GEOMETRY OF JETS AND FIELDS
BANACH CENTER PUBLICATIONS, VOLUME 110
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2016

DOUBLE EXTENSION FOR COMMUTATIVE n-ARY SUPERALGEBRAS WITH A SKEW-SYMMETRIC INVARIANT FORM

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Abstract. The method of double extension, introduced by A. Medina and Ph. Revoy, is a procedure which decomposes a Lie algebra with an invariant symmetric form into elementary pieces. Such decompositions were developed for other algebras, for instance for Lie superalgebras and associative algebras, Filippov n-algebras and Jordan algebras.

The aim of this note is to find a unified approach to such decompositions using the derived bracket formalism. More precisely, we show that any commutative n-ary superalgebra with a skew-symmetric invariant form can be obtained inductively by taking orthogonal sums and generalized double extensions.

1. Preliminaries. Let $\mathfrak{a} = \mathfrak{a}_{\bar 0} \oplus \mathfrak{a}_{\bar 1}$ be a finite dimensional $\mathbb{Z}_2$-graded vector space over $\mathbb{K}$, where we assume for simplicity that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\bar a \in \mathbb{Z}_2$ the parity of a homogeneous element $a \in \mathfrak{a}_{\bar i}$. A bilinear form $(\,,\,)$ on $\mathfrak{a}$ is called *even* if the corresponding linear map $\mathfrak{a} \otimes \mathfrak{a} \to \mathbb{K}$ is even. A bilinear form is called *skew-symmetric* if

$$(a, b) = -(-1)^{\bar a \bar b}(b, a)$$

for any homogeneous elements $a, b \in \mathfrak{a}$. From now on we assume that $(\,,\,)$ is an even non-degenerate skew-symmetric form on $\mathfrak{a}$. This is

- $(\,,\,)|_{\mathfrak{a}_{\bar 0} \times \mathfrak{a}_{\bar 0}}$ is a non-degenerate skew-symmetric form;
- $(\,,\,)|_{\mathfrak{a}_{\bar 1} \times \mathfrak{a}_{\bar 1}}$ is a non-degenerate symmetric form;
- $(\,,\,)|_{\mathfrak{a}_{\bar 0} \times \mathfrak{a}_{\bar 1}} = 0$.

DEFINITION 1.1. • An *n-ary superalgebra* is a vector space $\mathfrak{a}$ together with an n-linear map $\mathfrak{a} \times \cdots \times \mathfrak{a} \to \mathfrak{a}$. We denote this map by $(a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}$.

2010 *Mathematics Subject Classification*: 17B20, 17B01.
Key words and phrases: double extension, n-ary superalgebras, invariant form.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc110-0-18

• An n-ary superalgebra is called *commutative* if

$$\{a_1,\dots,a_i,a_{i+1},\dots,a_n\} = (-1)^{\bar a_i \bar a_{i+1}}\{a_1,\dots,a_{i+1},a_i,\dots,a_n\} \tag{1}$$

for all homogeneous $a_i, a_{i+1} \in \mathfrak{a}$.

• A commutative n-ary superalgebra is called *invariant with respect to the form* $(\,,)$ if the following holds:

$$(a_0,\{a_1,\dots,a_n\}) = (-1)^{\bar a_0 \bar a_1}(a_1,\{a_0,a_2,\dots,a_n\}) \tag{2}$$

for all homogeneous $a_i \in \mathfrak{a}$.

We will write *a commutative invariant n-ary superalgebra* as a shorthand for *a commutative n-ary superalgebra that is invariant with respect to the form* $(\,,)$. Let $\mathfrak{a}$ be a commutative n-ary superagebra.

Definition 1.2. • An *n-ary subalgebra* in $\mathfrak{a}$ is a vector subspace $\mathfrak{b}\subset\mathfrak{a}$ such that $\{\mathfrak{b},\dots,\mathfrak{b}\}\subset\mathfrak{b}$. An *ideal* in $\mathfrak{a}$ is a vector subspace $\mathfrak{i}\subset\mathfrak{a}$ such that $\{\mathfrak{a},\dots,\mathfrak{a},\mathfrak{i}\}\subset\mathfrak{i}$.

• Let $\mathfrak{a}$ and $\mathfrak{b}$ be two n-ary algebras. An even linear map $\phi:\mathfrak{a}\to\mathfrak{b}$ is called a *homomorphism* of n-ary algebras if

$$\phi(\{a_1,\dots,a_n\}_{\mathfrak{a}}) = \{\phi(a_1),\dots,\phi(a_n)\}_{\mathfrak{b}}.$$

The vector space $\ker\phi\subset\mathfrak{a}$ is an ideal in $\mathfrak{a}$.

• A commutative n-ary superalgebra is called *simple* if it is not trivial one dimensional and it does not have any proper ideals.

• An invariant commutative n-ary superalgebra is called *irreducible* if it is not a direct sum of two non-degenerate ideals. (An ideal $\mathfrak{i}$ is called *non-degenerate* if $(\,,)|_{\mathfrak{i}}$ is non-degenerate.)

The main tool that we use in this paper is the derived braket construction. Let $\mathfrak{a}$ be a $\mathbb{Z}_2$-graded vector space and $(\,,)$ be as above. We denote by $S^n\mathfrak{a}$ the n-th symmetric power of $\mathfrak{a}$ and we put $S^*\mathfrak{a}=\bigoplus_n S^n\mathfrak{a}$. The superspace $S^*\mathfrak{a}$ possesses a natural structure $[\,,]$ of a Poisson superalgebra that is defined in the following way:

$$[x,y] := (x,y),\quad x,y\in\mathfrak{a};$$
$$[v,w_1\cdot w_2] := [v,w_1]\cdot w_2 + (-1)^{vw_1}w_1\cdot[v,w_2],$$
$$[v,w] = -(-1)^{vw}[w,v],$$

where v,w,w_i are homogeneous elements in $S^*\mathfrak{a}$. The super-Jacobi identity has the following form:

$$[v,[w_1,w_2]] = [[v,w_1],w_2] + (-1)^{\bar v \bar w_1}[w_1,[v,w_2]].$$

Let us take any element $\mu\in S^{n+1}\mathfrak{a}$. Then we can define an n-ary superalgebra on $\mathfrak{a}$ in the following way:

$$\{a_1,\dots,a_n\} := [a_1,[\dots,[a_n,\mu]\dots]],\qquad a_i\in\mathfrak{a}. \tag{3}$$

Clearly, $\{a_1,\dots,a_n\}\in\mathfrak{a}$. We will denote the corresponding superalgebra by $(\mathfrak{a},\mu)$ and we will call the element μ the *derived potential* of $(\mathfrak{a},\mu)$. The n-ary superalgebras $(\mathfrak{a},\mu)$ are commutative and invariant with respect to the form $(\,,)$, see [Vor] and [V]. We will need the following observation (see [V] for details).

PROPOSITION 1.3. *Assume that $(,)$ is not degenerate. Any commutative invariant n-ary superalgebras on $\mathfrak{a}$ can be obtained by construction (3).*

2. Double extension for invariant n-ary superalgebras. Let $\mathfrak{g}$ be an invariant commutative n-ary superalgebra and $\mu \in S^{n+1}\mathfrak{g}$ be its derived potential. Let $\mathfrak{h}$ be any commutative n-ary superalgebra with the multiplication $\nu \in S^n\mathfrak{h}^* \otimes \mathfrak{h}$. We can identify the vector spaces $S^n\mathfrak{h}^* \otimes \mathfrak{h}$ with the subspace $S^n\mathfrak{h}^* \cdot \mathfrak{h} \subset S^*(\mathfrak{h} \oplus \mathfrak{h}^*)$ and consider ν as a derived potential for the invariant superalgebra $\mathfrak{h} \oplus \mathfrak{h}^*$. The even non-degenerate skew-symmetric invariant form $(,)$ on $\mathfrak{h} \oplus \mathfrak{h}^*$ is given by:

$$(\alpha, x) := \alpha(x), \quad (x, \alpha) := -(-1)^{\bar{\alpha}\bar{x}}(\alpha, x),$$

$\alpha \in \mathfrak{h}^*$, $x \in \mathfrak{h}$ are homogeneous elements.

Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ be a vector space with the non-degenerate skew-symmetric bilinear form $(,)$ that is the sum of the non-degenerate skew-symmetric bilinear forms on $\mathfrak{g}$ and $\mathfrak{h} \oplus \mathfrak{h}^*$.

DEFINITION 2.1. The commutative invariant n-ary superalgebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ with the derived potential

$$\begin{gathered} L = \mu + \nu + \sum_{i=1}^{n+1} \psi_i, \quad \text{where} \quad \mu \in S^{n+1}\mathfrak{g}, \\ \nu \in S^n\mathfrak{h}^* \cdot \mathfrak{h}, \quad \psi_i \in S^i\mathfrak{h}^* \cdot S^{n-i+1}\mathfrak{g} \end{gathered} \tag{4}$$

is called a *generalized double extension* of $\mathfrak{g}$ by $\mathfrak{h}$ via ψ_i, $i = 1, \dots, n+1$.

The main observation of this section is:

MAIN THEOREM 2.2. *Assume that $\mathfrak{a}$ is an irreducible but not simple commutative invariant n-ary superalgebra. Then $\mathfrak{a}$ is isomorphic to a certain generalized double extension.*

Proof. Let us take a maximal non-trivial ideal $\mathfrak{i}$ of $\mathfrak{a}$. Clearly, $\mathfrak{i}^\perp$ is a minimal ideal in $\mathfrak{a}$. Futhermore, since $\mathfrak{i}^\perp \cap \mathfrak{i}$ is also an ideal and $\mathfrak{a}$ is irreducible, we see that $\mathfrak{i}^\perp \subset \mathfrak{i}$. Therefore, $\mathfrak{i}^\perp$ is a minimal isotropic ideal in $\mathfrak{a}$. Let us take a subspace $\mathfrak{h}$ in $\mathfrak{a}$ such that $\mathfrak{h}$ is isotropic, $\mathfrak{h} \cap \mathfrak{i}^\perp = \{0\}$ and $(,)|_{\mathfrak{h}\oplus\mathfrak{i}^\perp}$ is non-degenerate. Since, $(,)|_{\mathfrak{h}\oplus\mathfrak{i}^\perp}$ is non-degenerate, we have $\mathfrak{a} = \mathfrak{i} \oplus \mathfrak{h}$. Consider the vector space $\mathfrak{w} = \mathfrak{i}^\perp \oplus \mathfrak{h}$. We have the decompositions $\mathfrak{a} = \mathfrak{w} \oplus \mathfrak{w}^\perp$ and $\mathfrak{i} = \mathfrak{i}^\perp \oplus \mathfrak{w}^\perp$.

Denote by $\lambda \in S^{n+1}\mathfrak{a}$ the derived potential of $\mathfrak{a}$. Since, $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{i}^\perp \oplus \mathfrak{w}^\perp$, we have:

$$S^{n+1}\mathfrak{a} = \bigoplus_{i+j+k=n+1} S^i(\mathfrak{h}) \cdot S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp).$$

Therefore, we have the following decomposition of the derived potential:

$$\lambda = \sum_{i+j+k=n+1} \lambda_{ijk}, \quad \text{where} \quad \lambda_{ijk} \in S^i(\mathfrak{h}) \cdot S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp).$$

Furthermore, for any $b \in \mathfrak{i}^\perp$ we have

$$[b, \lambda] \in \bigoplus_{i+j+k=n+1} S^{i-1}(\mathfrak{h}) \cdot S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp).$$

Since $\mathfrak{i}^\perp$ is an ideal we have

$$[a_1, \dots, [a_{n-1}, [b, \lambda]]] \in \mathfrak{i}^\perp \quad \text{for } a_p \in \mathfrak{a}.$$

Therefore, λ_{ijk} can be non-trivial only in the following two cases:

- $i = 0$;
- $i = 1$ and $k = 0$.

In other words, we get

$$\lambda \in \big(\mathfrak{h} \cdot S^n(\mathfrak{i}^\perp)\big) \oplus \Big(\bigoplus_{j+k=n+1} S^j(\mathfrak{i}^\perp) \cdot S^k(\mathfrak{w}^\perp)\Big). \tag{5}$$

We put

$$\mu := \lambda_{0,0,n+1} \in S^{n+1}(\mathfrak{w}^\perp); \quad \nu := \lambda_{1,n,0} \in \mathfrak{h} \cdot S^n(\mathfrak{i}^\perp);$$

$$\psi_i := \lambda_{0,i,n-i+1} \in S^i(\mathfrak{i}^\perp) \cdot S^{n+1-i}(\mathfrak{w}^\perp), \ \ i = 1, \dots, n+1;$$

$$\mathfrak{g} := \mathfrak{w}^\perp.$$

We also can identify $\mathfrak{i}^\perp$ with $\mathfrak{h}^*$. We see that $(\mathfrak{a}, \lambda)$ is a double extension of $(\mathfrak{g}, \mu)$ by $(\mathfrak{h}, \nu)$ via ψ_i. The result follows. ∎

Proposition 2.3. *The n-ary superalgebra $(\mathfrak{g}, \mu)$ is isomorphic to $\mathfrak{i}/\mathfrak{i}^\perp$.*

Proof. Indeed, assume that $a_i \in \mathfrak{i}$. Then

$$[a_1, \dots, [a_n, \nu]] = 0, \quad [a_1, \dots, [a_n, \psi_i]] \in \mathfrak{i}^\perp.$$

Hence,

$$\{\tilde{a}_1, \dots, \tilde{a}_n\}_{\mathfrak{i}/\mathfrak{i}^\perp} = \{a_1, \dots, a_n\}_{(\mathfrak{g},\mu)},$$

where $\tilde{a}_i$ is the image of a_i in $\mathfrak{i}/\mathfrak{i}^\perp$. ∎

Corollary 2.4. *Assume that $\mathfrak{a}$ is irreducible and not simple, $\mathfrak{i}$ is a maximal non-trivial ideal and $\mathfrak{h}$ is an isotropic subalgebra in $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{i} \oplus \mathfrak{h}$. Then $\mathfrak{a}$ is isomorphic to a certain generalized double extension with $\psi_n = \psi_{n+1} = 0$. In this case the generalized double extension is called* double extension.

Proof. Consider (5) holds. Let us take $x_i \in \mathfrak{h}$. The result follows from the following observations:

$$[x_1, \dots, [x_n, \mu + \sum_{i=1}^{n-1} \psi_i]] = 0$$

$$[x_1, \dots, [x_n, \nu]] \in \mathfrak{h}, \quad [x_1, \dots, [x_n, \psi_n]] \in \mathfrak{g}, \quad [x_1, \dots, [x_n, \psi_{n+1}]] \in \mathfrak{h}^*.$$

Since $\mathfrak{h}$ is a subalgebra, we have $\psi_n = \psi_{n+1} = 0$. The proof is complete. ∎

Corollary 2.5. *Assume that $\mathfrak{a}$ is an irreducible but not simple skew-symmetric invariant n-ary algebra and $\mathfrak{i}$ is a maximal non-trivial ideal of codimension 1. Then $\mathfrak{a}$ is isomorphic to a certain double extension with $\nu = \psi_i = 0$ for all $i \neq 1$.*

Proof. In this case the statement (5) has the following form:

$$\lambda \in \mathfrak{i}^\perp \cdot S^n(\mathfrak{w}^\perp) \oplus S^{n+1}(\mathfrak{w}^\perp). \ ∎$$

3. Lie algebras. In this section we show how to use our definition to obtain the notion of a generalized double extension for particular type of algebras: for Lie algebras. We also show that our definition coincides with the definition given in [MR] for Lie algebras.

DEFINITION 3.1. A *derivation* of an n-ary algebra (V,μ) is a linear map $D: V \to V$ such that

$$D(\{v_1,\ldots,v_n\}) = \sum_j \{v_1,\ldots,D(v_j),\ldots,v_n\}.$$

We denote by $\mathrm{IDer}(\mu)$ the vector space of all derivations of the algebra (V,μ) preserving the form $(\,,)$ on V. A proof of the following well-known proposition can be found for example in [V]:

PROPOSITION 3.2. *Let V be a vector space with a non-degenerate form $(\,,)$. Let us take any $\mu \in S^{n+1}(V)$. We have:*

$$\mathrm{IDer}(\mu) = \{w \in S^2(V) \mid ad_w(\mu) = 0\}$$

where $ad_w(\mu) := [w,\mu]$.

In [MR] the following theorem was proven:

THEOREM 3.3 (Medina, Revoy). *Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant symmetric form B. Let us take any Lie algebra $\mathfrak{h}$ with a homomorphism θ of $\mathfrak{h}$ to $\mathcal{D}er(\mathfrak{g})$ such that $\theta(\mathfrak{h})$ preserves the form B. Then $\mathfrak{d} = \mathfrak{h}^* \oplus \mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra with respect to the following multiplication:*

$$[(f_1,w_1,s_1),(f_2,w_2,s_2)] =$$
$$(ad^*(s_2)f_1 - ad^*(s_1)f_2 + \omega(w_1,w_2), [w_1,w_2] + \theta(s_1)(w_2) - \theta(s_2)(w_1), [s_1,s_2]).$$

Here $\omega(w_1,w_2)(s) := B(\theta(s)w_1,w_2)$. The Lie algebra $\mathfrak{d}$ possesses an invariant form given by the sum of B and the natural symmetric form on $\mathfrak{h}^* \oplus \mathfrak{h}$. (In [MR] the algebra $\mathfrak{d}$ is called a double extension *of $\mathfrak{g}$ by $\mathfrak{h}$.)*

Conversely, any Lie algebra with a non-degenerate invariant symmetric form can be obtained inductively by direct sums and double extensions.

We can simplify the definition of the double extension from [MR] using the derived bracket construction. Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant symmetric form. Consider $\mathfrak{g}$ as a pure odd vector space with a non-degenerate skew-symmetric form. In this section we will use the notation $\bigwedge^* W$ instead of S^*W for a pure odd vector space W. By Proposition 1.3, we can assign the derived potential μ to the Lie algebra structure on $\mathfrak{g}$. Further, let us take a Lie algebra $\mathfrak{h}$ with the multiplication $\nu \in \bigwedge^2 \mathfrak{h}^* \otimes \mathfrak{h}$. We have a non-degenerate symmetric form on $\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$: it is the sum of the form on $\mathfrak{g}$ and the natural non-degenerate symmetric form on $\mathfrak{h}^* \oplus \mathfrak{h}$. As above we denote by $[\,,]$ the corresponding Poisson bracket on $\bigwedge^*(\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g})$. (Again we can consider $\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$ as a pure odd vector space with a non-degenerate skew-symmetric form.)

There is a one-to-one correspondence between elements $\psi \in \mathfrak{h}^* \otimes \bigwedge^2 \mathfrak{g}$ and linear maps $\theta : \mathfrak{h} \to \bigwedge^2 \mathfrak{g} \simeq \mathfrak{so}(\mathfrak{g})$. This correspondence is given by $\psi \longmapsto \theta_\psi$, where $\theta_\psi(x) = [x,\psi]$.

THEOREM 3.4. *Let $\mathfrak{d} = \mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$, and $\mathfrak{g}$, $\mathfrak{h}$ and $\theta = \theta_\psi$ be a double extension of $\mathfrak{g}$ by $\mathfrak{h}$ via θ_ψ in the sense of Theorem 3.3. Then in terms of Proposition 1.3 the Lie algebra $\mathfrak{d}$*

has the derived potential

$$\mu + \nu + \psi,$$

and we have

$$[\mu + \nu + \psi, \mu + \nu + \psi] = 0. \tag{6}$$

Conversely, if the condition (6) holds then the Lie algebra $\mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{g}$ *is a double extension of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *via* θ_ψ *in the sense of Theorem 3.3.*

Proof. Assume that $\mathfrak{d}$ is a double extension of $\mathfrak{g}$ by $\mathfrak{h}$ via θ_ψ. An easy computation shows that the derived potential of $\mathfrak{d}$ is equal to $\mu+\nu+\psi$. Since $\mathfrak{d}$ is a Lie algebra, we have (6).

Conversely, let us take Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and an element ψ as above such that (6) holds. We need to show that $\theta = \theta_\psi$ is a homomorphism of $\mathfrak{h}$ to $\mathrm{Der}(\mathfrak{g})$ that preserves the form on $\mathfrak{g}$. Since $\mathfrak{g}$ and $\mathfrak{h} \oplus \mathfrak{h}^*$ are Lie algebras we have $[\mu, \mu] = 0$ and $[\nu, \nu] = 0$. Therefore, we also have

$$[\mu + \nu + \psi, \mu + \nu + \psi] = 2[\mu, \psi] + 2[\psi, \nu] + [\psi, \psi] = 0. \tag{7}$$

Note that $[\mu, \nu] = 0$. Since $[\mu, \psi] \in \mathfrak{h}^* \otimes \bigwedge^3 \mathfrak{g}$ and $[\psi, \nu], [\psi, \psi] \in \bigwedge^2 \mathfrak{h}^* \otimes \bigwedge^2 \mathfrak{g}$ we see that (7) is equivalent to

$$[\mu, \psi] = 0, \quad 2[\psi, \nu] + [\psi, \psi] = 0 \tag{8}$$

Let us show that

- from $[\mu, \psi] = 0$ it follows that $\theta(x)$ is a derivation of $\mathfrak{g}$ preserving $(\, ,)$;
- from $2[\psi, \nu] + [\psi, \psi] = 0$ it follows that the map θ is a homomorphism from $\mathfrak{h}$ to $\mathrm{Der}(\mathfrak{g})$

We have for any $x \in \mathfrak{h}$:

$$0 = [x, [\mu, \psi]] = [[x, \mu], \psi] - [\mu, [x, \psi]] = -[\mu, [x, \psi]] = -[\mu, \theta(x)].$$

Now we apply Proposition 3.2.

Let us study the second equation in (8). We need to show that the following holds:

$$\theta([x, [y, \nu]]) = [\theta(x), \theta(y)] \text{ for all } x, y \in \mathfrak{h}.$$

Indeed,

$$\theta([x, [y, \nu]]) = [\psi, [x, [y, \nu]]] = [[\psi, x], [y, \nu]] - [x, [[\psi, y], \nu]] + [x, [y, [\psi, \nu]]], \; x, y \in \mathfrak{h}.$$

Notice that $[\psi, x] \in \bigwedge^2 \mathfrak{g}$. Therefore, $[[\psi, x], [y, \nu]] = 0$. Similary, $[x, [[\psi, y], \nu]] = 0$. Therefore,

$$\theta([x, [y, \nu]]) = [x, [y, [\psi, \nu]]], \; x, y \in \mathfrak{h}.$$

Our statement follows from the following observation:

$$-\frac{1}{2}[x, [y, [\psi, \psi]]] = [x, [\psi, [y, \psi]]] = [[x, \psi], [y, \psi]] + 0 = [\theta(x), \theta(y)]. \; \blacksquare$$

A similar idea may be used to define a double extension for other types of invariant commutative algebras.

4. Solvable skew-symmetric invariant n-ary algebras. Define by induction the following subalgebras in $\mathfrak{a}$:

$$\mathfrak{a}^{(1)} := \{\mathfrak{a}, \ldots, \mathfrak{a}\}, \quad \mathfrak{a}^{(2)} := \{\mathfrak{a}^{(1)}, \ldots, \mathfrak{a}^{(1)}\}, \ldots$$

By induction, we see that $\mathfrak{a}^{(k+1)} \subset \mathfrak{a}^{(k)}$ and more precisely that $\mathfrak{a}^{(k+1)}$ is an ideal in $\mathfrak{a}^{(k)}$. We call an n-ary superalgebra $\mathfrak{a}$ *solvable* if there exists an integer K such that $\mathfrak{a}^{(K)} = \{0\}$.

PROPOSITION 4.1. *Subalgebras and homomorphic images of a solvable commutative n-ary superalgebra are solvable.*

Proof. Let $\mathfrak{b}$ be a subalgebra in a solvable n-ary superalgebra $\mathfrak{a}$. Then we see

$$\mathfrak{b}^{(1)} = \{\mathfrak{b}, \ldots, \mathfrak{b}\} \subset \{\mathfrak{a}, \ldots, \mathfrak{a}\} = \mathfrak{a}^{(1)}.$$

By induction we have $\mathfrak{b}^{(k)} \subset \mathfrak{a}^{(k)}$ for all k. Hence, $\mathfrak{b}$ is solvable.

Let $\phi : \mathfrak{a} \to \mathfrak{h}$ be a homomorphism. Denote by $\mathfrak{b}$ the image of ϕ. Then,

$$\mathfrak{b}^{(1)} = \{\phi(\mathfrak{a}), \ldots, \phi(\mathfrak{a})\} = \phi(\{\mathfrak{a}, \ldots, \mathfrak{a}\}) = \phi(\mathfrak{a}^{(1)}).$$

Again by induction we have $\mathfrak{b}^{(k)} = \phi(\mathfrak{a}^{(k)})$. Hence, $\mathfrak{b}$ is solvable. The proof is complete. ∎

All solvable skew-symmetric n-ary algebras can be obtained inductively by generalized double extensions via 1-dimensional superalgebras. Clealy, $\mathfrak{a}^{(1)} = \{\mathfrak{a}, \ldots, \mathfrak{a}\}$ is an ideal in $\mathfrak{a}$ and any proper subspace $\mathfrak{i}$ in $\mathfrak{a}$ such that $\mathfrak{a}^{(1)} \subset \mathfrak{i}$ is an ideal in $\mathfrak{a}$. Hence, there exists a maximal ideal of codimension 1. The result follows from Theorem 2.2. Since, $\mathfrak{g} \simeq \mathfrak{i}/\mathfrak{i}^{\perp}$, we see that $\mathfrak{g}$ is also solvable and we can repeat the procedure.

Acknowledgments. This research was supported by AFR-grant, University of Luxembourg and MPI Bonn.

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