads

$$\det\left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - f^{i,j}_{\sigma\nu}\right) \neq 0\,. \tag{33}$$

It is worth noting that by construction, explicit formulae for the Hamiltonian and momenta come from the *integration procedure* using the Poincaré Lemma, and *are determined by α rather than by a particular Lagrangian.*

Before presenting the formulae, recall a convenient modification of the Poincaré homotopy operator, adapted to the contact structure [21]. Denote by $\mathcal{P}$ the Poincaré homotopy operator. Then $\mathcal{A}$ defined by $\mathcal{A}p_0\omega = 0$, $\mathcal{A}p_k\omega = p_{k-1}\mathcal{P}\omega$, $k \geq 1$, satisfies $\pi^*_{r+1,r}\omega = \mathcal{A}d(\pi^*_{r+1,r}\omega) + d\mathcal{A}(\pi^*_{r+1,r}\omega)$, and is adapted to the decomposition of forms into contact components: if (locally) $\omega = d\rho$, then $\pi^*_{r+1,r}\rho = \mathcal{A}(\pi^*_{r+1,r}\omega) = \sum_{k=0}^q \mathcal{A}p_{k+1}\omega$, hence $p_k\rho = \mathcal{A}p_{k+1}\omega$. Compared to $\mathcal{P}$, the operator $\mathcal{A}$ concerns vertical curves (curves in the fibres over X) only. Now, a direct computation yields [38]:

$$\rho_0 = \mathcal{A}\alpha - d(f^i\omega_i) = -H\omega_0 + p^j_\sigma dy^\sigma \wedge \omega_j\,, \tag{34}$$

where with the help of the mappings $\chi : (u, (x^i, y^\sigma, y^\sigma_j)) \to (x^i, uy^\sigma, uy^\sigma_j)$ for $u \in [0,1]$, and $\bar{\chi} : (v, (x^i, y^\sigma, y^\sigma_j)) \to (x^i, y^\sigma, vy^\sigma_j)$ for $v \in [0,1]$,

$$f^i = y^\nu_j y^\sigma \int_0^1 \left(\int_0^1 \left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - f^{i,j}_{\sigma\nu}\right) \circ \chi\, u\, du\right) \circ \bar{\chi}\, v\, dv + \varphi^i(x^p, y^\rho), \tag{35}$$

and

$$\begin{aligned} p^i_\sigma &= -y^\nu \int_0^1 \left(\frac{\partial E_\sigma}{\partial y^\nu_i} - d_k f^{i,k}_{\sigma\nu}\right) \circ \chi\, u\, du - y^\nu_j \int_0^1 \left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - f^{i,j}_{\sigma\nu}\right) \circ \chi\, u\, du - \frac{\partial f^i}{\partial y^\sigma}\,, \\ H &= -y^\sigma \int_0^1 (A_\sigma \circ \chi) du + p^i_\sigma y^\sigma_i + \frac{\partial f^i}{\partial x^i}\,. \end{aligned} \tag{36}$$

Obviously $d\rho_0 = d\mathcal{A}\alpha = \alpha$. The functions $\varphi^i(x^p, y^\rho)$ defined on an open subset of Y play the role of *admissible gauge functions*, parametrizing the family of Hamiltonians and momenta: we have $\bar{H} = H + \partial\varphi^i/\partial x^i$, $\bar{p}^i_\sigma = p^i_\sigma - \partial\varphi^i/\partial y^\sigma$, and $\bar{\rho}_0 = \rho_0 - d(\varphi^i\omega_i)$. Accordingly, $\lambda_0 = h\rho_0 = h\mathcal{A}\alpha - hd(f^i\omega_i)$ are *distinguished* local first order Lagrangians for $E = p_1\alpha$ (equivalent to the second order Tonti Lagrangian $h\mathcal{A}\alpha = \mathcal{A}E$). They are determined up to the gauge $hd(\varphi^i\omega_i) = d_i\varphi^i\omega_0$.

Given a regular form α, we can see that $(x^i, y^\sigma, y^\sigma_j) \to (x^i, y^\sigma, p^j_\sigma)$ is a local coordinate transformation on J^1Y. In view of the next theorem, it can be called *Legendre transformation*:

THEOREM 4.5 ([38]). *If H and p^i_σ are a Hamiltonian and momenta of α defined by the canonical form of α (30) then locally there exists a first order Lagrangian for ε such that $H = -L + p^i_\sigma y^\sigma_i$ and $p^\sigma_i = \frac{\partial L}{\partial y^\sigma_i}$. Any such Lagrangian is of the form $\lambda = \lambda_0 + hd\varphi$, where $\lambda_0 = h\rho_0$ and φ is a local horizontal $(n-1)$-form on Y. Moreover, on the domain of λ it holds $\alpha = d\theta_\lambda$.*

If α is regular then there exists a covering of J^1Y by Legendre coordinates that are defined by the canonical form (30) of α. Moreover, every Lagrangian $\lambda = h\rho_0 + hd\varphi$

satisfies the regularity condition

$$\det\left(\frac{\partial p^i_\sigma}{\partial y^\nu_j}\right) = \det\left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j}\right) \neq 0\,. \tag{37}$$

It should be stressed that, in contrast with mechanics, *not every* Lagrangian for E is admissible in the sense that it should satisfy the above theorem. Indeed, momenta $\hat{p}^j_\sigma$ and Hamiltonian $\hat{H}$ defined by means of a Lagrangian $\hat{\lambda}$ such that $p_1\alpha = E_{\hat{\lambda}}$ need not satisfy $\alpha = -d\hat{H} \wedge \omega_0 + d\hat{p}^j_\sigma \wedge dy^\sigma \wedge \omega_j$ (thus for $\hat{\lambda}$, $d\theta_{\hat{\lambda}} \neq \alpha$); this is a case discussed in the example above. We can conclude that not a particular Lagrangian but rather the form α carries the information about momenta and energy of a Hamiltonian system, and once α associated with a Lagrangian system (represented by the Euler–Lagrange form) is chosen, the energy and momenta (as well as a subclass of 'privileged' Lagrangians giving the 'right' momenta and Hamiltonian) are determined uniquely up to a gauge function $\varphi(x^i, y^\nu)$ (or even $\varphi(y^\nu)$, if the Euler–Lagrange expressions do not depend explicitly on (x^i), giving us *a distinguished Hamiltonian density*).

We also note that, as expected, in the Legendre coordinates the Hamilton equations of α take the De Donder form (21).

Finally, the meaning of the regularity condition can be summarized as follows:

Theorem 4.6 ([40], [38]). *On a Lepage manifold (π_1, α) where α is at most 2-contact, $\{\omega^\sigma\}$-generated and regular, the Euler–Lagrange and Hamilton equations are equivalent.*

Explicitly, if γ is an extremal then $J^1\gamma$ is a Hamilton extremal, and, conversely, every Hamilton extremal is of the form $J^1\gamma$ where γ is an extremal.

4.4. A new look at the duality problem. With the revised concept of regularity and Legendre transformation it is possible to extend substantially the family of Lagrangians admitting the multisymplectic description in the dual bundle.

As above, let $J^\dagger Y$ and $J^\star Y$ be the affine dual and the reduced dual of $J^1 Y$, respectively, $\Omega = dP \wedge \omega_0 + dP^i_\sigma \wedge dy^\sigma \wedge \omega_i$ the canonical multisymplectic form on $J^\dagger Y$, and $\rho : J^\dagger Y \to J^\star Y$ the quotient projection. If h is a local Hamiltonian section of ρ defined on $U \subset J^\star Y$, let $\Omega_h = h^*\Omega = -d\mathcal{H} \wedge \omega_0 + dP^i_\sigma \wedge dy^\sigma \wedge \omega_i$ be the form on U related to Ω via h, and $\mathcal{D}_h$ be the corresponding exterior differential system.

Consider a Lepage manifold (π_1, α) where α is at most 2-contact, $\{\omega^\sigma\}$-generated and regular. The relationship between the concrete Lagrangian system on $J^1 Y$ represented by α and an abstract Hamiltonian system on $J^\star Y$ is obtained via Legendre maps $\mathrm{Leg}_\alpha : J^1 Y \to J^\dagger Y$ and $\mathrm{leg}_\alpha : J^1 Y \to J^* Y$ *defined* by means of the following *duality equations*:

$$\mathrm{Leg}^*_\alpha\, \Omega = \alpha \quad \text{and} \quad \mathrm{leg}^*_\alpha\, \Omega_h = \alpha\,, \tag{38}$$

where h is given by $\mathrm{Leg}_\alpha = h \circ \mathrm{leg}_\alpha$. We call Leg_α and leg_α *extended Legendre map* and *reduced Legendre map*, respectively. Since α satisfies the regularity condition (33) we can choose Legendre coordinates of α on $J^1 Y$. Note that in the Legendre coordinates on $J^1 Y$ and the canonical coordinates on $J^\star Y$, leg_α is represented by the identity mapping, giving us *formulae* for the Legendre maps. Obviously, one obtains in this way for $\mathcal{H}$ and P^i_σ the same formulae as (36) above.

THEOREM 4.7 ([40]). *If α is regular then every extended Legendre map is an immersion and every corresponding reduced Legendre map is a local diffeomorphism.*

α is called *hyper-regular* if there is an extended Legendre map Leg_α defined globally, and such that the corresponding reduced Legendre map leg_α is a diffeomorphism.

The following 'Duality theorem' summarizes the meaning of regularity in this case:

THEOREM 4.8 ([40]). *Let α be regular. Then (on the corresponding domain)*

(1) $\mathrm{Leg}_\alpha^*\Omega = \mathrm{leg}_\alpha^* h^*\Omega = \alpha$.

(2) $\mathrm{rank}\, h^*\Omega = \mathrm{rank}\, \mathcal{D}_h = \mathrm{rank}\, \alpha = \mathrm{rank}\, \mathcal{D}_\alpha = m + nm$.

(3) $\mathrm{leg}_\alpha^* \mathcal{D}_h = \mathcal{D}_\alpha$.

(4) *If $\psi : X \to J^*Y$ is an integral section of $\mathcal{D}_h$ then $\mathrm{leg}_\alpha^{-1} \circ \psi = J^1\gamma$ where γ is a section of $\pi : Y \to X$, and it is an integral section of $\mathcal{D}_\alpha$.*

(5) *Every integral section of $\mathcal{D}_\alpha$ is of the form $J^1\gamma$, and $\psi = \mathrm{leg}_\alpha \circ J^1\gamma$ is an integral section of $\mathcal{D}_h$.*

4.5. Some applications. In practical situations, given a Lagrangian system, we are interested if it has an *equivalent dual representation.*

As we have seen, a Lagrangian system is globally represented by its Euler–Lagrange form E. All Hamiltonian systems related to E arise from the Lepage equivalents of E. The question now means that we are searching for a *regular* Lepage equivalent α of E defined on J^1Y, at most 2-contact, and $\{\omega^\sigma\}$-generated. Thus, we have a *closed* α in the form (32), satisfying the regularity condition (33), where now the functions $f^{j,k}_{\sigma\nu}$ play the role of parameters *to be specified*, and different choices correspond to different Lepage $(n+1)$-forms α associated to E.

If we are happy to know a Lagrangian λ for E satisfying the regularity condition (18) then putting $\alpha = d\theta_\lambda$ we obtain a regular Lepage manifold (π_1, α) which by the standard Legendre map constructed by means of λ is related to the dual $(J^\dagger Y, \Omega)$, and putting $h = \mathrm{Leg} \circ \mathrm{leg}^{-1}$ we get Ω_h on $J^\star Y$, as desired.

There are, however, many important Lagrangians which do not satisfy condition (18), like, for example, Lagrangians affine in the first derivatives (Dirac field), some Lagrangians quadratic in the first derivatives (electromagnetic type Lagrangians—Maxwell and Yang–Mills fields), or some second order Lagrangians affine in the second derivatives (Einstein–Hilbert Lagrangian (scalar curvature)). It has been shown that *all the mentioned Lagrangian systems admit a regular De Donder–Hamilton theory*, i.e. a dual description on $J^\star Y$; in other words, they can be treated *without* the Dirac constraint formalism. The point is that one can find a *regular* Lepage $(n+1)$-form α as above, such that $E = p_1\alpha$ is the given Euler–Lagrange form. The class of proper (regular) Lagrangians for E, as well as proper momenta and Hamiltonians, then are defined, indeed, by the canonical form of α. The 'original' (given) degenerate Lagrangian λ is, of course, equivalent to the new one, λ_0, so that $L_0 = L + l$ where l is a null Lagrangian ('divergence'). In [33] l is called a *satellite* of L. Thus we can say that the problem is to find to a degenerate Lagrangian a satellite in such a way that the 'corrected' Lagrangian would be regular. Due to non-uniqueness of α, we may get distinct satellite Lagrangians dependent upon auxiliary parameters the number of which is determined by dimensions of the base and the fibres. This is typically the case of Lagrangians affine in the first derivatives (Dirac field); in other interesting cases (electrodynamics, gravity) apart from considering the parameters in the Hamiltonian theory, we have a *canonical choice* of a satellite, giving us a distinguished De Donder–Hamilton system equivalent to the given Euler–Lagrange system, together with new momenta and energy (which, compared to the Hamiltonian and momenta determined from the usual Lagrangian are different, being corrected by the satellite term). For details we refer to [24], [19] for the scalar curvature Lagrangian, [32] and [33] for Lagrangians affine in the first derivatives (and the Dirac Lagrangian in particular) and for all quadratic first order Lagrangians (in particular the electromagnetic field), and [37] for the Yang–Mills field.

Let us mention here explicitly the case of Einstein equations and Maxwell equations.

(1) A Lepage manifold for relativity [24]. Consider the scalar curvature Lagrangian. Being affine in the second derivatives it does not satisfy the conventional regularity condition for second order Lagrangians. Also it is important to notice that the conventional formulae for momenta and Hamiltonian computed from this Lagrangian are second order functions, hence one is forced to consider constraint Hamilton equations on J^2Y. On the other hand, within the setting of Lepage manifolds, we come to a surprising conclusion that *gravity admits a regular first order Hamiltonian theory*, and fully fits with the *dual multisymplectic formulation*. In this situation we have a fibred manifold $\pi : Y \to X$ where X is a four-dimensional manifold (space-time) and Y is the bundle of metrics over X. The Lagrangian is $\lambda = R\sqrt{|\det g|}\omega_0$, and the Lepage 5-form $\alpha = d\theta_\lambda$ is projectable onto J^1Y and *regular*. This means that a 'right' background is the regular Lepage manifold (π_1, α). The theorem on the canonical form of α yields locally $\alpha = -dH \wedge \omega_0 + dp^{rs,i} \wedge dg_{rs} \wedge \omega_i$, where momenta are closely related to the Levi-Civita connection of the metric g,

$$p^{rs,i} = -\sqrt{|\det g|}\left(\frac{1}{2}g^{rs}(g^{pq}\Gamma^i_{pq} + g^{iq}\Gamma^p_{pq}) - g^{rq}g^{ps}\Gamma^i_{pq} - \frac{1}{2}(g^{ir}g^{qs} + g^{is}g^{qr})\Gamma^p_{pq}\right), \quad (39)$$

and the Hamiltonian is quadratic in momenta,

$$H = \frac{1}{6}\frac{1}{\sqrt{|\det g|}}(g_{jk}g_{ab}g_{rs} - 4g_{aj}g_{kb}g_{rs} + 4g_{rj}g_{kb}g_{as})p^{ab,j}p^{rs,k}\,. \quad (40)$$

The momenta and Hamiltonian are *first order* functions, and we have the Legendre transformation $(x^i, g_{rs}, g_{rs,j}) \to (x^i, g_{rs}, p^{rs,j})$ on J^1Y. The De Donder–Hamilton equations for gravity then take the form

$$\frac{\partial H}{\partial g_{rs}} + \frac{\partial p^{rs,k}}{\partial x^k} = 0, \quad \frac{\partial H}{\partial p^{rs,k}} - \frac{\partial g_{rs}}{\partial x^k} = 0$$

and they have the equivalent counterpart on the dual bundle $J^\star Y$, arising from $\Omega_h = -d\mathcal{H} \wedge \omega_0 + dP^{rs,i} \wedge dg_{rs} \wedge \omega_i$.

(2) A Lepage manifold for electromagnetism [33]. Let $\pi : Y \to X$ be a vector bundle, where $X = \mathbb{R}^4$ with the Lorenz metric $g = \mathrm{diag}(-1, 1, 1, 1)$, and the fibres are four-dimensional. Consider the 'standard' Lagrangian $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(y^\sigma_\nu y^\nu_\sigma - g^{\sigma\nu}g_{\mu\rho}y^\mu_\sigma y^\rho_\nu)$, where $y^\sigma = g^{\sigma\nu}A_\nu = A^\sigma$ and $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$. It is easy to see that L is degenerate. However, it has been shown that the more general regularity condition (33) can be satisfied [6], [32], [33]. Remarkably, as found in [33], there is a canonical,

parameter-independent regular Lepage manifold (π_1, α) such that $E = p_1\alpha$ is the Maxwell Euler–Lagrange form as follows: $\alpha = d\theta_{\lambda_0}$ where $L_0 = L - 2(\mathrm{Tr}\, A'^2 - (\mathrm{Tr}\, A')^2)$, giving the satellite Lagrangian to L in the form $l = 2(\mathrm{Tr}\, A'^2 - (\mathrm{Tr}\, A')^2) = 2(A^\mu_\nu A^\nu_\mu - A^\nu_\nu A^\mu_\mu)$; here A' denotes the matrix (A^μ_ν) with the entries $A^\mu_\nu = \partial_\nu A^\mu = \partial A^\mu/\partial x^\nu$. From α (hence from λ_0) we get *independent momenta* and the energy $H_0 = H - 2(\mathrm{Tr}\, A'^2 - (\mathrm{Tr}\, A')^2)$, where H is the 'standard' Hamiltonian derived from L.

Note that H_0 is not gauge invariant, however, under gauge transformations $\bar{A}_\mu = A_\mu + \partial_\mu\psi$ it transfers according to the rule $\bar{H}_0 = H_0 + \partial_\nu\varphi^\nu$ where $\varphi^\nu = 4(A^\nu g^{\rho\kappa}\partial^2_{\rho\kappa}\psi - A^\rho g^{\nu\kappa}\partial^2_{\kappa\rho}\psi) - 2(g^{\nu\lambda}g^{\kappa\rho}\partial^2_{\lambda\rho}\psi\,\partial_\kappa\psi - g^{\nu\kappa}g^{\lambda\rho}\partial^2_{\lambda\rho}\psi\,\partial_\kappa\psi)$. Since $\varphi^\nu = \varphi^\nu(x^\mu, A_\mu)$ (i.e. does not depend on the derivatives of A_μ), it is for any ψ, an admissible gauge function for the Hamiltonian and momenta of α. In other words, gauge transformations are invariance transformation of the *class* of *admissible Hamiltonians*. Consequently, they do not change the $(n+1)$-form α and the De Donder–Hamilton equations of α.

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