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NOETHER'S THEOREMS IN A GENERAL SETTING. REDUCIBLE GRADED LAGRANGIANS

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Abstract. Noether's first and second theorems are formulated in a general setting of reducible degenerate Grassmann-graded Lagrangian theory of even and odd variables on graded bundles. Such Lagrangian theory is characterized by a hierarchy of higher-stage Noether identities described by a Koszul–Tate chain complex. Noether's second theorems associate to this complex a cochain sequence whose ascent operator defines higher-stage gauge symmetries of the Grassmann-graded Lagrangian system. This operator is extended to a nilpotent BRST operator that provides a BRST extension of the original Lagrangian theory. Noether's first theorem is formulated as a straightforward corollary of the global variational formula. It associates to any gauge symmetry a conserved current which is proved to be a total differential on-shell.

1. Introduction. Noether's theorems are well known to treat symmetries of Lagrangian systems. We refer the reader to the brilliant volume [17] for the history of this subject. We aim to formulate them in a very general setting of reducible degenerate Grassmann-graded Lagrangian theory of even and odd variables on graded bundles [13, 14, 15, 25].

Lagrangian theory of even (commutative) variables on an n-dimensional smooth manifold X is conventionally formulated in terms of smooth fibre bundles over X and jet manifolds of their sections [20, 24, 29] in the framework of general technique of nonlinear differential operators and equations [5, 18, 12]. At the same time, different geometric models of odd variables either on graded manifolds or supermanifolds are discussed [6, 7, 8, 19, 25]. Both graded manifolds and supermanifolds are described in terms of sheaves of graded commutative rings [3, 25]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing sheaves on supervector spaces. We follow Serre–Swan Theorem 3.2 for graded manifolds.

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It states that, if a graded commutative ring is generated by a projective $C^\infty(X)$-module of finite rank, it is isomorphic to a ring of graded functions on a graded manifold with a body X. Accordingly, we describe odd variables in terms of graded bundles (Section 3).

Lagrangian theory on a fibre bundle $Y \to X$ can be appropriately formulated in algebraic terms of a variational bicomplex of exterior forms on the infinite order jet manifold $J^\infty Y$ [1, 20, 24, 29]. This technique is extended to Lagrangian theory on graded bundles [2, 4, 13, 25] where Lagrangians and Euler–Lagrange operators are defined as the elements (17) and (18) of the Grassmann-graded variational bicomplex of graded exterior forms on a graded infinite order jet manifold $(J^\infty Y, \mathfrak{A}_{J^\infty F})$ (Section 5).

The cohomology of a variational bicomplex (Theorem 5.1) provides the global variational formula (20) whose straightforward corollary is Noether's first theorem 6.6. It associates to any symmetry (24) of a Lagrangian L the conserved current (27) whose total differential vanishes on-shell (Section 6). One can show that a conserved current along a gauge symmetry itself is a total differential on-shell (Theorem 7.2).

Noether's second theorems are well known to provide the correspondence between Noether identities (henceforth, NI) and gauge symmetries of a Lagrangian system [17]. A problem is that any Euler–Lagrange operator satisfies NI, which therefore must be separated into the trivial and non-trivial ones. These NI can obey first-stage NI, which in turn are subject to the second-stage ones, and so on. Thus, there is a hierarchy of NI and higher-stage NI which characterizes the degeneracy of a Lagrangian theory (Section 8). A Lagrangian system is called degenerate if it admits non-trivial NI, and reducible if there exists non-trivial higher-stage NI. We follow the general analysis of NI and higher-stage NI of differential operators on fibre bundles when trivial and non-trivial NI are represented by boundaries and cycles of a chain complex [15, 21, 27]. If a certain homology condition holds, one can associate to a Grassmann-graded Lagrangian system the exact Koszul–Tate (henceforth, KT) complex (47) possessing a boundary KT operator whose nilpotence is equivalent to all complete non-trivial NI (39) and higher-stage NI (48) [4, 25, 27].

Noether's inverse second theorem 9.3 associates to this KT complex the cochain sequence (56) whose ascent gauge operator (57) defines gauge and higher-stage gauge symmetries of Lagrangian theory [4, 25, 27]. Conversely, given these symmetries, Noether's direct second theorem 9.4 states that the corresponding NI and higher-stage NI hold.

The gauge operator can be extended to a nilpotent BRST operator (67) that turns the cochain sequence (56) into the BRST complex (68). The KT and BRST complexes provide the BRST extension (69) of the original Lagrangian theory [14, 15, 25].

2. Mathematical preliminaries. Smooth manifolds throughout are Hausdorff, second-countable and, consequently, paracompact. Given a smooth manifold X, its tangent and cotangent bundles TX and T^*X are endowed with bundle coordinates $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$ with respect to holonomic frames $\{\partial_\lambda\}$ and $\{dx^\lambda\}$, respectively. A multi-index Λ of length $|\Lambda| = k$ denotes a collection of indices $(\lambda_1 \dots \lambda_k)$ modulo permutations. By $\lambda + \Lambda$ is meant the multi-index $(\lambda\lambda_1 \dots \lambda_k)$. We use the compact notation $\partial_\Lambda = \partial_{\lambda_k} \cdots \partial_{\lambda_1}$.

Unless otherwise stated, by a graded structure is meant a Grassmann ($\mathbb{Z}_2$-) one. The symbol [.] stands for Grassmann parity.

A real algebra $\mathcal{A}$ is called graded if it is a graded vector space such that $[aa'] = [a]+[a']$. It is said to be graded commutative if $aa' = (-1)^{[a][a']}a'a$. If $\mathcal{A}$ is a ring, then $[\mathbf{1}] = 0$. Given a graded commutative algebra $\mathcal{A}$, a graded $\mathcal{A}$-module Q is defined as a bimodule where $[aq] = [a] + [q]$ and $qa = (-1)^{[a][q]}aq$.

A Grassmann ring of rank m is defined as the exterior algebra $\Lambda = \wedge\mathbb{R}^m$.

The differential calculus over a graded commutative ring is defined similarly to that over commutative rings [24, 25]. For instance, a graded derivation of a graded commutative ring $\mathcal{A}$ is a first order differential operator Δ so that the graded Leibniz rule

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b), \quad a, b \in \mathcal{A},$$

holds. The graded derivations of $\mathcal{A}$ constitute a graded $\mathcal{A}$-module $\mathfrak{d}\mathcal{A}$. It is a real Lie superalgebra with respect to the superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']}u' \circ u, \qquad u, u' \in \mathcal{A}. \tag{1}$$

Then one can consider the Chevalley–Eilenberg complex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ for $\mathfrak{d}\mathcal{A}$ [9, 15, 25]:

$$0 \to \mathbb{R} \to \mathcal{A} \xrightarrow{d} \mathcal{O}^1[\mathfrak{d}\mathcal{A}] \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{O}^k[\mathfrak{d}\mathcal{A}] \xrightarrow{d} \cdots, \tag{2}$$

where $\mathcal{O}^k[\mathfrak{d}\mathcal{A}] = \operatorname{Hom}_{\mathcal{A}}(\wedge^k \mathfrak{d}\mathcal{A}, \mathcal{A})$ are the $\mathfrak{d}\mathcal{A}$-modules of $\mathcal{A}$-linear graded morphisms of graded exterior products $\wedge^k \mathfrak{d}\mathcal{A}$ to $\mathcal{A}$. The complex (2) is provided with the structure of a differential bigraded algebra (henceforth, DBGA) with respect to the graded exterior product $\wedge$ and the Chevalley–Eilenberg coboundary operator d which obey the relations

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'|+[\phi][\phi']}\phi' \wedge \phi, \qquad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|}\phi \wedge d\phi'. \tag{3}$$

In particular, $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$ is the dual of $\mathfrak{d}\mathcal{A}$. This duality is extended to $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$u\rfloor(bda) = (-1)^{[u][b]}bu(a), \qquad u\rfloor(\phi \wedge \phi') = (u\rfloor\phi) \wedge \phi' + (-1)^{|\phi|+[\phi][u]}\phi \wedge (u\rfloor\phi').$$

As a consequence, a graded derivation $u \in \mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields the graded Lie derivative

$$\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \qquad \mathbf{L}_u(\phi \wedge \phi') = \mathbf{L}_u(\phi) \wedge \phi' + (-1)^{[u][\phi]}\phi \wedge \mathbf{L}_u(\phi'),$$

of the DBGA $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$. The minimal graded differential calculus $\mathcal{O}^*\mathcal{A} \subset \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a graded commutative ring $\mathcal{A}$ consists of the monomials $a_0da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$.

3. Graded manifolds and bundles. A graded manifold is defined as a local-ringed space $(Z, \mathfrak{A})$ where Z is a smooth manifold and the structure sheaf $\mathfrak{A}$ is a sheaf of Grassmann rings Λ [3, 24, 25]. Sections of a sheaf $\mathfrak{A}$ are called graded functions on the graded manifold $(Z, \mathfrak{A})$. They form a graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$.

By virtue of Batchelor's theorem [3], graded manifolds possess the following structure.

Theorem 3.1. *Let $(Z, \mathfrak{A})$ be a graded manifold. There exists a vector bundle $E \to Z$ with a typical fibre $V = \mathbb{R}^m$ so that the structure sheaf of $(Z, \mathfrak{A})$ is isomorphic to the sheaf $\mathfrak{A}_E$ of sections of the exterior bundle $\wedge E^*$ whose typical fibre is the Grassmann ring $\wedge V^*$.*

Theorem 3.1 and the well-known Serre–Swan theorem lead to the following [15, 25].

Theorem 3.2. *Let Z be a smooth manifold. A graded commutative $C^\infty(Z)$-ring $\mathcal{A}$ is isomorphic to the structure ring of a graded manifold with body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank.*

We call $(Z,\mathfrak{A}_E)$ in Theorem 3.1 a simple graded manifold (henceforth, SGM) modelled over a vector bundle $E\to Z$. Accordingly, the structure ring $\mathfrak{A}_E(Z)$ of $(Z,\mathfrak{A}_E)$ is the structure module $\mathcal{A}_E=\wedge E^*(Z)$ of sections of the exterior bundle $\wedge E^*$.

REMARK 3.3. A local-ringed space $(Z,\mathfrak{A}_0=C^\infty_Z)$ exemplifies a trivial SGM modelled over the trivial bundle $E=Z\times\{0\}$. Its structure module is the ring $C^\infty(Z)$.

Every trivialization chart $(U;z^A,q^a)$ of a vector bundle $E\to Z$ yields a splitting domain $(U;z^A,c^a)$ of the SGM $(Z,\mathfrak{A}_E)$ where $\{c^a\}$ is the corresponding local fibre basis for $E^*\to X$. Graded functions on such a chart are Λ-valued functions

$$f=\sum_{k=0}^m \frac{1}{k!}f_{a_1\ldots a_k}(z)c^{a_1}\cdots c^{a_k}, \tag{4}$$

where $f_{a_1\cdots a_k}$ are smooth functions on U. One calls $\{z^A,c^a\}$ a generating basis for $(Z,\mathfrak{A}_E)$.

Let us consider the graded derivation module $\mathfrak{d}\mathcal{A}_E$ of the graded commutative ring $\mathcal{A}_E$. Its elements are called graded vector fields on the SGM $(Z,\mathfrak{A}_E)$. The following holds.

LEMMA 3.4. *Graded vector fields $u\in\mathfrak{d}\mathcal{A}_E$ on the SGM $(Z,\mathfrak{A}_E)$ are represented by sections of a certain vector bundle $\mathcal{V}_E$ locally isomorphic to $\wedge E^*\otimes_Z(E\oplus_Z TZ)$* [22, 24].

Graded vector fields on a splitting domain $(U;z^A,c^a)$ of $(Z,\mathfrak{A}_E)$ read

$$u=u^A\partial_A+u^a\partial_a, \qquad \partial_a\circ\partial_b=-\partial_b\circ\partial_a, \qquad \partial_A\circ\partial_a=\partial_a\circ\partial_A,$$

where u^A,u^a are local graded functions on U. They act on $f\in\mathfrak{A}_E(U)$ (4) by the rule

$$u(f_{a\ldots b}c^a\cdots c^b)=u^A\partial_A(f_{a\ldots b})c^a\cdots c^b+u^kf_{a\ldots b}\partial_k\rfloor(c^a\cdots c^b).$$

Given the structure ring $\mathcal{A}_E$ of the SGM $(Z,\mathfrak{A}_E)$ and the Lie superalgebra $\mathfrak{d}\mathcal{A}_E$ of its graded derivations, let us consider the graded differential calculus

$$\mathcal{S}^*[E;Z]=\mathcal{O}^*[\mathfrak{d}\mathcal{A}_E] \tag{5}$$

over $\mathcal{A}_E$ where $\mathcal{S}^0[E;Z]=\mathcal{A}_E$. Since the graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module of sections of the vector bundle $\mathcal{V}_E\to Z$ in Lemma 3.4, elements of $\mathcal{S}^*[E;Z]$ are represented by sections of the exterior bundle $\wedge\overline{\mathcal{V}}_E$ of the $\mathcal{A}_E$-dual $\overline{\mathcal{V}}_E\to Z$ of $\mathcal{V}_E$. Relative to dual fibre bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for E^*, they take the form

$$\phi=\phi_Adz^A+\phi_adc^a.$$

The duality isomorphism $\mathcal{S}^1[E;Z]=\mathfrak{d}\mathcal{A}_E^*$ is given by the graded interior product

$$u\rfloor\phi=u^A\phi_A+(-1)^{[\phi_a]}u^a\phi_a.$$

Elements of $\mathcal{S}^*[E;Z]$ are called graded exterior forms on the SGM $(Z,\mathfrak{A}_E)$.

Seen as an $\mathcal{A}_E$-algebra, the DBGA $\mathcal{S}^*[E;Z]$ (5) on a splitting domain $(U;z^A,c^a)$ is locally generated by the graded one-forms dz^A, dc^i. Accordingly, the graded Chevalley–Eilenberg coboundary operator d, called the graded exterior differential, reads

$$d\phi=dz^A\wedge\partial_A\phi+dc^a\wedge\partial_a\phi.$$

LEMMA 3.5. *The DBGA $\mathcal{S}^*[E;Z]$ (5) is the minimal differential calculus over $\mathcal{A}_E$, i.e., it is generated by df, $f\in\mathcal{A}_E$. Given the differential graded algebra $\mathcal{O}^*(Z)$ of exterior forms on Z, there is a canonical monomorphism $\mathcal{O}^*(Z)\to\mathcal{S}^*[E;Z]$* [22, 24].

A morphism of graded manifolds $(Z,\mathfrak{A}) \to (Z',\mathfrak{A}')$ is that of local-ringed spaces

$$\phi : Z \to Z', \qquad \widehat{\Phi} : \mathfrak{A}' \to \phi_*\mathfrak{A}, \tag{6}$$

where ϕ is a manifold morphism and $\widehat{\Phi}$ is a sheaf morphism of $\mathfrak{A}'$ to the direct image $\phi_*\mathfrak{A}$ of $\mathfrak{A}$ onto Z'. The morphism (6) is said to be an epimorphism if ϕ is a surjection and $\widehat{\Phi}$ is a monomorphism. It is called a graded bundle if $Z \to Z'$ is a fibre bundle [11, 26, 28]. In this case, there is a pull-back monomorphism of structure rings $\mathfrak{A}'(Z') \to \mathfrak{A}(Z)$ of graded functions on the graded manifolds $(Z',\mathfrak{A}')$ and $(Z,\mathfrak{A})$.

In particular, let $(Y,\mathfrak{A})$ be a graded manifold whose body is a fibre bundle $Y \to X$. Then we have a graded bundle $(Y,\mathfrak{A}) \to (X, C^\infty_X)$ over the trivial graded manifold (X, C^∞_X) (Remark 3.3) We call it the graded bundle over the smooth manifold X. Let us denote it by $(X,Y,\mathfrak{A})$. Its generating basis can be brought into the form (x^λ, y^i, c^a) where (x^λ, y^i) are bundle coordinates on $Y \to X$.

Remark 3.6. Let $Y \to X$ be a fibre bundle. Then the trivial graded manifold (Y, C^∞_Y) together with the ring monomorphism $C^\infty(X) \to C^\infty(Y)$ is the graded bundle (X,Y,C^∞_Y).

Remark 3.7. A graded manifold $(X,\mathfrak{A})$ itself can be treated as the graded bundle $(X,X,\mathfrak{A})$ associated to the identity smooth bundle $X \to X$.

Let $E \to Z$ and $E' \to Z'$ be vector bundles and $\Phi : E \to E'$ a bundle morphism. It yields a morphism of SGMs

$$(Z,\mathfrak{A}_E) \to (Z',\mathfrak{A}_{E'}). \tag{7}$$

In particular, the graded manifold morphism (7) is a graded bundle if Φ is a fibre bundle. Let $\mathcal{A}_{E'} \to \mathcal{A}_E$ be the corresponding pull-back monomorphism of structure rings. By virtue of Lemma 3.5 it yields a monomorphism of DBGAs

$$\mathcal{S}^*[E';Z'] \to \mathcal{S}^*[E;Z]. \tag{8}$$

Let $(Y,\mathfrak{A}_F)$ be a SGM modelled over a vector bundle $F \to Y$. This is a graded bundle $(X,Y,\mathfrak{A}_F)$ modelled over the composite bundle

$$F \to Y \to X. \tag{9}$$

The structure ring of the SGM $(Y,\mathfrak{A}_F)$ is the graded commutative $C^\infty(X)$-ring $\mathcal{A}_F = \wedge F^*(Y)$. Let the composite bundle (9) be provided with adapted bundle coordinates (x^λ, y^i, q^a). Then (x^λ, y^i, c^a) is the corresponding generating basis for $(Y,\mathfrak{A}_F)$.

4. Graded jet manifolds. Given a fibre bundle $Y \to X$, its jet manifolds J^*Y are fibre bundles over X and, therefore, they can be seen as trivial graded bundles $(X, J^kY, C^\infty_{J^kY})$. Let us define their counterparts in the case of graded bundles $(Y,\mathfrak{A}_F) \to (X, C^\infty_X)$ as follows.

Let $(X,\mathfrak{A}_E)$ be a SGM modelled over a vector bundle $E \to X$. Let us consider the k-order jet manifold J^kE of E. It is a vector bundle over X. Then let $(X,\mathfrak{A}_{J^kE})$ be a SGM modelled over $J^kE \to X$. We call $(X,\mathfrak{A}_{J^kE})$ the graded k-order jet manifold of the SGM $(X,\mathfrak{A}_E)$. Given a splitting domain $(U; x^\lambda, c^a)$ of a SGM $(Z,\mathfrak{A}_E)$, we have the splitting domain $(U; x^\lambda, c^a, c^a_\lambda, c^a_{\lambda_1\lambda_2}, \ldots, c^a_{\lambda_1\ldots\lambda_k})$ of the graded jet manifold $(X,\mathfrak{A}_{J^kE})$.

Since a SGM is a particular graded bundle over its body (Remark 3.7), the definition of graded jet manifolds is generalized to graded bundles over smooth manifolds as follows. Let $(X, Y, \mathfrak{A}_F)$ be a graded bundle modelled over the composite bundle (9). It is readily observed that the jet manifold J^rF of $F \to X$ is a vector bundle $J^rF \to J^rY$ coordinatized by $(x^\lambda, y^i_\Lambda, q^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $(J^rY, \mathfrak{A}_r = \mathfrak{A}_{J^rF})$ be a SGM modelled over this vector bundle. Its generating basis is $(x^\lambda, y^i_\Lambda, c^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. We call $(J^rY, \mathfrak{A}_r)$ the graded r-order jet manifold of the graded bundle $(X, Y, \mathfrak{A}_F)$.

In particular, let $Y \to X$ be a smooth bundle seen as the trivial graded bundle (X, Y, C^∞_Y) modelled over the composite bundle $Y \times \{0\} \to Y \to X$. Then its graded jet manifold is the trivial graded bundle $(X, J^rY, C^\infty_{J^rY)})$, i.e., the jet manifold J^rY of Y. Thus, the above notion of jets of graded bundles is compatible with the conventional one.

The jet manifolds J^*Y of a fibre bundle $Y \to X$ form the inverse sequence

$$Y \xleftarrow{\pi} J^1Y \longleftarrow \cdots \longleftarrow J^{r-1}Y \xleftarrow{\pi^r_{r-1}} J^rY \longleftarrow \cdots \tag{10}$$

of affine bundles π^r_{r-1}. Its projective limit $J^\infty Y$, called the infinite order jet manifold, is a paracompact Fréchet manifold [22, 25, 29]. A bundle coordinate atlas (x^λ, y^i) of Y provides $J^\infty Y$ with the adapted manifold coordinate atlas

$$(x^\lambda, y^i_\Lambda), \qquad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial {x'}^\lambda} d_\mu {y'}^i_\Lambda, \qquad d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + \sum_{0<|\Lambda|} y^i_{\lambda+\Lambda} \partial^\Lambda_i. \tag{11}$$

The inverse sequence (10) of jet manifolds yields a direct sequence

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^{1}_0{}^*} \mathcal{O}^*_1 \longrightarrow \cdots \longrightarrow \mathcal{O}^*_{r-1} \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \longrightarrow \cdots \tag{12}$$

of the graded differential algebras $\mathcal{O}^*_r$ of exterior forms on finite order jet manifolds. Its direct limit $\mathcal{O}^*_\infty$ consists of all these exterior forms modulo pull-back identification.

The fibre bundles $J^{r+1}F \to J^rF$ over the fibrations $J^{r+1}Y \to J^rY$ yield an inverse sequence

$$(Y, \mathfrak{A}_F) \longleftarrow (J^1Y, \mathfrak{A}_{J^1F}) \longleftarrow \cdots \longleftarrow (J^{r-1}Y, \mathfrak{A}_{J^{r-1}F}) \longleftarrow (J^rY, \mathfrak{A}_{J^rF}) \longleftarrow \cdots$$

of graded bundles (7), including pull-back monomorphisms of the structure rings

$$\mathcal{S}^0_r[F; Y] \to \mathcal{S}^0_{r+1}[F; Y] \tag{13}$$

of graded functions on $(J^rY, \mathfrak{A}_r)$ and $(J^{r+1}Y, \mathfrak{A}_{r+1})$. Its inverse limit $(J^\infty Y, \mathfrak{A}_\infty)$ is a graded Fréchet manifold whose body is the infinite order jet manifold $J^\infty Y$, and $\mathfrak{A}_\infty$ is the sheaf of germs of graded functions on the SGMs $(J^*Y, \mathfrak{A}_{J^*F})$ [22, 25, 27].

By virtue of Lemma 3.5, the differential calculus $\mathcal{S}^*_r[F; Y]$ is minimal. Therefore, monomorphisms of structure rings (13) yield a direct system

$$\mathcal{S}^*[F; Y] \xrightarrow{\pi^*} \mathcal{S}^*_1[F; Y] \longrightarrow \cdots \longrightarrow \mathcal{S}^*_{r-1}[F; Y] \xrightarrow{\pi^{r*}_{r-1}} \mathcal{S}^*_r[F; Y] \longrightarrow \cdots \tag{14}$$

of pull-back monomorphisms (8) of the DBGAs $\mathcal{S}^*_r[F; Y] \to \mathcal{S}^*_{r+1}[F; Y]$. Its direct limit $\mathcal{S}^*_\infty[F; Y]$ consists of all graded exterior forms $\phi \in \mathcal{S}^*[F_r; J^rY]$ on the SGMs $(J^rY, \mathfrak{A}_r)$ modulo pull-back identification. Its elements obey the relations (3).

Cochain monomorphisms $\mathcal{O}^*_r \to \mathcal{S}^*_r[F; Y]$ yield a monomorphism of the direct system (12) to the direct system (14) and, accordingly, a monomorphism $\mathcal{O}^*_\infty \to \mathcal{S}^*_\infty[F; Y]$.

One can think of elements of $\mathcal{S}^*_\infty[F;Y]$ as being graded exterior forms on the infinite order jet manifold $J^\infty Y$ [22, 25]. Restricted to the coordinate chart (11) of $J^\infty Y$, an $\mathcal{O}^0_\infty$-algebra $\mathcal{S}^*_\infty[F;Y]$ is locally generated by the elements

$$(c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}dx^\lambda, \theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda), \qquad 0 \leq |\Lambda|,$$

where $[c^a_\Lambda] = [\theta^a_\Lambda] = 1$ and $[dx^\lambda] = [\theta^i_\Lambda] = 0$. We call (y^i, c^a) a generating basis for $\mathcal{S}^*_\infty[F;Y]$. Let a common symbol s^A stand for its elements. We further denote $[A] = [s^A]$.

5. Graded Lagrangian formalism. Let $(X, Y, \mathfrak{A}_F)$ be a graded bundle modelled over the composite bundle (9) over an n-dimensional smooth manifold X, and let $\mathcal{S}^*_\infty[F;Y]$ be the DBGA of graded exterior forms on graded jet manifolds of $(X, Y, \mathfrak{A}_F)$. As was mentioned above, Grassmann-graded Lagrangian theory on a graded bundle is formulated in terms of the variational bicomplex which the DBGA $\mathcal{S}^*_\infty[F;Y]$ splits into [4, 13, 22, 27].

The DBGA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into the $\mathcal{S}^0_\infty[F;Y]$-modules $\mathcal{S}^{k,r}_\infty[F;Y]$ of k-contact and r-horizontal graded forms together with the corresponding projections

$$h_k : \mathcal{S}^*_\infty[F;Y] \to \mathcal{S}^{k,*}_\infty[F;Y], \qquad h^m : \mathcal{S}^*_\infty[F;Y] \to \mathcal{S}^{*,m}_\infty[F;Y].$$

Accordingly, the graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ decomposes into a sum $d = d_V + d_H$ of nilpotent vertical and total graded differentials

$$d_V \circ h^m = h^m \circ d \circ h^m, \qquad d_V(\phi) = \theta^A_\Lambda \wedge \partial^\Lambda_A \phi, \qquad \phi \in \mathcal{S}^*_\infty[F;Y],$$
$$d_H \circ h_k = h_k \circ d \circ h_k, \qquad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \qquad d_\lambda = \partial_\lambda + \sum s^A_{\lambda+\Lambda}\partial^\Lambda_A.$$

The DBGA $\mathcal{S}^*_\infty[F;Y]$ is also provided with the graded projection endomorphism

$$\varrho = \sum_{k>0} \frac{1}{k}\overline{\varrho} \circ h_k \circ h^n : \mathcal{S}^{*>0,n}_\infty[F;Y] \to \mathcal{S}^{*>0,n}_\infty[F;Y],$$
$$\overline{\varrho}(\phi) = \sum (-1)^{|\Lambda|}\theta^A \wedge [d_\Lambda(\partial^\Lambda_A \rfloor \phi)], \qquad \phi \in \mathcal{S}^{>0,n}_\infty[F;Y],$$

such that $\varrho \circ d_H = 0$, and with the nilpotent graded variational operator

$$\delta = \varrho \circ d : \mathcal{S}^{*,n}_\infty[F;Y] \to \mathcal{S}^{*+1,n}_\infty[F;Y].$$

With these operators the DBGA $\mathcal{S}^*_\infty[F;Y]$ splits into a Grassmann-graded variational bicomplex [22, 25, 27]. We restrict our consideration to a short variational subcomplex

$$0 \to \mathbb{R} \to \mathcal{S}^0_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty[F;Y] \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \varrho(\mathcal{S}^{1,n}_\infty[F;Y]), \tag{15}$$

of this bicomplex and its subcomplex of one-contact graded forms

$$0 \to \mathcal{S}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty[F;Y] \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{S}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \varrho(\mathcal{S}^{1,n}_\infty[F;Y]) \to 0. \tag{16}$$

They possess the following cohomology [13, 25, 27].

THEOREM 5.1. *The cohomology of the complex (15) equals the de Rham cohomology of Y. The complex (16) is exact.*

Decomposed into a variational bicomplex, the DBGA $\mathcal{S}^*_\infty[F;Y]$ describes graded Lagrangian theory on the graded bundle $(X, Y, \mathfrak{A}_F)$. Its Lagrangian is defined as the element

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_\infty[F;Y], \qquad \omega = dx^1 \wedge \cdots \wedge dx^n, \tag{17}$$

of the graded variational complex (15). Accordingly, the graded exterior form

$$\delta L=\theta^A\wedge\mathcal{E}_A\omega=\sum(-1)^{|\Lambda|}\theta^A\wedge d_\Lambda(\partial^\Lambda_A L)\omega\in\varrho(\mathcal{S}^{1,n}_\infty[F;Y]) \qquad (18)$$

is a graded Euler–Lagrange operator. Its kernel yields the Euler–Lagrange equation

$$\delta L=0, \qquad \mathcal{E}_A=\sum(-1)^{|\Lambda|}\theta^A\wedge d_\Lambda(\partial^\Lambda_A L)=0. \qquad (19)$$

We call $(\mathcal{S}^*_\infty[F;Y],L)$ the graded Lagrangian system and $\mathcal{S}^*_\infty[F;Y]$ its structure algebra.

The following is a corollary of Theorem 5.1 [13, 22, 25].

Corollary 5.2. *Given a graded Lagrangian L, there is the global variational formula*

$$dL=\delta L-d_H\Xi_L, \qquad \Xi\in\mathcal{S}^{n-1}_\infty[F;Y], \qquad (20)$$

$$\Xi_L=L+\sum_{s=0}\theta^A_{\nu_s\ldots\nu_1}\wedge F^{\lambda\nu_s\ldots\nu_1}_A\omega_\lambda, \qquad (21)$$

$$F^{\nu_k\ldots\nu_1}_A=\partial^{\nu_k\ldots\nu_1}_A\mathcal{L}-d_\lambda F^{\lambda\nu_k\ldots\nu_1}_A+\sigma^{\nu_k\ldots\nu_1}_A, \qquad k=1,2,\ldots,$$

where the local graded functions σ obey the relations $\sigma^\nu_A=0$, $\sigma^{(\nu_k\nu_{k-1})\ldots\nu_1}_A=0$.

The form Ξ_L (21) provides a global Lepage equivalent of the graded Lagrangian L.

6. Noether's first theorem. Given a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$, by its infinitesimal transformations are meant contact graded derivations of the graded commutative ring $\mathcal{S}^0_\infty[F;Y]$ [13, 15]. Its derivations constitute an $\mathcal{S}^0_\infty[F;Y]$-module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ which is a real Lie superalgebra relative to the Lie superbracket (1). The following holds.

Theorem 6.1. *The derivation module $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ is isomorphic to the $\mathcal{S}^0_\infty[F;Y]$-dual $\mathcal{S}^1_\infty[F;Y]^*$ of the module of graded one-forms $\mathcal{S}^1_\infty[F;Y]$.*

In particular, it follows that the DBGA $\mathcal{S}^*_\infty[F;Y]$ is the minimal differential calculus over the graded commutative ring $\mathcal{S}^0_\infty[F;Y]$. Restricted to the coordinate chart (11) of $J^\infty Y$, the algebra $\mathcal{S}^*_\infty[F;Y]$ is the free $\mathcal{S}^0_\infty[F;Y]$-module generated by the one-forms dx^λ, θ^A_Λ.

Due to the isomorphism in Theorem 6.1, a graded derivation $\vartheta\in\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ reads

$$\vartheta=\vartheta^\lambda\partial_\lambda+\vartheta^A\partial_A+\sum_{0<|\Lambda|}\vartheta^A_\Lambda\partial^\Lambda_A. \qquad (22)$$

Given $\vartheta\in\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ and $\phi\in\mathcal{S}^1_\infty[F;Y]$, let $\vartheta\rfloor\phi$ denote the corresponding interior product. Extended to the DBGA $\mathcal{S}^*_\infty[F;Y]$, it obeys the rule

$$\vartheta\rfloor(\phi\wedge\sigma)=(\vartheta\rfloor\phi)\wedge\sigma+(-1)^{|\phi|+[\phi][\vartheta]}\phi\wedge(\vartheta\rfloor\sigma), \qquad \phi,\sigma\in\mathcal{S}^*_\infty[F;Y].$$

Every graded derivation ϑ (22) of the ring $\mathcal{S}^0_\infty[F;Y]$ yields the Lie derivative

$$\mathbf{L}_\vartheta\phi=\vartheta\rfloor d\phi+d(\vartheta\rfloor\phi), \qquad \mathbf{L}_\vartheta(\phi\wedge\sigma)=\mathbf{L}_\vartheta(\phi)\wedge\sigma+(-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma),$$

of the DBGA $\mathcal{S}^*_\infty[F;Y]$. The graded derivation ϑ (22) is called contact if the Lie derivative $\mathbf{L}_\vartheta$ preserves the ideal of $\mathcal{S}^*_\infty[F;Y]$ of contact graded forms.

Lemma 6.2. *With respect to the generating basis (s^A) for the DBGA $\mathcal{S}^*_\infty[F;Y]$, any contact graded derivation of it takes the form*

$$\vartheta=\vartheta_H+\vartheta_V=\upsilon^\lambda d_\lambda+\Big[\upsilon^A\partial_A+\sum_{|\Lambda|>0}d_\Lambda(\upsilon^A-s^A_\mu\upsilon^\mu)\partial^\Lambda_A\Big], \qquad (23)$$

where ϑ_H and ϑ_V denotes the horizontal and vertical parts of ϑ [13].

A glance at the expression (23) shows that a contact graded derivation ϑ is the infinite order jet prolongation $\vartheta = J^\infty \upsilon$ of its restriction

$$\upsilon = \upsilon^\lambda \partial_\lambda + \upsilon^A \partial_A = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + (u^A \partial_A - s^A_\lambda \partial^\lambda_A) \tag{24}$$

to the graded commutative ring $S^0[F;Y]$. We call υ (24) a generalized vector field on the SGM $(Y, \mathfrak{A}_F)$. This fails to be a graded vector field on $(Y, \mathfrak{A}_F)$ in general because its component may depend on jets of elements of the generating basis for $(Y, \mathfrak{A}_F)$.

In particular, the vertical contact graded derivation (23) reads

$$\vartheta = \upsilon^A \partial_A + \sum_{|\Lambda|>0} d_\Lambda \upsilon^A \partial^\Lambda_A.$$

It is said to be nilpotent if

$$\mathbf{L}_\vartheta(\mathbf{L}_\vartheta \phi) = \sum (\upsilon^B_\Sigma \partial^\Sigma_B(\upsilon^A_\Lambda)\partial^\Lambda_A + (-1)^{[s^B][\upsilon^A]} \upsilon^B_\Sigma \upsilon^A_\Lambda \partial^\Sigma_B \partial^\Lambda_A)\phi = 0$$

for any horizontal graded form $\phi \in S^{0,*}_\infty$. It is nilpotent only if it is odd.

Remark 6.3. If there is no danger of confusion, the common symbol υ stands for the generalized vector field υ (24), the contact graded derivation ϑ determined by υ, and the Lie derivative $\mathbf{L}_\vartheta$. We call all these operators, briefly, graded derivations.

Remark 6.4. For the sake of convenience, right graded derivations $\overleftarrow{\upsilon} = \overleftarrow{\partial}_A \upsilon^A$ are also considered. They act on graded functions and exterior forms ϕ on the right by the rules

$$\overleftarrow{\upsilon}(\phi) = d\phi \lfloor \overleftarrow{\upsilon} + d(\phi \lfloor \overleftarrow{\upsilon}), \qquad \overleftarrow{\upsilon}(\phi \wedge \phi') = (-1)^{[\phi']} \overleftarrow{\upsilon}(\phi) \wedge \phi' + \phi \wedge \overleftarrow{\upsilon}(\phi').$$

Given a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$, the generalized vector field υ (24) is called a symmetry of the Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L$ of L along the contact graded derivation $\vartheta = J^\infty \upsilon$ (23) is d_H-exact, i.e., $\mathbf{L}_\vartheta L = d_H \sigma$. It follows from the global variational formula (20) that the Lie derivative of the graded Lagrangian along any contact graded derivation (23) admits a decomposition

$$\mathbf{L}_\vartheta L = \upsilon_V \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + d_V(\upsilon_H \rfloor \omega)\mathcal{L}, \tag{25}$$

called the first variational formula. A glance at this expression shows the following.

Lemma 6.5. *(i) A generalized vector field υ is a symmetry only if it projects onto X.*

(ii) A generalized vector field υ is a symmetry iff so is its vertical part υ_V (24).

(iii) υ is a symmetry iff the graded density $\upsilon_V \rfloor \delta L$ is d_H-exact.

An immediate corollary of the first variational formula (25) is Noether's first theorem.

Theorem 6.6. *If the generalized vector field υ (24) is a symmetry of the graded Lagrangian L, the first variational formula (25) leads to the weak (on-shell) conservation law*

$$0 \approx -d_H(-h_0(\vartheta \rfloor \Xi_L) + \sigma) \tag{26}$$

of the symmetry current

$$\mathcal{J}_\upsilon = \mathcal{J}^\mu_\vartheta \omega_\mu = -h_0(\vartheta \rfloor \Xi_L) + \sigma. \tag{27}$$

7. Gauge symmetries. Treating gauge symmetries of Lagrangian theory, one usually follows Yang–Mills gauge theory on principal bundles. This notion of gauge symmetries has been generalized to Lagrangian theory on an arbitrary fibre bundle $Y \to X$. Gauge symmetry is defined as a differential operator on sections of some vector bundle $E \to X$ with values in the space of symmetries of the Lagrangian L [14, 15].

To define gauge symmetries in graded Lagrangian formalism, one considers an extension of a simple graded manifold $(Y, \mathcal{A}_F)$ modelled over a composite bundle $F \to Y \to X$ to $(E^0 \times_X Y, \mathcal{A}_{E\times_X F})$ modelled over the fibre bundle $F \times_X E$, where

$$E = E_1 \oplus_X E_0 \to E_0 \to X$$

is some graded vector bundle over X. Let us consider the corresponding DBGA $\mathcal{S}^*_\infty[E \times_X F; E^0 \times_X Y]$. Given a Lagrangian $L \in \mathcal{S}^{0,n}_\infty[F; Y]$, let us define its pull-back

$$L \in \mathcal{S}^{0,n}_\infty[F; Y] \subset \mathcal{S}^*_\infty[E \times_X F; E^0 \times_X Y], \tag{28}$$

and consider the extended Lagrangian system

$$(\mathcal{S}^*_\infty[E \times_X F; E^0 \times_X Y], L) \tag{29}$$

provided with the local generating basis (s^A, c^r).

Definition 7.1. A gauge transformation of the Lagrangian L (28) is defined to be a contact graded derivation ϑ of the ring $\mathcal{S}^0_\infty[E \times_X F; E^0 \times_X Y]$ such that ϑ equals zero on the subring $\mathcal{S}^0_\infty[E; E^0] \subset \mathcal{S}^0_\infty[E \times_X F; E^0 \times_X Y]$.

In view of the condition in Definition 7.1, the variables c^r of the extended Lagrangian system (29) can be treated as gauge parameters of the gauge transformation ϑ. Furthermore, we additionally assume that ϑ is linear in the gauge parameters c^r and their jets c^r_Λ. Then the generalized vector field υ (24) reads

$$\upsilon = \sum_{0\leq|\Lambda|\leq m} \upsilon^{\lambda\Lambda}_r(x^\mu) c^r_\Lambda \partial_\lambda + \sum_{0\leq|\Lambda|\leq m} \upsilon^{A\Lambda}_r(x^\mu, s^B_\Sigma) c^r_\Lambda \partial_A. \tag{30}$$

This is called a gauge symmetry if it is a symmetry of the Lagrangian L (28).

By virtue of item (ii) of Lemma 6.5, the generalized vector field υ (30) is a gauge symmetry iff its vertical part is. If υ (30) is a gauge symmetry, the corresponding conserved current $\mathcal{J}_\upsilon$ (27) is reduced to the superpotential as follows [15, 23, 27].

Theorem 7.2. *If υ (30) is a gauge symmetry of a Lagrangian L, the corresponding conserved current $\mathcal{J}_\upsilon$ (27) takes the form*

$$\mathcal{J}_\upsilon = W + d_H U = (W^\mu + d_\nu U^{\nu\mu})\omega_\mu, \tag{31}$$

where the term W vanishes on-shell, and $U = U^{\nu\mu}\omega_{\nu\mu}$ is called the superpotential.

8. Noether and higher-stage Noether identities. Without loss of generality, let a Lagrangian L be even and its Euler–Lagrange operator δL (18) at least of first order.

Remark 8.1. Let us introduce the following notation. If $E \to X$ is a vector bundle, we call $\overline{E} = E^* \otimes_X \wedge^n T^*X$ the density dual of E. Given a fibre bundle $Y \to X$, by the density-dual of its vertical tangent bundle VY is meant the fibre bundle

$$\overline{VY} = V^*Y \otimes_Y \wedge^n T^*X. \tag{32}$$

If $Y = E$ is a vector bundle, then $\overline{VE} = \overline{E} \times_X E$. Let $E = E^0 \oplus_X E^1$ be a graded vector bundle over X. Its graded density-dual is defined to be $\overline{E} = \overline{E}^1 \oplus \overline{E}^0$ with even part $\overline{E}^1 \to X$ and odd part $\overline{E}^0 \to X$. Given a graded vector bundle E, we consider the product $(X, E^0 \times_X Y, \mathfrak{A}_{E\times_X F})$ of graded bundles over the product of composite bundles F (9) and $E \to E^0 \to X$. The corresponding DBGA reads $\mathcal{S}^*_\infty[F \times_X E; Y \times_X E^0]$. In particular, we treat the composite bundle F (9) as a graded vector bundle over Y only with an odd part. The density-dual (32) of the vertical tangent bundle VF of $F \to X$ is $\overline{VF} = V^*F \otimes_F \wedge^n T^*X$. However it is not a vector bundle over Y. Therefore, we focus on the case of the pull-back bundle $F = Y \times_X F^1$ where $F^1 \to X$ is a vector bundle. Then

$$\overline{VF} = \overline{F}^1 \oplus_Y ((V^*Y \otimes_Y \wedge^n T^*X) \oplus_Y F^1) \tag{33}$$

is a graded vector bundle over Y. It can be seen as a product of composite bundles

$$\overline{VF^1} = \overline{F}^1 \oplus_X F^1 \to \overline{F}^1 \to X, \qquad \overline{VY} \to Y \to X.$$

Let us consider the corresponding graded bundle and DBGA

$$\mathcal{P}^*_\infty[\overline{VF}; Y] = \mathcal{S}^*_\infty[\overline{VF}; Y \times_X \overline{F}^1] = \mathcal{S}^*_\infty[\overline{VF}^1 \times_X \overline{VY}; Y \times_X \overline{F}^1]. \tag{34}$$

One can associate to any graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ the chain complex (35) whose one-boundaries vanish on-shell. Let us consider the density-dual $\overline{VF}$ (33) of the vertical tangent bundle $VF \to F$, and let us enlarge the original DBGA $\mathcal{S}^*_\infty[F;Y]$ to the DBGA $\mathcal{P}^*_\infty[\overline{VF}; Y]$ (34) with the generating basis $(s^A, \overline{s}_A)$, $[\overline{s}_A] = [A] + 1$. Following the terminology of Lagrangian BRST theory [2, 16], we call its elements $\overline{s}_A$ antifields of antifield number $\mathrm{Ant}[\overline{s}_A] = 1$. The DBGA $\mathcal{P}^*_\infty[\overline{VF}; Y]$ is endowed with the nilpotent right graded derivation $\overline{\delta} = \overleftarrow{\partial}^A \mathcal{E}_A$, where $\mathcal{E}_A$ are the variational derivatives (18). Then we have a chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF}; Y]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF}; Y]_2 \tag{35}$$

of graded densities of antifield number ≤ 2. Its one-boundaries $\overline{\delta}\Phi$, $\Phi \in \mathcal{P}^{0,n}_\infty[\overline{VF}; Y]_2$, by the very definition, vanish on-shell. Any one-cycle

$$\Phi = \sum \Phi^{A,\Lambda} \overline{s}_{\Lambda A} \omega \in \mathcal{P}^{0,n}_\infty[\overline{VF}; Y]_1 \tag{36}$$

of the complex (35) is a differential operator on the bundle $\overline{VF}$ which is linear on the fibres of $\overline{VF} \to F$ and its kernel contains the graded Euler–Lagrange operator δL (18), i.e.,

$$\overline{\delta}\Phi = 0, \qquad \sum \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A \omega = 0. \tag{37}$$

Referring to the notion of NI of a differential operator, we say that the one-cycles Φ (36) define the NI (37) of the Euler–Lagrange operator δL [4, 21].

In particular, one-chains Φ (36) are necessarily NI if they are boundaries. Therefore, these NI are called trivial. Accordingly, non-trivial NI modulo trivial ones are associated to elements of the first homology $H_1(\overline{\delta})$ of the complex (35).

Non-trivial NI can obey first-stage NI. To describe them, let us assume that the module $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)} \subset H_1(\overline{\delta})$ of finite rank with a basis $\{\Delta_r \omega\}$ so that elements $\Phi \in H_1(\overline{\delta})$

factorize as

$$\Phi = \sum \Phi^{r,\Xi} d_\Xi \Delta_r \omega, \qquad \Phi^{r,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{38}$$

through elements $\Delta_r \omega$ of $\mathcal{C}_{(0)}$. Thus, all non-trivial NI (37) result from the NI

$$\overline{\delta}\Delta_r = \sum \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{39}$$

called the complete NI. A Lagrangian system with finitely generated non-trivial NI is called finitely degenerate. Hereafter, only Lagrangian systems of this type are considered.

Then the complex (35) can be extended to the chain complex (41) with a boundary operator whose nilpotency conditions are equivalent to the complete NI (39). By virtue of Theorem 3.2, a graded module $\mathcal{C}_{(0)}$ is isomorphic to that of sections of the density-dual $\overline{E}_0$ of some graded vector bundle $E_0 \to X$. Let us enlarge $\mathcal{P}^*_\infty[\overline{VF};Y]$ to a DBGA

$$\overline{\mathcal{P}}^*_\infty\{0\} = \mathcal{P}^*_\infty[\overline{VF} \times_X \overline{E}_0; Y] = \mathcal{S}^*_\infty[\overline{VF} \times_X \overline{E}_0; \overline{E}_0^1 \times_X \overline{F}^1 \times_X Y] \tag{40}$$

with a generating basis $(s^A, \overline{s}_A, \overline{c}_r)$ where $\overline{c}_r$ are antifields of Grassmann parity $[\overline{c}_r] = [\Delta_r] + 1$ and antifield number $\mathrm{Ant}[\overline{c}_r] = 2$. The DBGA (40) admits an odd right graded derivation $\delta_0 = \overline{\delta} + \overleftarrow{\partial}^r \Delta_r$ which is nilpotent iff the complete NI (39) hold. Then δ_0 is a boundary operator of a chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \overset{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \overset{\delta_0}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \overset{\delta_0}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_3 \tag{41}$$

of graded densities of antifield number ≤ 3. We can show that its cohomology $H_1(\delta_0)$ is 0.

Let us consider the second homology $H_2(\delta_0)$ of the complex (41). Its two-chains read

$$\Phi = G + H = \sum G^{r,\Lambda} \overline{c}_{\Lambda r} \omega + \sum H^{(A,\Lambda)(B,\Sigma)} \overline{s}_{\Lambda A} \overline{s}_{\Sigma B} \omega. \tag{42}$$

Its two-cycles define the first-stage NI

$$\delta_0 \Phi = 0, \qquad \sum G^{r,\Lambda} d_\Lambda \Delta_r \omega = -\overline{\delta} H. \tag{43}$$

Conversely, let the equality (43) hold. Then it is a cycle condition of the two-chain (42).

The first-stage NI (43) are trivial either if the two-cycle Φ (42) is a δ_0-boundary or its summand G vanishes on-shell. Therefore, non-trivial first-stage NI fail to exhaust the second homology $H_2(\delta_0)$ of the complex (41) in general. One can show that non-trivial first-stage NI modulo trivial ones are identified with elements of the homology $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$ is a δ_0-boundary.

Non-trivial first-stage NI can obey second-stage NI, and so on. Iterating the arguments, we say that a degenerate graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ is N-stage reducible if it admits finitely generated non-trivial N-stage NI, but no non-trivial $(N+1)$-stage ones. It is characterized as follows [4, 25, 27].

There are graded vector bundles $E_0, \ldots, E_N$ over X, and a DBGA $\mathcal{P}^*_\infty[\overline{VF};Y]$ is enlarged to a DBGA

$$\overline{\mathcal{P}}^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{VF} \times_X \overline{E}_0 \times_X \cdots \times_X \overline{E}_N; Y] \tag{44}$$

with a generating basis $(s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$ where $\overline{c}_{r_k}$ are k-stage antifields of antifield number $\mathrm{Ant}[\overline{c}_{r_k}] = k + 2$. It is provided with a nilpotent right graded derivation

$$\delta_{\mathrm{KT}} = \delta_N = \overline{\delta} + \sum \overleftarrow{\partial}^r \Delta_r^{A,\Lambda} \overline{s}_{\Lambda A} + \sum_{1\leq k\leq N} \overleftarrow{\partial}^{r_k} \Delta_{r_k}, \tag{45}$$

$$\Delta_{r_k}\omega = \sum \Delta_{r_k}^{r_{k-1},\Lambda} \overline{c}_{\Lambda r_{k-1}}\omega + \sum (h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} + \cdots)\omega \in \overline{\mathcal{P}}_\infty^{0,n}\{k-1\}_{k+1}, \tag{46}$$

of antifield number -1, where $k = -1$ stands for $\overline{s}_A$. It is called the KT operator. With this graded derivation, a module $\overline{\mathcal{P}}_\infty^{0,n}\{N\}_{\leq N+3}$ of densities of antifield number $\leq (N+3)$ decomposes into the exact KT chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF};Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}_\infty^{0,n}\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}_\infty^{0,n}\{1\}_3 \cdots \xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}_\infty^{0,n}\{N-1\}_{N+1} \xleftarrow{\delta_{\mathrm{KT}}} \overline{\mathcal{P}}_\infty^{0,n}\{N\}_{N+2} \xleftarrow{\delta_{\mathrm{KT}}} \overline{\mathcal{P}}_\infty^{0,n}\{N\}_{N+3} \tag{47}$$

which satisfies the homology regularity condition that any $\delta_{k<N}$-cycle $\phi \in \overline{\mathcal{P}}_\infty^{0,n}\{k\}_{k+3} \subset \overline{\mathcal{P}}_\infty^{0,n}\{k+1\}_{k+3}$ is a δ_{k+1}-boundary. The nilpotence $\delta_{\mathrm{KT}}^2 = 0$ of the KT operator (45) is equivalent to the complete non-trivial NI (39) and the complete non-trivial $(k \leq N)$-stage NI

$$\sum \Delta_{r_k}^{r_{k-1},\Lambda} d_\Lambda \Big(\sum \Delta_{r_{k-1}}^{r_{k-2},\Sigma} \overline{c}_{\Sigma r_{k-2}}\Big) = -\overline{\delta}\Big(\sum h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A}\Big). \tag{48}$$

Any $(k+2)$-chain $\Phi \in \mathcal{P}_\infty^{0,n}\{k\}_{k+2}$ takes the form

$$\Phi = G + H = \sum G^{r_k,\Lambda} \overline{c}_{\Lambda r_k}\omega + \sum (H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} + \cdots)\omega. \tag{49}$$

If it is a δ_k-cycle, then

$$\sum G^{r_k,\Lambda} d_\Lambda \Big(\sum \Delta_{r_k}^{r_{k-1},\Sigma} \overline{c}_{\Sigma r_{k-1}}\Big) + \overline{\delta}\Big(\sum H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}}\Big) = 0 \tag{50}$$

are the k-stage NI.

9. Noether's second theorems. Different variants of the second Noether theorem have been suggested in order to relate reducible NI and gauge symmetries [2, 4, 10]. The inverse second Noether theorem (Theorem 9.3), which we formulate in homology terms, associates to the KT complex (47) of non-trivial NI the cochain sequence (56) with the ascent operator **u** (57) whose components are gauge and higher-stage gauge symmetries of a Lagrangian system. Let us start with the following notation.

REMARK 9.1. Given the DBGA $\overline{\mathcal{P}}_\infty^*\{N\}$ (44), we consider a DBGA

$$P_\infty^*\{N\} = \mathcal{S}_\infty^*[F \times_X E_0 \times_X \cdots \times_X E_N; Y \times_X E_0^0 \times_X \cdots \times_X E_N^0], \tag{51}$$

possessing the generating basis $(s^A, c^r, c^{r_1}, \ldots, c^{r_N})$, $[c^{r_k}] = [\overline{c}_{r_k}] + 1$, and a DBGA

$$\mathcal{P}_\infty^*\{N\} = \mathcal{P}_\infty^*[\overline{VF} \times_X E_0 \times_X \cdots \times_X E_N \times_X \overline{E}_0 \times_X \cdots \times_X \overline{E}_N; Y] \tag{52}$$

with a generating basis $(s^A, \overline{s}_A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$. Their elements c^{r_k} are called k-stage ghosts of ghost number $\mathrm{gh}[c^{r_k}] = k+1$ and antifield number $\mathrm{Ant}[c^{r_k}] = -(k+1)$. The $C^\infty(X)$-module $\mathcal{C}^{(k)}$ of k-stage ghosts is the density-dual of the module $\mathcal{C}_{(k+1)}$ of $(k+1)$-stage antifields. The DBGAs $\overline{\mathcal{P}}_\infty^*\{N\}$ (44) and the DBGA $P^*(N)$ (51) are subalgebras of the DBGA $\mathcal{P}_\infty^*\{N\}$ (52). The KT operator δ_{KT} (45) is naturally extended to $\mathcal{P}_\infty^*\{N\}$.

REMARK 9.2. Any graded differential form $\phi \in \mathcal{S}^*_\infty[F;Y]$ and any finite tuple (f^Λ), $0 \leq |\Lambda| \leq k$, of local graded functions $f^\Lambda \in \mathcal{S}^0_\infty[F;Y]$ obey the following relations:

$$\sum_{0\leq|\Lambda|\leq k} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda\phi) = \sum_{0\leq|\Lambda|\leq k} \eta(f)^\Lambda d_\Lambda\phi, \tag{53}$$

$$\eta(f)^\Lambda = \sum_{0\leq|\Sigma|\leq k-|\Lambda|} (-1)^{|\Sigma+\Lambda|} \frac{(|\Sigma+\Lambda|)!}{|\Sigma|!|\Lambda|!} d_\Sigma f^{\Sigma+\Lambda}, \tag{54}$$

$$\eta(\eta(f))^\Lambda = f^\Lambda. \tag{55}$$

THEOREM 9.3. *Given the KT complex (47), a module of graded densities $P^{0,n}_\infty\{N\}$ decomposes into a cochain sequence*

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{u}} P^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{u}} P^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{u}} \cdots, \tag{56}$$

$$\mathbf{u} = u + u^{(1)} + \cdots + u^{(N)} = u^A\frac{\partial}{\partial s^A} + u^r\frac{\partial}{\partial c^r} + \cdots + u^{r_{N-1}}\frac{\partial}{\partial c^{r_{N-1}}}, \tag{57}$$

graded by ghost number. Its ascent operator $\mathbf{u}$ *(57) is an odd graded derivation of ghost number* 1 *where* u *(62) is a gauge symmetry of a graded Lagrangian* L *and the graded derivations* $u_{(k)}$ *(65),* $k = 1,\ldots,N$*, obey the relations (64).*

Proof. Given the KT operator (45), let us extend the original Lagrangian L to

$$L_e = L + L_1 = L + \sum_{0\leq k\leq N} c^{r_k}\Delta_{r_k}\omega = L + \delta_{\rm KT}\Big(\sum_{0\leq k\leq N} c^{r_k}\overline{c}_{r_k}\omega\Big) \tag{58}$$

of zero antifield number. It is readily observed that the KT operator $\delta_{\rm KT}$ is an exact symmetry of the extended Lagrangian $L_e \in \mathcal{P}^{0,n}_\infty\{N\}$ (58). Since the graded derivation $\delta_{\rm KT}$ is vertical, it follows from the decomposition (25) that

$$\left[\frac{\delta^\leftarrow\mathcal{L}_e}{\delta\overline{s}_A}\mathcal{E}_A + \sum_{0\leq k\leq N}\frac{\delta^\leftarrow\mathcal{L}_e}{\delta\overline{c}_{r_k}}\Delta_{r_k}\right]\omega = \left[\upsilon^A\mathcal{E}_A + \sum_{0\leq k\leq N}\upsilon^{r_k}\frac{\delta\mathcal{L}_e}{\delta c^{r_k}}\right]\omega = d_H\sigma, \tag{59}$$

$$\upsilon^A = \frac{\delta^\leftarrow\mathcal{L}_e}{\delta\overline{s}_A} = u^A + w^A = \sum c^r_\Lambda\eta(\Delta^A_r)^\Lambda + \sum_{1\leq i\leq N}\sum c^{r_i}_\Lambda\eta(\overleftarrow{\partial}{}^A(h_{r_i}))^\Lambda,$$

$$\upsilon^{r_k} = \frac{\delta^\leftarrow\mathcal{L}_e}{\delta\overline{c}_{r_k}} = u^{r_k} + w^{r_k} = \sum c^{r_{k+1}}_\Lambda\eta(\Delta^{r_k}_{r_{k+1}})^\Lambda + \sum_{k+1<i\leq N}\sum c^{r_i}_\Lambda\eta(\overleftarrow{\partial}{}^{r_k}(h_{r_i}))^\Lambda.$$

The equality (59) splits into a set of equalities

$$\frac{\delta^\leftarrow(c^r\Delta_r)}{\delta\overline{s}_A}\mathcal{E}_A\omega = u^A\mathcal{E}_A\omega = d_H\sigma_0, \tag{60}$$

$$\left[\frac{\delta^\leftarrow(c^{r_k}\Delta_{r_k})}{\delta\overline{s}_A}\mathcal{E}_A + \sum_{0\leq i<k}\frac{\delta^\leftarrow(c^{r_k}\Delta_{r_k})}{\delta\overline{c}_{r_i}}\Delta_{r_i}\right]\omega = d_H\sigma_k, \tag{61}$$

where $k = 1,\ldots,N$. A glance at the equality (60) shows that, by virtue of the decomposition (25), the odd graded derivation

$$u = u^A\partial_A, \qquad u^A = \sum c^r_\Lambda\eta(\Delta^A_r)^\Lambda, \tag{62}$$

of $P^0_\infty\{0\}$ is a symmetry of the graded Lagrangian L. It satisfies Definition 7.1 and, thus, it is a gauge symmetry of L associated to the complete non-trivial NI (39). Every equality (61) splits into a set of equalities graded by the polynomial degree in antifields. Let us consider the one that is linear in the antifields $\overline{c}_{r_{k-2}}$. We obtain the equality

$$\sum c^{r_k} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} d_\Xi \mathcal{E}_A \omega + u^{r_{k-1}} \sum \Delta^{r_{k-2},\Xi}_{r_{k-1}} \overline{c}_{\Xi r_{k-2}} \omega = d_H \sigma'_k. \tag{63}$$

The variational derivative of both sides with respect to $\overline{c}_{r_{k-2}}$ leads to the relation

$$\sum d_\Sigma u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}_\Sigma} u^{r_{k-2}} = \overline{\delta}(\alpha^{r_{k-2}}), \tag{64}$$
$$\alpha^{r_{k-2}} = -\sum \eta(h^{(r_{k-2})(A,\Xi)}_{r_k})^\Sigma d_\Sigma (c^{r_k} \overline{s}_{\Xi A}),$$

satisfied by the odd graded derivation

$$u^{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} = \sum c^{r_k}_\Lambda \eta(\Delta^{r_{k-1}}_{r_k})^\Lambda \frac{\partial}{\partial c^{r_{k-1}}}, \quad k = 1, \ldots, N. \tag{65}$$

The relation (64) for $k = 1$ takes the form

$$\sum d_\Sigma u^r \partial^\Sigma_r u^A = \overline{\delta}(\alpha^A)$$

of a first-stage gauge symmetry condition on-shell, satisfied by the non-trivial gauge symmetry u (62). Therefore, one can treat the odd graded derivation

$$u^{(1)} = u^r \partial_r, \qquad u^r = \sum c^{r_1}_\Lambda \eta(\Delta^r_{r_1})^\Lambda,$$

as a first-stage gauge symmetry associated to the complete first-stage NI

$$\sum \Delta^{r,\Lambda}_{r_1} d_\Lambda \Big(\sum \Delta^{A,\Sigma}_r \overline{s}_{\Sigma A}\Big) = -\overline{\delta}\Big(\sum h^{(B,\Sigma)(A,\Xi)}_{r_1} \overline{s}_{\Sigma B} \overline{s}_{\Xi A}\Big).$$

Iterating the arguments, one comes to the relation (64) which provides a k-stage gauge symmetry condition which is associated to the complete non-trivial k-stage NI (48). The odd graded derivation $u_{(k)}$ (65) is called a k-stage gauge symmetry. ■

Thus, components of the ascent operator $\mathbf{u}$ (57) in Theorem 9.3 are non-trivial gauge and higher-stage gauge symmetries. Therefore, we call this operator the gauge operator.

The correspondence of gauge and higher-stage gauge symmetries to NI and higher-stage NI in Theorem 9.3 is unique due to the following direct second Noether theorem.

THEOREM 9.4. *(i) If u (62) is a gauge symmetry, the variational derivative of the d_H-exact density $u^A \mathcal{E}_A \omega$ (60) with respect to the ghosts c^r leads to the equality*

$$\begin{aligned}\delta_r(u^A \mathcal{E}_A \omega) &= \sum (-1)^{|\Lambda|} d_\Lambda [u^{A\Lambda}_r \mathcal{E}_A] \\ &= \sum (-1)^{|\Lambda|} d_\Lambda (\eta(\Delta^A_r)^\Lambda \mathcal{E}_A) = \sum (-1)^{|\Lambda|} \eta(\eta(\Delta^A_r))^\Lambda d_\Lambda \mathcal{E}_A = 0,\end{aligned} \tag{66}$$

which reproduces the complete NI (39) by means of the relation (55).

(ii) Given the k-stage gauge symmetry condition (64), the variational derivative of the equality (63) with respect to the ghosts c^{r_k} leads to an equality reproducing the k-stage NI (48) by means of the relations (53)–(55).

10. Lagrangian BRST theory. The gauge operator $\mathbf{u}$ (56) need not be nilpotent. Let us study its extension to a nilpotent graded derivation

$$\mathbf{b} = \mathbf{u} + \gamma = \mathbf{u} + \sum_{1\leq k\leq N+1} \gamma^{(k)} = \mathbf{u} + \sum_{1\leq k\leq N+1} \gamma^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}}$$
$$= \left(u^A \frac{\partial}{\partial s^A} + \gamma^r \frac{\partial}{\partial c^r}\right) + \sum_{0\leq k\leq N-1} \left(u^{r_k} \frac{\partial}{\partial c^{r_k}} + \gamma^{r_{k+1}} \frac{\partial}{\partial c^{r_{k+1}}}\right) \tag{67}$$

of ghost number 1 by means of antifield-free terms $\gamma^{(k)}$ of higher polynomial degree in the ghosts c^{r_i} and their jets $c^{r_i}_\Lambda$, $0 \leq i < k$. We call $\mathbf{b}$ (67) the BRST operator, where k-stage gauge symmetries are extended to k-stage BRST transformations acting both on $(k-1)$-stage and k-stage ghosts [14, 25, 26].

If a BRST operator exists, the sequence (56) turns into a BRST complex

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{b}} P^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{b}} P^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{b}} \cdots . \tag{68}$$

The following is a necessary condition of the existence of a BRST extension [25, 26].

THEOREM 10.1. *The gauge operator (56) admits the nilpotent extension (67) only if the gauge symmetry conditions (64) and the higher-stage NI (48) are satisfied off-shell.*

The KT complex (47) and the BRST complex (68) provide the BRST extension

$$L_E = L + \mathbf{b}\Big(\sum_{0\leq k\leq N} c^{r_{k-1}} \overline{c}_{r_{k-1}}\Big)\omega + d_H\sigma, \tag{69}$$

of the original Lagrangian theory by the graded ghosts c^{r_k} and the antifields $\overline{c}_{r_k}$.

This extension is a preliminary step towards the BV quantization of reducible degenerate Lagrangian theories [2, 16].

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